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Chaos in a simple snap system with only one nonlinearity, its adaptive control and real circuit design

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We study an elegant snap system with only one nonlinear term, which is a quadratic nonlinearity. The snap system displays chaotic attractors, which are controlled easily by changing a system parameter. By using analysis, simulations and a real circuit, the dynamics of such a snap system has been investigated. We also investigate backstepping based adaptive control schemes for the new snap system with unknown parameters.

Key words: chaos, chaotic systems, hyperjerk systems, adaptive control, backstepping control, circuit design

1. Introduction

There is great interest in the chaos literature in the modelling of new chaotic systems with special properties [1, 2]. Chaotic systems are identified as nonlinear dynamical systems with characterizing properties such as sensitivity to initial conditions, topologically mixing and also with dense periodic orbits [3]. Chaotic systems have many applications in science such as lasers [4,5], neural networks [6, 7], robotics [8,9], oscillators [10–14], secure communications [15–17],

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weather systems [18,19], neurons [20], circuits [21–23], memristors [24], finance systems [25], etc.

A jerk differential equation in mechanics is defined as

$$\frac{\mathrm{d}^3 x}{\mathrm{d}t^3} = f\left(x,\ \frac{\mathrm{d}x}{\mathrm{d}t},\ \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\right). \tag{1}$$

To simplify notations, we use $D = \dfrac{\mathrm{d}}{\mathrm{d}t}$. Then the ODE (1) has the compact form

$$D^3 x = f\left(x,\ Dx,\ D^2 x\right). \tag{2}$$

The ODE (2) has a single variable x and a nonlinear function $f(.)$ (named the "jerk" [26,27].

Several jerk equations displaying chaotic behavior have been reported in the chaos literature [28–34].

A hyperjerk system can be defined by

$$D^n x = f\left(x, Dx, D^2 x, \ldots, D^{n-1} x\right) \tag{3}$$

for $n \geqslant 4$ [35]. In Eq. (3), $D^2 x = \dfrac{\mathrm{d}^2 x}{\mathrm{d}t^2}$, $D^3 x = \dfrac{\mathrm{d}^3 x}{\mathrm{d}t^3}$ and higher order derivatives can be similarly defined.

Especially, the hyperjerk system (3) for $n = 4$ is called a *snap system* [35]. Previous research has established that chaos has been discovered in snap systems [36].

Simple chaotic snap systems attracted considerable attention due to its simplicity [37,38]. Five elementary chaotic snap flows were found and studied in [38]. Bao et al. contructed a snap system by applying the concept of memory element [39]. Dalkiran and Sprott proposed a snap system with an exponential nonlinear function [37]. Dalkiran and Sprott used a Field-Programmable Analog Array (FPAA) to realize the hyperjerk system with exponential term.

Finding new simple snap systems with chaotic behavior is still an attractive topic in research because chaos has been applied in steganography [40], random number generator [41], multiuser communication [42], and cryptosystems [43–48], etc. Moreover, further studies should be done about hyperjerk systems with quadratic nonlinear terms because they are elegant and possess algebraic simplicity [35].

A novel snap system with only one nonlinearity is studied in our work. Section 2 presents dynamics of the snap system. In Sec. 3, we realize the simple snap system in a real circuit. Adaptive control of the snap system via backstepping control method is detailed in Sec. 4. We report the adaptive synchronization of the identical snap systems via backstepping control method in Sec. 5. We remark noticeable conclusions in Sec. 6.

2. The snap system and its dynamics

2.1. The snap system with only one nonlinearity

In order to design a new snap system, we consider the following hyperjerk equation

$$f\left(x, Dx, D^2x, D^3x\right) = -ax - Dx - D^3x + bxD^2x. \tag{4}$$

It is simple to observe that the proposed hyperjerk function f consists of just one quadratic nonlinear term.

We define new phase variables as

$$x_1 = x, \quad x_2 = Dx, \quad x_3 = D^2x, \quad x_4 = D^3x. \tag{5}$$

As a result, we propose the new snap system:

$$\begin{cases} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= -ax_1 - x_2 - x_4 + bx_1x_3. \end{cases} \tag{6}$$

Here a, b are positive parameters.

The proposed system (6) includes only one quadratic nonlinear term (x_1x_3).

It is simple to confirm that $E_0 = \mathbf{0}$ is a single equilibrium of the snap system (6).

Interestingly, for the set of parameters $a = 0.2, b = 1$, the calculated Lyapunov exponents of system (6) are $L_1 = 0.0849$, $L_2 = 0$, $L_3 = -0.3014$ and $L_4 = -0.7834$ (for initial conditions $(x_1(0), x_2(0), x_3(0), x_4(0)) = (0, 0.1, 0, 0)$). Therefore, snap system (6) generates chaotic attractor as presented in Fig. 1.

2.2. Equilibrium point analysis

We start with the linearization matrix of the snap system (6) at $E_0 = \mathbf{0}$:

$$J_E = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a & -1 & 0 & -1 \end{bmatrix}. \tag{7}$$

Therefore, the spectrum of the system (6) at E is characterized by the spectral equation

$$|\lambda I_d - J_E| = \lambda^4 + \lambda^3 + \lambda + a = 0. \tag{8}$$

For the chaotic case, $(a, b) = (0.2, 1)$, the spectral equation (8) takes the particular form

