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Combined modified function projective synchronization of different systems through adaptive control

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This work describes a new study to achieve a combination of modified function projective synchronization between three different chaotic systems through adaptive control. Using the Lyapunov function theory, the asymptotic stability of the error dynamics is obtained and discussed. Further, we set some appropriate initial conditions for the state variables and assigning specific values to the parameters and obtain the graphical results, which shows the efficiencies of the new method. Finally, we summarized our work with conclusion and references.

Key words: Modified Function Projective Synchronization, Novel Chaotic System, Four-Scroll Chaotic System, adaptive control

1. Introduction

The phenomenon of synchronization is an active research field in science and engineering, this include, secure communication, digital secure communication, design information process and chemical reactions, plasma technologies, information science, biological systems, etc. [1–12]. Besides this, it is also applied in physics, and particularly, in secure communication. A number of synchronization are investigated, for example, anti and complete synchronization [13, 14], projective and generalized synchronization [7, 8, 15, 16], lag synchronization [17], partial synchronization [18] etc. In all of these, the projective synchronization is regarded one of the most hot research area. It characterized both drive and response systems synchronization up to a constant scaling factor. The author's in [19], for the first time introduced the idea of the projective synchronization.

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Moreover, on function projective synchronization (FPS), many ideas have been presented by the researchers, see for example [20, 21]. It is proven up to a scaling function the synchronization of drive and response systems in the above mentioned work.

In recent decades, a modified projective synchronization (MPS) has been introduced [22, 23]. It is shown by the researchers that the chaos systems up to a constant scaling matrix could be synchronized. Most recently, a new synchronization method has emerged known as modified function projective synchronization (MFPS), that generalizes FPS and MPS [24,25], where the scaling function matrix has been considered, respectively, by a constant matrix as well as scaling function. For security purposes, this scaling matrix can be used, which will improve the secure communication. Thus, it is important to study further, the modified function projective synchronization.

A typical chaotic synchronization consists of one drive and one response system; however, for a secure communication, a stronger anti-attack ability system is necessary, so, for this purpose, a new method (combined synchronization) was proposed by Luo *et al.* in 2011 [26]. In this new method, there is one response system while having more than two drive systems [27, 28]. In recent years, the idea of (CS) and (CPS) has been extended further by many researchers [29].

Motivated by the above studies, we use an adaptive control method to investigate the combined modified function projective synchronization with two drive and one response systems. A detail discussion on the previous work has been presented in section 1. The remaining work is organized is as follows: Systems and their description is given in section 2. Section 3, is devoted to the basic definitions associated to combined modified function projective synchronization. We briefly discuss the stability result for the synchronization of chaotic systems by the modified combined function projective by using the Lyapunov function theory, in section 4. To illustrate the results, numerical experiments in the form of graphs are given in section 5 while the work is concluded in section 6.

2. Description of the systems

Here, we provide some systems which are chaotic for the specified set of parameters. In order to discuss each system with details and the parameters that the system could be a chaotic attractor is discussed below one by one. The set of differential equation that describes a novel chaotic system is given in [30], which are the following:

$$\dot{l} = \left(-a + \frac{1}{b}\right) l + lm + n, \qquad \dot{m} = -bm - l^2, \qquad \dot{n} = -l - cn. \tag{1}$$

In equation (1), the state vectors are given by l, m and n whereas c, a and b are the associated parameters. The given system (1) for the set of values $a = 2$, $b = 0.1$ and $c = 1$, is a chaotic system. This fact is illustrated in Fig. 1.