$$\lambda^4 + \lambda^3 + \lambda + 0.2 = 0 \tag{9}$$

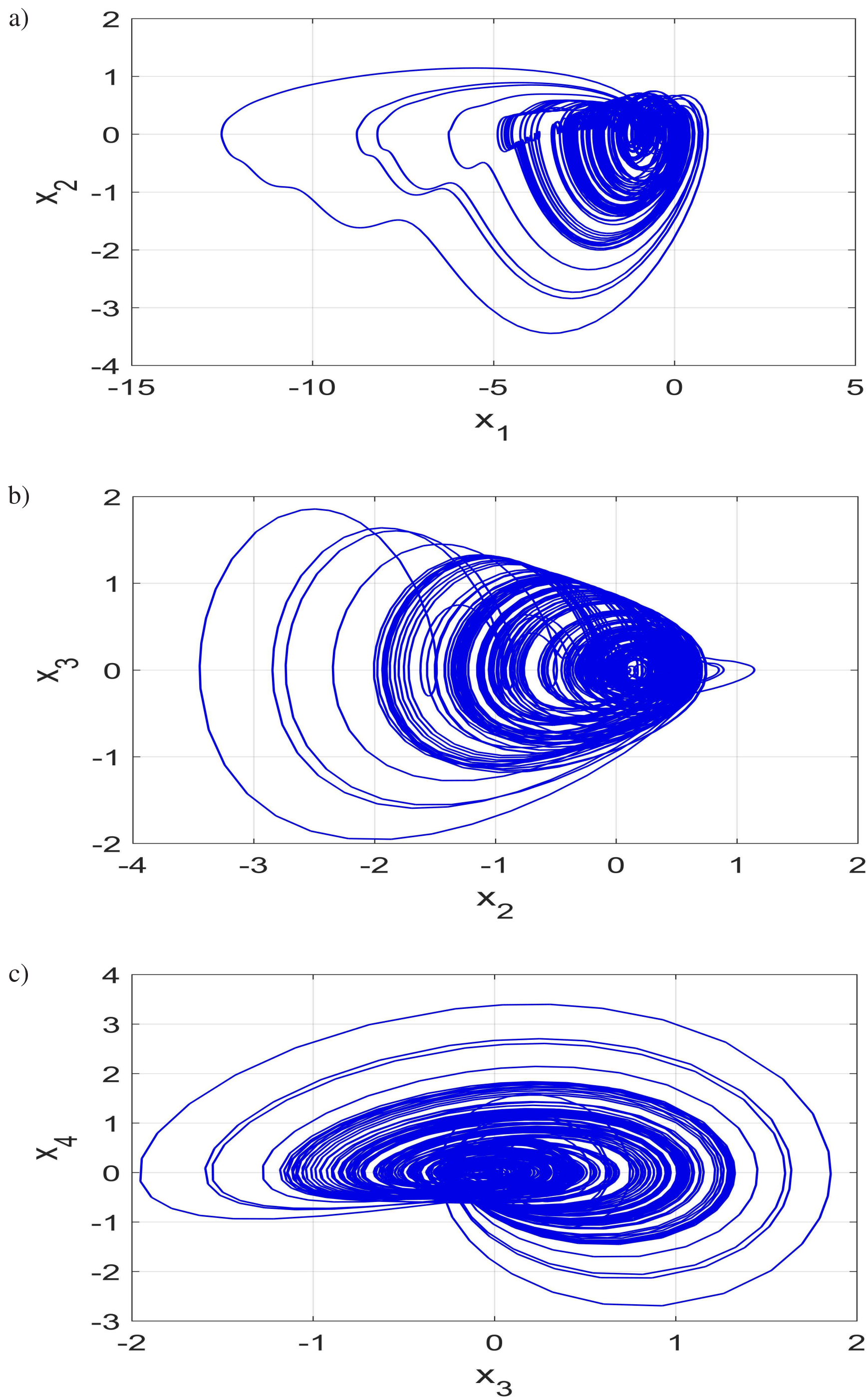

Figure 1: Phase portraits of the snap system for the set of parameters $(a, b) = (0.2, 1)$ and the initial conditions $\mathbf{x}(0) = (0, 0.1, 0, 0)$

which yields the spectral values as

$$\lambda_1 = -1.4239, \quad \lambda_2 = -0.1941, \quad \lambda_{3,4} = 0.309 \pm 0.7925i. \tag{10}$$

We determine that the rest point E_0 is an unstable point with saddle-focus type.

2.3. Dissipativity

In order to check the dissipativity of the snap system (6), we compute

$$\nabla V = \frac{\partial \dot{x}_1}{\partial x_1} + \frac{\partial \dot{x}_2}{\partial x_2} + \frac{\partial \dot{x}_3}{\partial x_3} + \frac{\partial \dot{x}_4}{\partial x_4} = -1. \tag{11}$$

Hence, it is quite straightforward to confirm that the snap system (6) is dissipative.

2.4. Dynamics of the snap system

Different dynamics have been observed in system (6) when varying the parameter a. As can be seen from the bifurcation diagram (Fig. 2) and Lyapunov exponents (Fig. 3), system (6) exhibits periodic and chaotic dynamics.

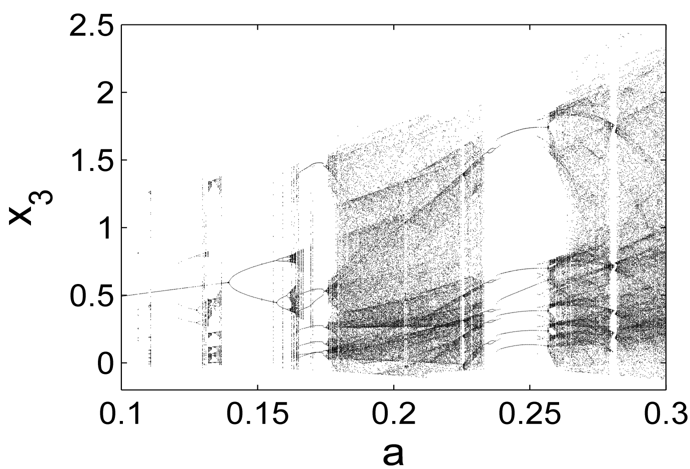

Figure 2: Bifurcation diagram of system (6) for $a \in [0.1, 0.3]$ and $b = 1$

Moreover, we can take the parameter b as a control parameter for varying the size of attractor [49–51]. As illustrated in Fig. 4, chaotic attractors are adjusted by changing b.

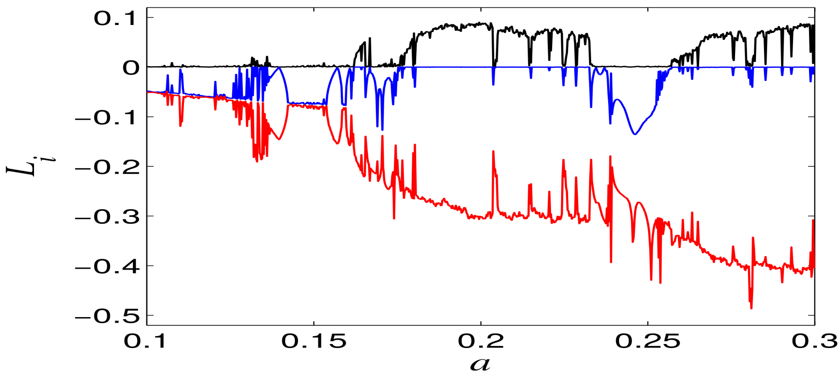

Figure 3: Calculation of Lyapunov exponents L_1 (black color), L_2 (blue color) and L_3 (red color) of snap system (6) for $a \in [0.1, 0.3]$ and $b = 1$

3. Circuit of the simple snap system

In chaos applications, a vital topic is the hardware realization of mathematical models [52, 53]. We unveil an electronic circuit for implementing the proposed snap system (6) in this section.

Figure 5 illustrates the schematic of the circuit for the snap system (6), which is based on four integrators ($U_1, \ldots, U_4$).

Equations of the designed circuit of Fig. 5 are

$$\begin{cases} \dot{x}_1 = \dfrac{1}{RC} x_2, \\ \dot{x}_2 = \dfrac{1}{RC} x_3, \\ \dot{x}_3 = \dfrac{1}{RC} x_4, \\ \dot{x}_4 = \dfrac{1}{RC} \left(-\dfrac{R}{R_a} x_1 - x_2 - x_4 + \dfrac{R}{R_b} x_1 x_3 \right), \end{cases} \tag{12}$$

where the variables x_1, x_2, x_3 and x_4 are the voltages of the integrators.

The circuit has been realized on a breadboard for $R = 10$ kΩ, $R1 = 90$ kΩ, $Rb = 50$ kΩ, and $C = 10$ nF. A digital oscilloscope (HMO1002 of ROHDE & SCHWARZ) for observing the measurements and capturing the experimental results has been used.

In this way, the experimental phase portraits of circuit's behavior, for the same values of the parameters a, b as in the corresponding phase portraits of Fig. 1, are depicted as shown in Fig. 6. As expected, the obtained experimental results confirm the feasibility of the proposed snap system (6).

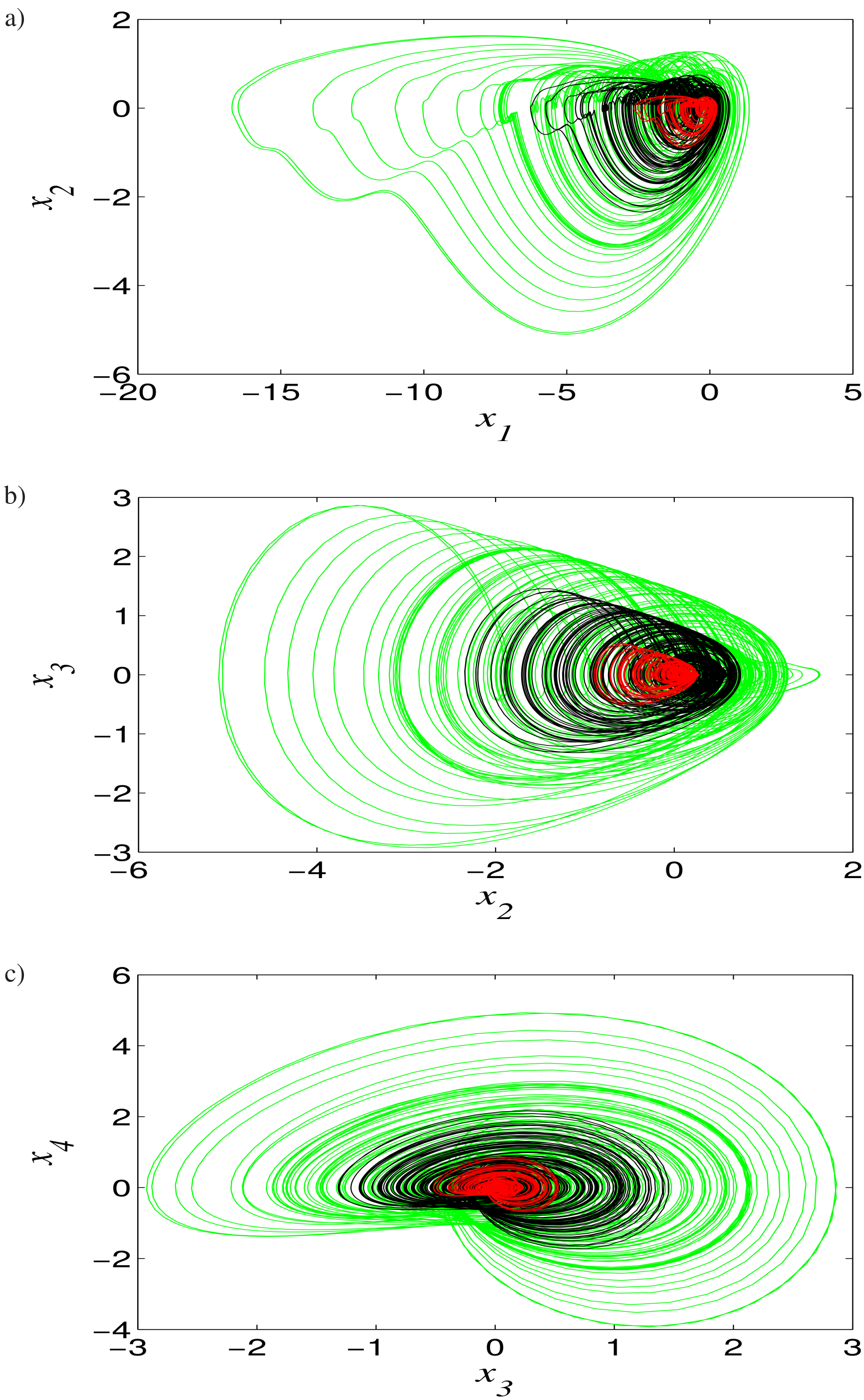

Figure 4: Phase portraits when changing the parameter b: green color for $b = 0.6$, black color for $b = 1$, red color for $b = 2$