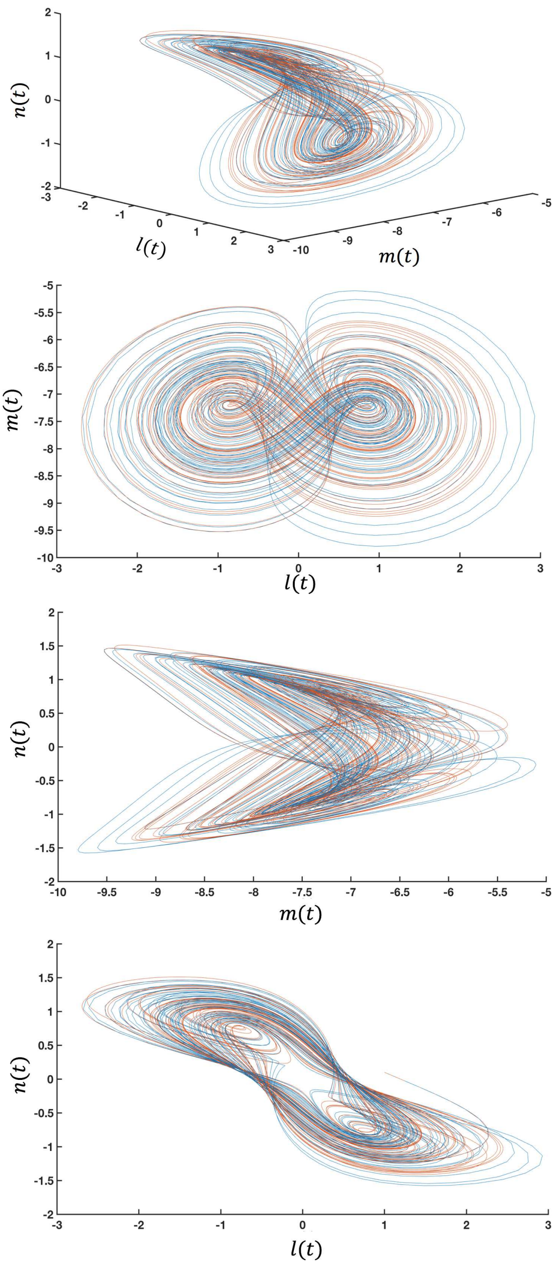

Figure 1: The plot shows a novel chaotic system when $a = 2$, $b = 0.1$ and $c = 1$

Consider the Four-Scroll chaotic system, given in [31] by the equations:

$$\dot{l} = \widetilde{a}m + \widetilde{b}l + \widetilde{c}mn, \qquad \dot{m} = \widetilde{d}m - n + \widetilde{e}ln, \qquad \dot{n} = \widetilde{f}n + \widetilde{g}lm. \tag{2}$$

In system (2), l, m and n are the state vectors while $\widetilde{a}$, $\widetilde{d}$, $\widetilde{c}$, $\widetilde{b}$, $\widetilde{e}$ and $\widetilde{g}$ are the corresponding parameters. This system is found to be chaotic for the proposed set of parameters $\widetilde{a} = 2.4$, $\widetilde{b} = -3$, $\widetilde{c} = 14$, $\widetilde{d} = -11$, $\widetilde{e} = 4$, $\widetilde{f} = 5.85$ and $\widetilde{g} = -1$, this fact can bee seen in Fig. 2.

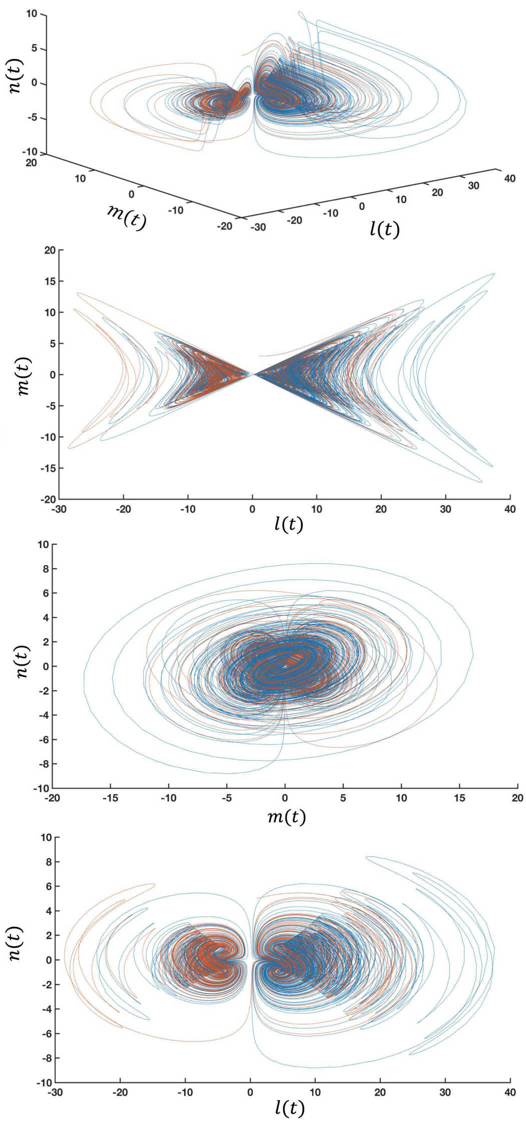

Figure 2: The plot shows the Four-Scroll chaotic system when $\widetilde{a} = 2.4$, $\widetilde{b} = -3$, $\widetilde{c} = 14$, $\widetilde{d} = -11$, $\widetilde{e} = 4$, $\widetilde{f} = 5.85$ and $\widetilde{g} = -1$

Further, we present in the following a new chaotic model as stated in [32]:

$$\dot{l} = -l - \hat{a}m + mn, \qquad \dot{m} = \hat{b}m - ln, \qquad \dot{n} = -\hat{c}n + lm, \tag{3}$$

in system (3), l, m, and n and $\hat{b}$, $\hat{a}$ and $\hat{c}$ respectively, show the system vectors and the associated parameters. For the proposed specific values assign to the parameters $\hat{a} = 1.5$, $\hat{b} = 2.5$ and $\hat{c} = 4.9$, is a chaotic system, which is illustrated in Fig. 3.

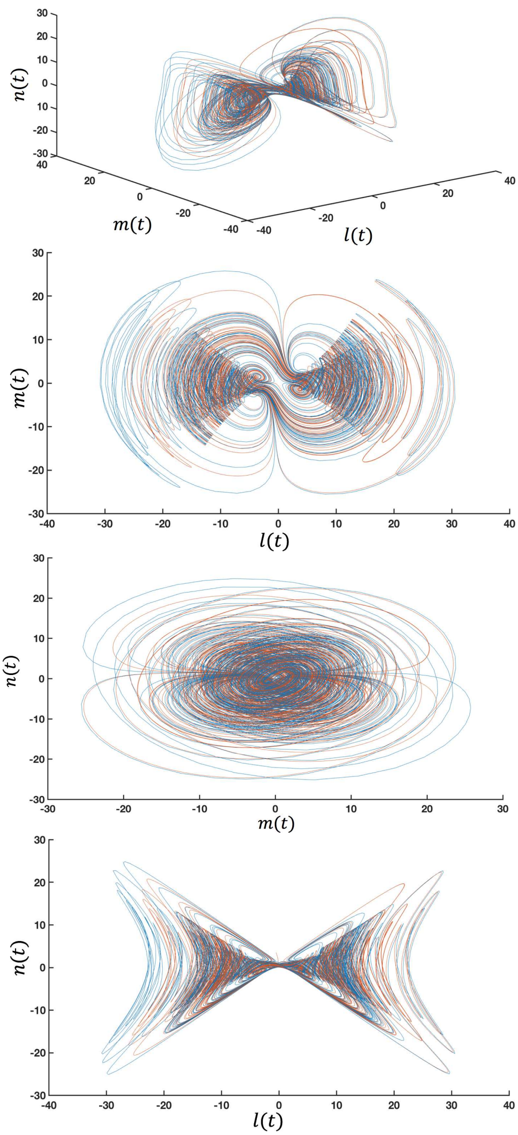

Figure 3: The plot shows the chaotic behavior when $\hat{a} = 1.5$, $\hat{b} = 2.5$ and $\hat{c} = 4.9$

3. Combined modified function projective synchronization (CMFPS)

Here, we give the details of the new proposed method. The following equations shows the representations of the chaotic drive system:

$$\dot{\mathcal{L}} = \mathcal{F}(\mathcal{L}), \tag{4}$$

the second drive system can be expressed as below:

$$\dot{\mathcal{M}} = \mathcal{G}(\mathcal{M}), \tag{5}$$

and the controller together with response system, $U \in R^n$, given by

$$\dot{\mathcal{N}} = \mathcal{H}(\mathcal{N}) + \mathcal{U}(\mathcal{L}, \mathcal{M}, \mathcal{N}, t), \tag{6}$$

where $\mathcal{L} = (l_1, l_2, \ldots, l_i)^{\mathrm{T}}$, $\mathcal{M} = (m_1, m_2, \ldots, m_i)^{\mathrm{T}}$, and $\mathcal{N} = (n_1, n_2, \ldots, n_i)^{\mathrm{T}} \in \mathcal{R}^n$ respectively represent the state vectors of the system (4)–(6), whereas $\mathcal{F}, \mathcal{G}, \mathcal{H} : \mathcal{R}^n \to \mathcal{R}^n$ and $\mathcal{U} : (\mathcal{R}^n, \mathcal{R}^n, \mathcal{R}^n) \to \mathcal{R}^n$ denotes the continues vector functions and the controller, respectively.