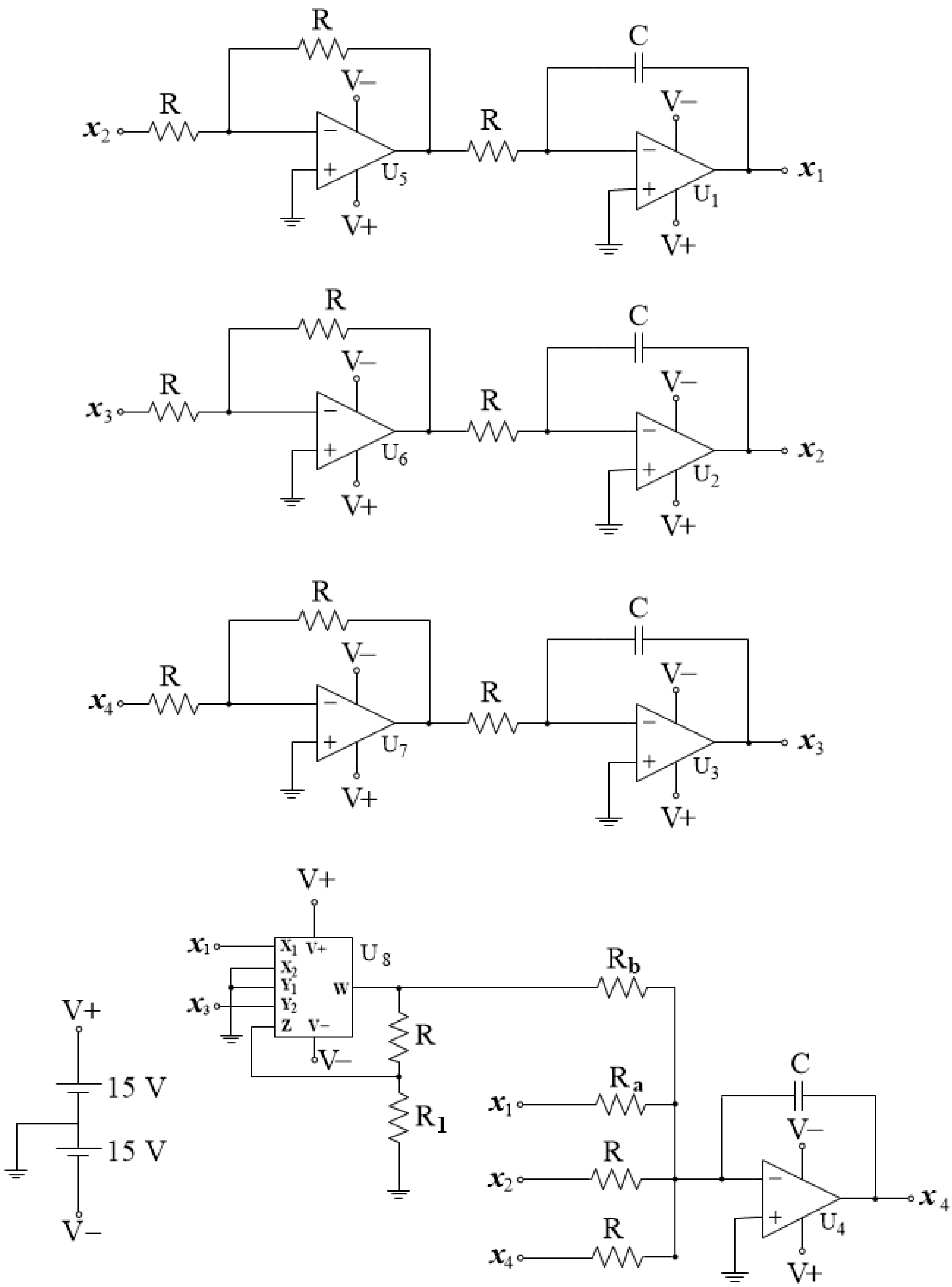

Figure 5: The circuit designed for the snap system (6)

a)
b)
c)

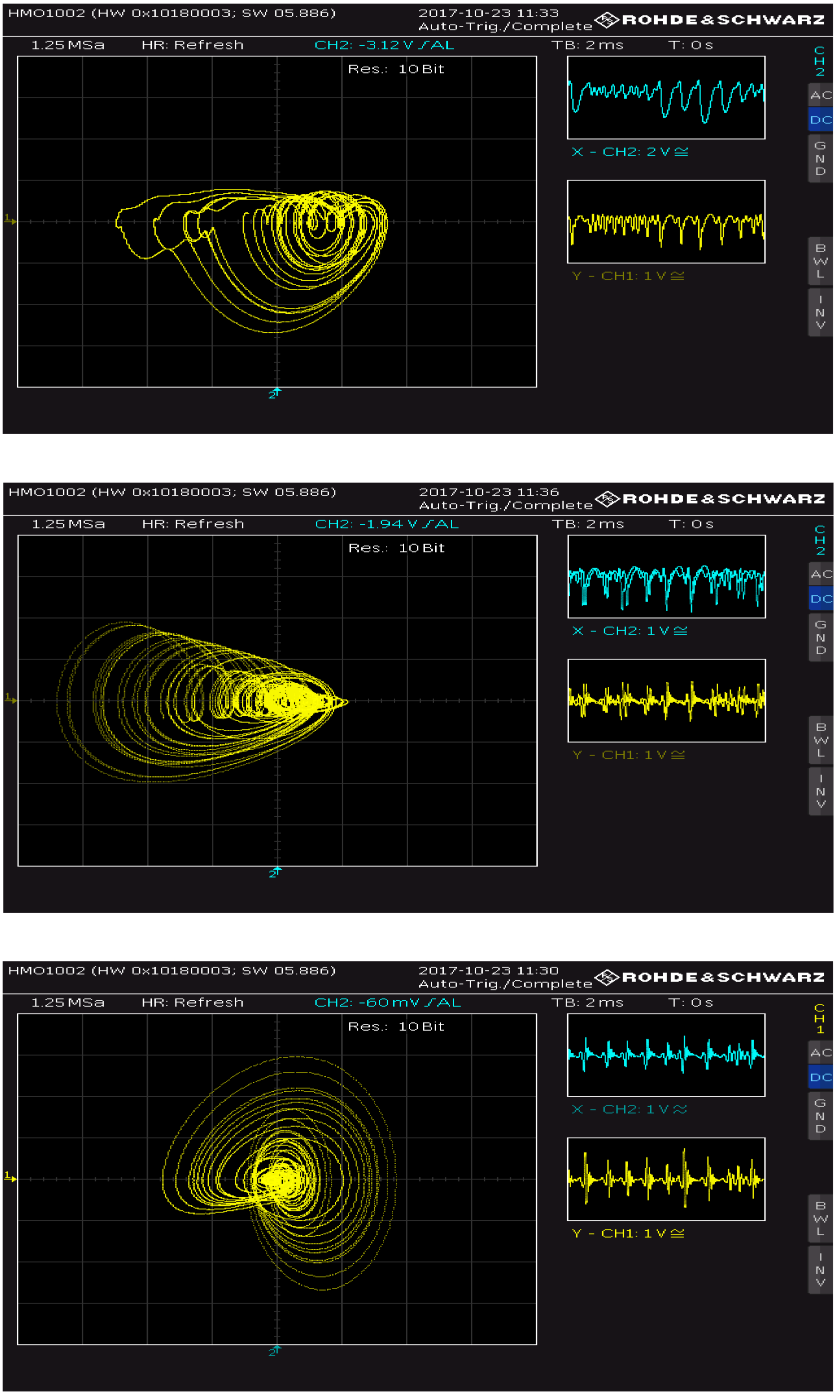

Figure 6: Experimental observation of circuit's chaotic behavior in different phase portraits

4. Control the simple snap system

The literature on chaos has highlighted the complex dynamics of chaotic systems, in which a tiny change of initial conditions leads completely different trajectories [54, 55]. Scientists developed many deterministic schemes to control chaotic dynamical systems [56]. In this section, we stabilize the new snap system via backstepping control method.