Definition 1 *For the master and slave systems given by (4), (5) and (6) respectively, then there is a combined modified function projective synchronization (CMFPS), if some vector function say, $\mathcal{U}(\mathcal{L}, \mathcal{M}, t)$, satisfies*

$$\lim_{t \to +\infty} \|\mathcal{N} - \Lambda(\mathcal{L})(\mathcal{L} + \mathcal{M})\| = 0.$$

Here, $\Lambda(\mathcal{L}) = diag\{\varphi_1(\mathcal{L}), \varphi_2(\mathcal{L}), \ldots, \varphi_n(\mathcal{L})\}$, the continuous functions, $\varphi_i(\mathcal{L})$, and $\|.\|$, is the vector induced norm.

Remark 1 *For the equations (4)–(6), we can define the error vectors for CMFPS, given by*

$$e = \mathcal{N} - \Lambda(\mathcal{L})(\mathcal{L} + \mathcal{M}),$$

where $e = (e_1, e_2, \ldots, e_n)^{\mathrm{T}}$ and $e_i = \mathcal{N}_i - \varphi_i(\mathcal{L})(\mathcal{L}_i + \mathcal{M}_i)$, $\varphi_i = \alpha_{i1} + (\mathcal{L}_i + \mathcal{M}_i)\alpha_{i2}$, $(i = 1, 2, \ldots, n)$.

Remark 2 *If*

$$\Lambda = \sigma \mathbf{I}, \qquad \sigma \in \mathbf{R},$$

then, it leads to combined projective synchronization, where $\mathbf{I}$ is the identity matrix of order $n \times n$. Specifically, if we put $\sigma = 1$, we obtain the combined complete synchronization and by using $\sigma = -1$, we obtain the anti-phase synchronization. The combined modified projective synchronization will appear, if $\Lambda = diag\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.

4. Combined modified function projective synchronization scheme

The novel chaotic system in terms of drive system can be shown through the following equations:

$$\begin{cases} \dot{l}_1 &= \left(-a + \dfrac{1}{b}\right) l_1 + l_1 m_1 + n_1, \\ \dot{m}_1 &= -b m_1 - l_1^2, \\ \dot{n}_1 &= -l_1 - c n_1, \end{cases} \tag{7}$$

where $[l_1, m_1, n_1]^{\mathrm{T}} \in \mathbb{R}^3$ is the state vector and the parameters a, b and c are positive real constants.

Let the Four–Scroll chaotic system be the second drive system, that can be shown by the following system:

$$\begin{cases} \dot{l}_2 &= \widetilde{a} m_2 + \widetilde{b} l_2 + \widetilde{c} m_2 n_2, \\ \dot{m}_2 &= \widetilde{d} m_2 - n_2 + \widetilde{e} l_2 n_2, \\ \dot{n}_2 &= \widetilde{f} n_2 + \widetilde{g} l_2 m_2, \end{cases} \tag{8}$$

where $[l_2, m_2, n_2]^{\mathrm{T}} \in \mathbb{R}^3$ is the state vector and the parameters $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, $\widetilde{d}$, $\widetilde{e}$, $\widetilde{f}$ and $\widetilde{g}$ are positive real constants.

The new chaotic model via response system, can be shown through the following system:

$$\begin{cases} \dot{l}_3 &= -l_3 - \hat{a} m_3 + m_3 n_3 + u_1, \\ \dot{m}_3 &= \hat{b} m_3 - l_3 n_3 + u_2, \\ \dot{n}_3 &= -\hat{c} n_3 + l_3 m_3 + u_3, \end{cases} \tag{9}$$

where $[l_3, m_3, n_3]^{\mathrm{T}} \in \mathbb{R}^3$ is the state vector and the parameters $\hat{a}$, $\hat{b}$ and $\hat{c}$ are positive real constants.