We take the new snap system with a single feedback control given by

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = -ax_1 - x_2 - x_4 + bx_1x_3 + u, \end{cases} \tag{13}$$

where a, b are unknown parameters. The control law u uses a time-varying parameter estimate $(A(t), B(t))$ in lieu of (a, b).

We have the parameter estimation errors:

$$e_a(t) = a - A(t), \qquad e_b(t) = b - B(t). \tag{14}$$

From (14), we get:

$$\dot{e}_a = \dot{A}, \qquad \dot{e}_b = \dot{B}. \tag{15}$$

Then, the main adaptive control result of this section is established.

Theorem 1 *The new snap system (13)is stabilized via the following feedback control law,*

$$u = -[5 - A(t)]x_1 - 9x_2 - 9x_3 - 3x_4 - B(t)x_1x_3 - Kz_4 \tag{16}$$

with a positive constant K,

$$z_4 = 3x_1 + 5x_2 + 3x_3 + x_4 \tag{17}$$

and the dynamics to update the parameter estimates as

$$\begin{cases} \dot{A}(t) = -z_4x_1, \\ \dot{B}(t) = z_4x_1x_3. \end{cases} \tag{18}$$

Proof. This result is proved by applied Lyapunov stability theory [57].

In backstepping control method, we start with a quadratic Lyapunov function

$$V_1(z_1) = \frac{1}{2} z_1^2, \tag{19}$$

where

$$z_1 = x_1 \,. \tag{20}$$

Differentiating V_1 along (13), we find that

$$\dot{V}_1 = z_1 \dot{z}_1 = x_1 x_2 = -z_1^2 + z_1(x_1 + x_2). \tag{21}$$

We set

$$z_2 = x_1 + x_2, \tag{22}$$

The equation (21) can be simplified as

$$\dot{V}_1 = -z_1^2 + z_1 z_2 \,. \tag{23}$$

Next, we take a quadratic Lyapunov function

$$V_2(z_1, z_2) = V_1(z_1) + \frac{1}{2} z_2^2 = \frac{1}{2}\left(z_1^2 + z_2^2\right). \tag{24}$$

Differentiating V_2 along (13), we find that

$$\dot{V}_2 = -z_1^2 - z_2^2 + z_2(2x_1 + 2x_2 + x_3). \tag{25}$$

We set

$$z_3 = 2x_1 + 2x_2 + x_3, \tag{26}$$

Using (26), the equation (25) can be simplified as

$$\dot{V}_2 = -z_1^2 - z_2^2 + z_2 z_3 \,. \tag{27}$$

Next, we take a quadratic Lyapunov function

$$V_3(z_1, z_2, z_3) = V_2(z_1, z_2) + \frac{1}{2} z_3^2 = \frac{1}{2}\left(z_1^2 + z_2^2 + z_3^2\right). \tag{28}$$

Differentiating V_3 along the dynamics (13), we find that

$$\dot{V}_3 = -z_1^2 - z_2^2 - z_3^2 + z_3(3x_1 + 5x_2 + 3x_3 + x_4). \tag{29}$$

We set

$$z_4 = 3x_1 + 5x_2 + 3x_3 + x_4 \,. \tag{30}$$

Using (30), the equation (29) can be simplified as

$$\dot{V}_2 = -z_1^2 - z_2^2 - z_3^2 + z_3 z_4 \,. \tag{31}$$

Finally, we set the quadratic Lyapunov function

$$V(z_1, z_2, z_3, z_4, e_a, e_b) = V_3(z_1, z_2, z_3) + \frac{1}{2}z_4^2 + \frac{1}{2}e_a^2 + \frac{1}{2}e_b^2 . \tag{32}$$

Differentiating V along (13), we get

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 - z_4^2 + z_4(z_4 + z_3 + \dot{z}_4) - e_a\dot{A} - e_b\dot{B}. \tag{33}$$

The equation (33) can be expressed in a compact manner as

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 - z_4^2 + z_4 S - e_a\dot{A} - e_b\dot{B}, \tag{34}$$

where

$$S = z_4 + z_3 + \dot{z}_4 = z_4 + z_3 + 3\dot{x}_1 + 5\dot{x}_2 + 3\dot{x}_3 + \dot{x}_4 . \tag{35}$$

It is easy to see that

$$S = (5-a)x_1 + 9x_2 + 9x_3 + 3x_4 + bx_1x_3 + u. \tag{36}$$

We substitute the adaptive control law (16) into (36) and get

$$S = -[a - A(t)]x_1 + [b - B(t)]x_1x_3 - Kz_4 . \tag{37}$$

Using (15), it is easy to simplify (37) as

$$S = -e_a x_1 + e_b x_1 x_3 - K z_4 . \tag{38}$$

Combining (38) and (34), we find that we obtain

$$\begin{aligned} \dot{V} = -z_1^2 - z_2^2 - z_3^2 - (1+K)z_4^2 + e_a\left[-z_4x_1 - \dot{A}\right] \\ + e_b\left[z_4x_1x_3 - \dot{B}\right]. \end{aligned} \tag{39}$$

It is simple to confirm that $\dot{V}$ is a negative semi-definite because of

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 - (1+K)z_4^2 . \tag{40}$$

Using Barbalat's lemma [57], it verifies that $\mathbf{z}(t) \to \mathbf{0}$ asymptotically as times goes to infinity for all initial conditions $\mathbf{z}(0)$ in $\mathbf{R}^4$.

As a consequence, it confirms that $\mathbf{x}(t)$ converges to $\mathbf{0}$ asymptotically as $t \to \infty$ for all values of $\mathbf{x}(0) \in \mathbf{R}^4$. ■

For our numerical example, we suppose that the parameters of the new snap system (13) are taken as $(a, b) = (0.2, 1)$, a chaotic case.

We choose the new snap system (13) with initial conditions $x_1(0) = 3.5$, $x_2(0) = 2.9$, $x_3(0) = 1.3$ and $x_4(0) = 4.8$.

Th initial conditions of the parameter estimates are $A(0) = 12.4$ and $B(0) = 9.5$. In addition we take the parameter $K = 25$.

The asymptotic convergence of the controlled state $\mathbf{x}(t)$ is exhibited in Fig. 7, when the controls (16) and (18) are implemented.

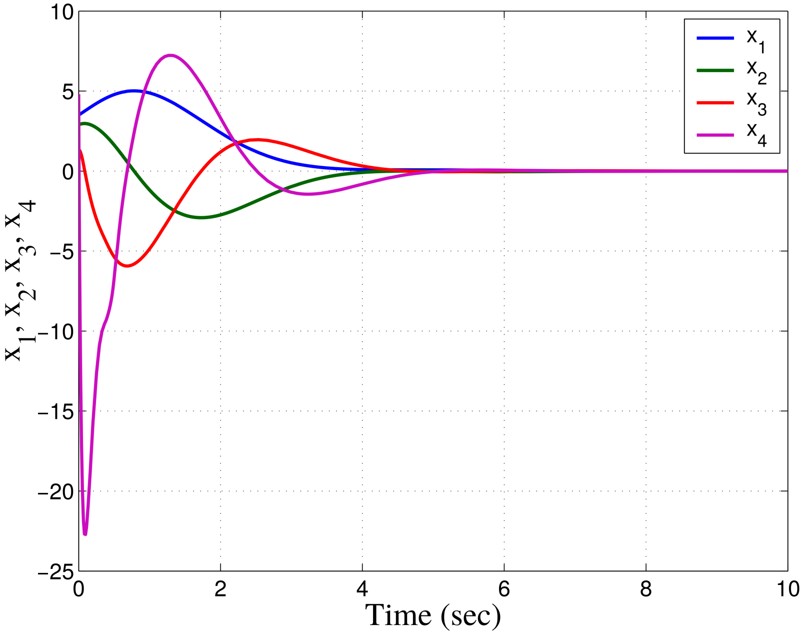

Figure 7: The states of the simple snap system (13) when are controlled

5. Synchronization of the identical snap systems

Chaos synchronization plays a vital role in chaos studies [58–60]. In this section, for synchronization between a pair of identical new snap systems (*master* and *slave* systems), we use backstepping control method.

We consider the master and slave snap systems (41), (42):

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = -ax_1 - x_2 - x_4 + bx_1x_3; \end{cases} \tag{41}$$

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = y_3, \\ \dot{y}_3 = y_4, \\ \dot{y}_4 = -ay_1 - y_2 - y_4 + by_1y_3 + u. \end{cases} \tag{42}$$

Here a, b are unknown parameters and u is an adaptive feedback control law which uses a time-varying parameter estimate $(A(t), B(t))$ in lieu of (a, b).

The complete synchronization error is

$$e_i(t) = y_i(t) - x_i(t) \qquad (i = 1,\ 2,\ 3,\ 4). \tag{43}$$

The synchronization error dynamics for the snap systems is obtained as follows:

$$\begin{cases} \dot{e}_1 &= e_2, \\ \dot{e}_2 &= e_3, \\ \dot{e}_3 &= e_4, \\ \dot{e}_4 &= -ae_1 - e_2 - e_4 + b(y_1 y_3 - x_1 x_3) + u. \end{cases} \tag{44}$$

We get the parameter estimation errors:

$$e_a(t) = a - A(t), \qquad e_b(t) = b - B(t). \tag{45}$$

It is easy to see that

$$\dot{e}_a = \dot{A}, \qquad \dot{e}_b = \dot{B}. \tag{46}$$

Then, the main synchronization result is presented.

Theorem 2 *The new simple snap systems (41) and (42) are synchronized via the control law,*

$$u = -[5 - A(t)]e_1 - 9e_2 - 9e_3 - 3e_4 - B(t)(y_1 y_3 - x_1 x_3) - Kz_4 \tag{47}$$

with a positive constant K,

$$z_4 = 3e_1 + 5e_2 + 3e_3 + e_4 \tag{48}$$

and the following dynamics to update the parameter estimates

$$\begin{cases} \dot{A}(t) &= -z_4 e_1, \\ \dot{B}(t) &= z_4(y_1 y_3 - x_1 x_3). \end{cases} \tag{49}$$

Proof. By using Lyapunov stability theory [57], this result is proved.

In backstepping control method, we start with a quadratic Lyapunov function

$$V_1(z_1) = \frac{1}{2} z_1^2, \tag{50}$$

where

$$z_1 = e_1\,. \tag{51}$$

Differentiating V_1 along (44), we find that

$$\dot{V}_1 = z_1 \dot{z}_1 = e_1 e_2 = -z_1^2 + z_1(e_1 + e_2). \tag{52}$$

We set

$$z_2 = e_1 + e_2\,. \tag{53}$$

The equation (52) can be simplified as

$$\dot{V}_1 = -z_1^2 + z_1 z_2\,. \tag{54}$$

Next, we take a quadratic Lyapunov function

$$V_2(z_1, z_2) = V_1(z_1) + \frac{1}{2}\,z_2^2 = \frac{1}{2}\left(z_1^2 + z_2^2\right). \tag{55}$$

Differentiating V_2 along (44), we find that

$$\dot{V}_2 = -z_1^2 - z_2^2 + z_2(2e_1 + 2e_2 + e_3). \tag{56}$$

We set

$$z_3 = 2e_1 + 2e_2 + e_3\,. \tag{57}$$

Using (57), the equation (56) can be simplified as

$$\dot{V}_2 = -z_1^2 - z_2^2 + z_2 z_3\,. \tag{58}$$

Next, we take a quadratic Lyapunov function

$$V_3(z_1, z_2, z_3) = V_2(z_1, z_2) + \frac{1}{2}\,z_3^2 = \frac{1}{2}\left(z_1^2 + z_2^2 + z_3^2\right). \tag{59}$$

Differentiating V_3 along the dynamics (44), we find that

$$\dot{V}_3 = -z_1^2 - z_2^2 - z_3^2 + z_3(3e_1 + 5e_2 + 3e_3 + e_4). \tag{60}$$

We set

$$z_4 = 3e_1 + 5e_2 + 3e_3 + e_4\,. \tag{61}$$

Using (61), the equation (60) can be simplified as

$$\dot{V}_2 = -z_1^2 - z_2^2 - z_3^2 + z_3 z_4\,. \tag{62}$$

Finally, the quadratic Lyapunov function is selected as a positive definite function on $\mathbf{R}^6$:

$$V(z_1, z_2, z_3, z_4, e_a, e_b) = V_3(z_1, z_2, z_3) + \frac{1}{2}z_4^2 + \frac{1}{2}e_a^2 + \frac{1}{2}e_b^2\,. \tag{63}$$