Here, the state vectors and the control functions are respectively shown by l_i, m_i and n_i $(i = 1, 2, 3)$, and u_1, u_2, and u_3 for achieving combined modified function projective synchronization between (7)–(9). Let the vector error states be as follows:

$$\begin{cases} e_l &= l_3 - [\alpha_{11}(l_1 + l_2) + \alpha_{12}](l_1 + l_2), \\ e_m &= m_3 - [\alpha_{21}(m_1 + m_2) + \alpha_{22}](m_1 + m_2), \\ e_n &= n_3 - [\alpha_{31}(n_1 + n_2) + \alpha_{32}](n_1 + n_2). \end{cases} \tag{10}$$

We can obtain the error dynamical system in the following:

$$
\begin{cases}
\dot{e}_l = -l_3 - \hat{a}m_3 + m_3n_3 + u_1 - 2\alpha_{11}\left[\left(-a+\dfrac{1}{b}\right)l_1^2 + l_1^2m_1 + l_1n_1 + \widetilde{a}l_2m_2\right. \\
\qquad + \widetilde{b}l_2^2 + \widetilde{c}l_2m_2n_2 + \left(-a+\dfrac{1}{b}\right)l_1l_2 + l_1l_2m_1 + l_2n_1 + \widetilde{a}l_1m_2 + \widetilde{b}l_1l_2 \\
\qquad \left. + \widetilde{c}l_1m_2n_2\right] - \alpha_{12}\left[\left(-a+\dfrac{1}{b}\right)l_1 + l_1m_1 + n_1 + \widetilde{a}m_2 + \widetilde{b}l_2 + \widetilde{c}m_2n_2\right], \\
\dot{e}_m = \hat{b}m_3 - l_3n_3 + u_2 - 2\alpha_{21}\left[-bm_1^2 - l_1^2m_1 + \widetilde{d}m_2^2 - m_2n_2 + \widetilde{e}l_2m_2n_2\right. \\
\qquad \left. - bm_1m_2 - l_1^2m_2 + \widetilde{d}m_1m_2 - m_1n_2 + \widetilde{e}m_1l_2n_2\right] - \alpha_{22}\left[-bm_1 - l_1^2\right. \\
\qquad \left. + \widetilde{d}m_2 - n_2 + \widetilde{e}l_2n_2\right], \\
\dot{e}_n = -\hat{c}n_3 + l_3m_3 + u_3 - 2\alpha_{31}\left[-l_1n_1 - cn_1^2 + \widetilde{f}n_2^2 + \widetilde{g}l_2m_2n_2 - l_1n_2\right. \\
\qquad \left. - cn_1n_2 + \widetilde{f}n_1n_2 + \widetilde{g}n_1l_2m_2\right] - \alpha_{32}\left[-l_1 - cn_1 + \widetilde{f}n_2 + \widetilde{g}l_2m_2\right].
\end{cases}
\tag{11}
$$

Our purpose here is to design such controllers, u_1, u_2, and u_3, that stabilize the error variables of (10). So, by defining the Lyapunov function as:

$$
V = \frac{1}{2}\left(e_l^2 + e_m^2 + e_n^2\right). \tag{12}
$$

$\dot{V}$ of (12) is

$$
\begin{aligned}
\dot{V} &= e_l\dot{e}_l + e_m\dot{e}_m + e_n\dot{e}_n \\
&= e_l\left[-l_3 - \hat{a}m_3 + m_3n_3 + u_1 - 2\alpha_{11}\left[\left(-a+\frac{1}{b}\right)l_1^2 + l_1^2m_1 + l_1n_1 + \widetilde{a}l_2m_2\right.\right. \\
&+ \widetilde{b}l_2^2 + \widetilde{c}l_2m_2n_2 + \left(-a+\frac{1}{b}\right)l_1l_2 + l_1l_2m_1 + l_2n_1 + \widetilde{a}l_1m_2 + \widetilde{b}l_1l_2 \\
&\left.\left. + \widetilde{c}l_1m_2n_2\right] - \alpha_{12}\left[\left(-a+\frac{1}{b}\right)l_1 + l_1m_1 + n_1 + \widetilde{a}m_2 + \widetilde{b}l_2 + \widetilde{c}m_2n_2\right]\right] \\
&+ e_m\left[\hat{b}m_3 - l_3n_3 + u_2 - 2\alpha_{21}\left[-bm_1^2 - l_1^2m_1 + \widetilde{d}m_2^2 - m_2n_2 + \widetilde{e}l_2m_2n_2 - bm_1m_2\right.\right. \\
&\left.\left. - l_1^2m_2 + \widetilde{d}m_1m_2 - m_1n_2 + \widetilde{e}m_1l_2n_2\right] - \alpha_{22}\left[-bm_1 - l_1^2 + \widetilde{d}m_2 - n_2 + \widetilde{e}l_2n_2\right]\right] \\
&+ e_n\left[-\hat{c}n_3 + l_3m_3 + u_3 - 2\alpha_{31}\left[-l_1n_1 - cn_1^2 + \widetilde{f}n_2^2 + \widetilde{g}l_2m_2n_2 - l_1n_2\right.\right. \\
&\left.\left. - cn_1n_2 + \widetilde{f}n_1n_2 + \widetilde{g}n_1l_2m_2\right] - \alpha_{32}\left[-l_1 - cn_1 + \widetilde{f}n_2 + \widetilde{g}l_2m_2\right]\right].
\end{aligned}
\tag{13}
$$