We differentiate V along (44):

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 - z_4^2 + z_4(z_4 + z_3 + \dot{z}_4) - e_a\dot{A} - e_b\dot{B}. \tag{64}$$

The equation (64) can be expressed in a compact manner as

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 - z_4^2 + z_4 S - e_a \dot{A} - e_b \dot{B}, \tag{65}$$

where

$$S = z_4 + z_3 + \dot{z}_4 = z_4 + z_3 + 3\dot{e}_1 + 5\dot{e}_2 + 3\dot{e}_3 + \dot{e}_4 \, . \tag{66}$$

It is easy to see that

$$S = (5 - a)e_1 + 9e_2 + 9e_3 + 3e_4 + b(y_1 y_3 - x_1 x_3) + u. \tag{67}$$

We substitute the adaptive control law (47) into (67) and get:

$$S = -[a - A(t)]e_1 + [b - B(t)](y_1 y_3 - x_1 x_3) - K z_4 \, . \tag{68}$$

Using (46), it is easy to simplify (68) as

$$S = -e_a e_1 + e_b (y_1 y_3 - x_1 x_3) - K z_4 \, . \tag{69}$$

Combining (69) and (65), we find that we obtain

$$\begin{aligned} \dot{V} = {} & -z_1^2 - z_2^2 - z_3^2 - (1 + K) z_4^2 + e_a \left[-z_4 e_1 - \dot{A} \right] \\ & + e_b \left[z_4 (y_1 y_3 - x_1 x_3) - \dot{B} \right] . \end{aligned} \tag{70}$$

Substituting (49) into (70), $\dot{V}$ is a negative semi-definite function on $\mathbf{R}^6$:

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 - (1 + K) z_4^2 \, . \tag{71}$$

Barbalat's lemma [57] confirms that $\mathbf{z}(t) \to \mathbf{0}$ asymptotically as times goes to infinity for all initial conditions $\mathbf{z}(0)$ in $\mathbf{R}^4$.

As a consequence, it verifies that $\mathbf{e}(t) \to \mathbf{0}$ asymptotically as $t \to \infty$ for all initial conditions $\mathbf{e}(0) \in \mathbf{R}^4$. ■

For our numerical example, we suppose that the parameters of the new snap systems are $(a, b) = (0.2, 1)$, the chaotic case.

We take initial conditions (72) for the master snap system (41):

$$x_1(0) = 1.2, \quad x_2(0) = 2.5, \quad x_3(0) = -4.3, \quad x_4(0) = 3.7. \tag{72}$$

For the slave snap system (42), initial conditions (73) are taken:

$$y_1(0) = 2.8, \quad y_2(0) = -4.2, \quad y_3(0) = 1.4, \quad y_4(0) = 2.5. \tag{73}$$

In addition, we select $A(0) = 12.7$, $B(0) = 6.3$ and $K = 25$.

The complete asymptotic synchronization of corresponding states is indicated in Figs. 8–11. The time-history of the complete synchronization error is displayed in Fig. 12.

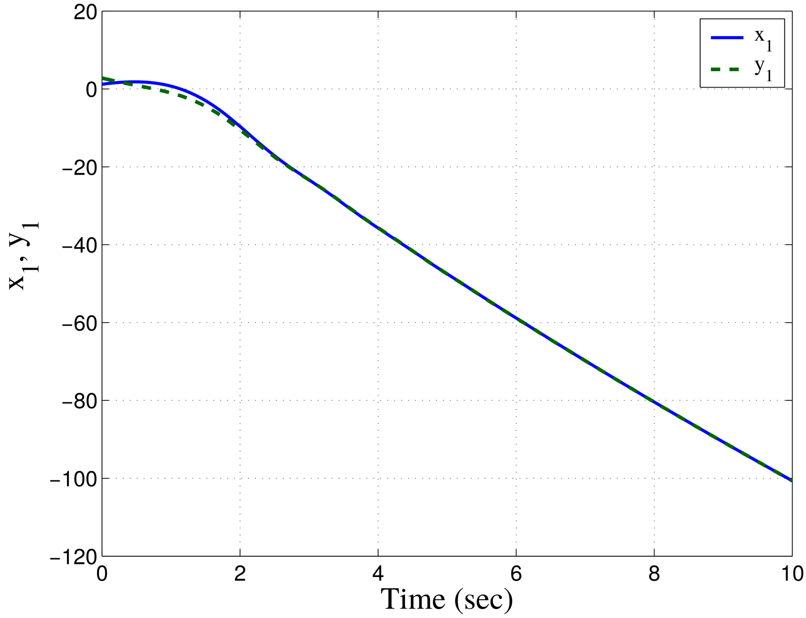

Figure 8: Simulation showing the asymptotic synchronization of the state variables x_1 and y_1 of the snap systems (41) and (42)

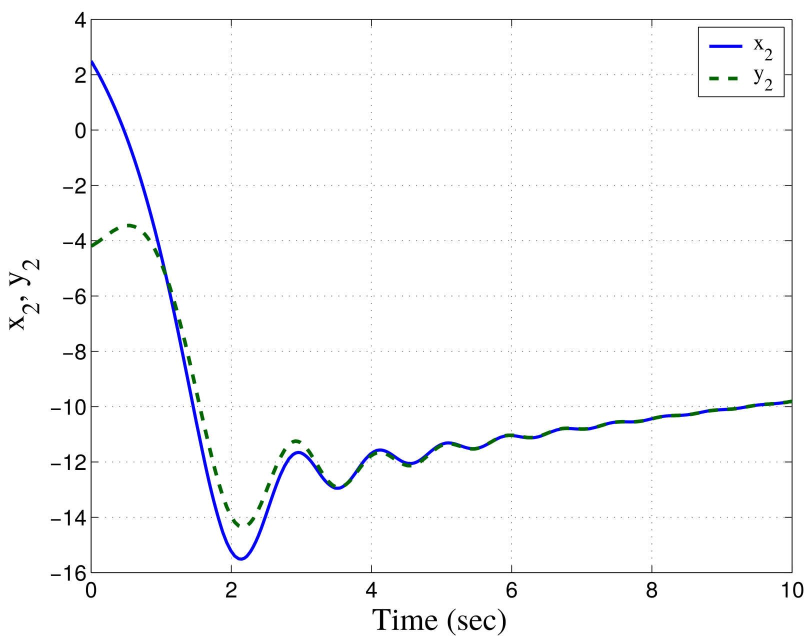

Figure 9: Simulation showing the asymptotic synchronization of the state variables x_2 and y_2 of the snap systems (41) and (42)

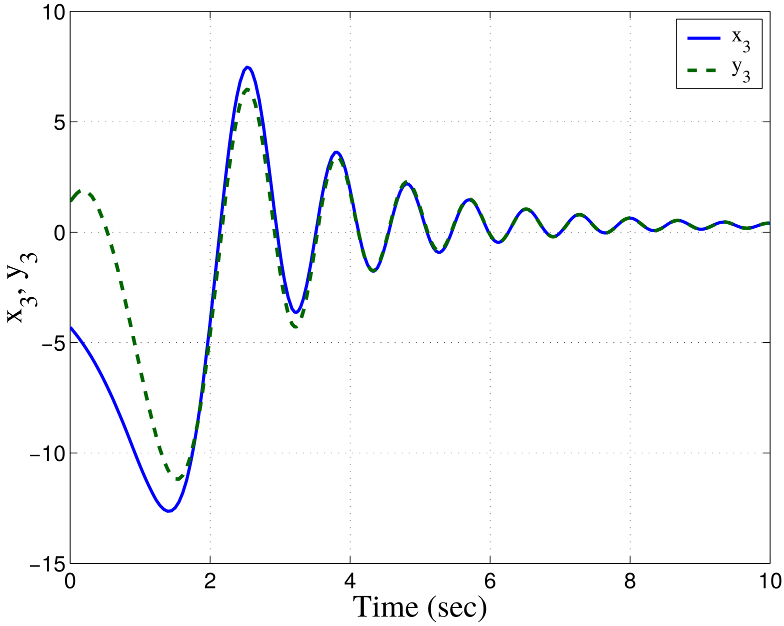

Figure 10: Simulation showing the asymptotic synchronization of the state variables x_3 and y_3 of the snap systems (41) and (42)

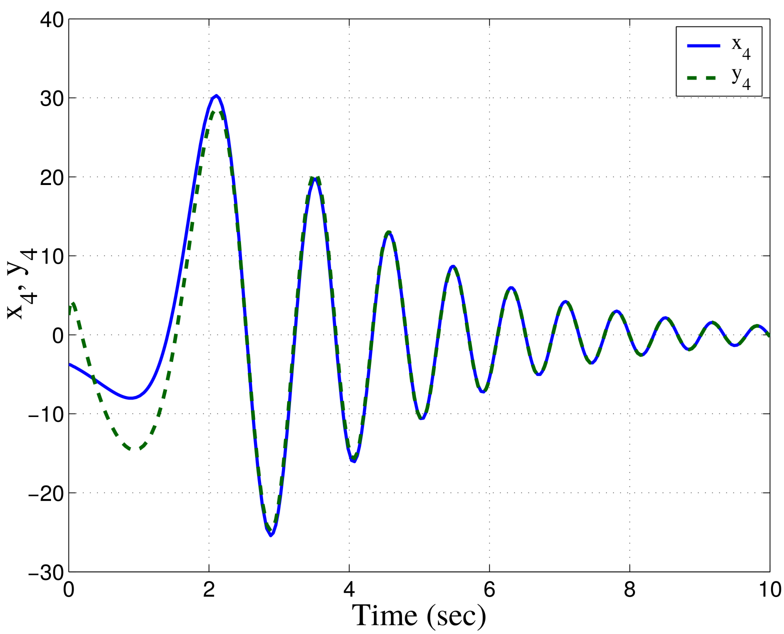

Figure 11: Simulation showing the asymptotic synchronization of the state variables x_4 and y_4 of the snap systems (41) and (42)

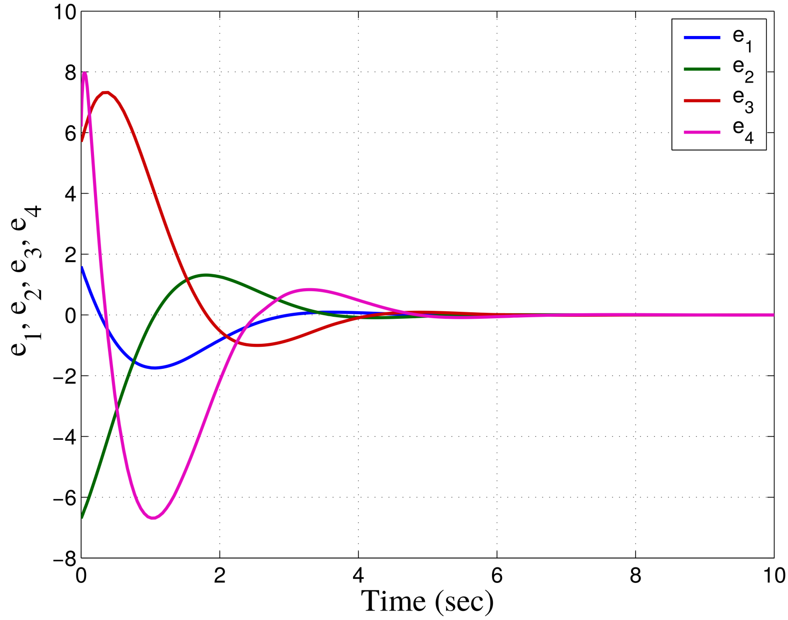

Figure 12: Simulation showing the synchronization error between the snap systems (41) and (42)

6. Conclusion

Great interest in investigation of hyperjerk systems having minimum number of nonlinearities and exhibiting chaos has witnessed recently. Our work derived a simple four-dimensional hyperjerk system, which is also known as a snap system, with only one nonlinear term. Explicitly, we used just a quadratic nonlinearity in our model of the new snap system. We showed that the snap-system displayed chaotic attractors, which are controlled easily by changing a system parameter. We presented detailed analysis, phase portraits, simulations and a real circuit of the new snap system. Control applications for control and synchronization of the new snap system were reported. Snap systems are simple and elegant because they describe the time evolution of a single scalar variable. Many four-dimensional mechanical systems can be conveniently presented as a snap system. Chaotic behaviors generated from snap systems have many potential applications in robotics, cryptosystems and chaos-based communication systems.

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