The control functions are chosen as:

$$\begin{aligned}
u_1 &= \hat{a}m_3 - m_3n_3 + 2\alpha_{11}\left[\left(\frac{1}{b}\right)l_1^2 + l_1^2m_1 + l_1n_1 + \widetilde{a}l_2m_2 + \widetilde{b}l_2^2 + \widetilde{c}l_2m_2n_2\right.\\
&+ \left.\left(\frac{1}{b}\right)l_1l_2 + l_1l_2m_1 + l_2n_1 + \widetilde{a}l_1m_2 + \widetilde{b}l_1l_2 + \widetilde{c}l_1m_2n_2\right] + \alpha_{12}\left[\left(\frac{1}{b}\right)l_1\right.\\
&+ \left. l_1m_1 + n_1 + \widetilde{a}m_2 + \widetilde{b}l_2 + \widetilde{c}m_2n_2\right] - al_3 - \alpha_{11}al_1^2 + \alpha_{11}l_1^2 + 2\alpha_{11}l_1l_2\\
&+ \alpha_{11}(a+1)l_2^2 + \alpha_{12}l_1 + \alpha_{12}(a+1)l_2,\\
u_2 &= -2\hat{b}m_3 - bm_3 + l_3n_3 + 2\alpha_{21}\left[-l_1^2m_1 + \widetilde{d}m_2^2 - m_2n_2 + \widetilde{e}l_2m_2n_2 - l_1^2m_2\right.\\
&+ \left.\widetilde{d}m_1m_2 - m_1n_2 + \widetilde{e}m_1l_2n_2\right] + \alpha_{22}\left[-l_1^2 + \widetilde{d}m_2 - n_2 + \widetilde{e}l_2n_2\right] - \alpha_{21}bm_1^2\\
&+ \hat{b}\alpha_{21}m_1^2 + 2\hat{b}\alpha_{21}m_1m_2 + \left(\hat{b}+b\right)\alpha_{12}m_2^2 + \alpha_{22}\hat{b}m_1 + \left(b+\hat{b}\right)\alpha_{22}m_2,\\
u_3 &= -cn_3 - c\alpha_{31}n_1^2 - l_3m_3 + 2\alpha_{31}\left[-l_1n_1 + \widetilde{f}n_2^2 + \widetilde{g}l_2m_2n_2 - l_1n_2\right.\\
&+ \left.\widetilde{f}n_1n_2 + \widetilde{g}n_1l_2m_2\right] + \alpha_{32}\left[-l_1 + \widetilde{f}n_2 + \widetilde{g}l_2m_2\right] + (c+\hat{c})\alpha_{31}n_2^2\\
&+ \alpha_{31}\hat{c}n_1^2 + 2\alpha_{31}\hat{c}n_1n_2 + \alpha_{32}\hat{c}(n_1+n_2) + \alpha_{32}cn_2 - \alpha_{31}cn_1^2.
\end{aligned} \tag{14}$$

Substituting (14) into (13), we get:

$$\begin{aligned}
\dot{V} = e_l&\left[-(a+1)\left[l_3 - \alpha_{11}\left(l_1^2 + 2l_1l_2 + l_2^2\right) - \alpha_{12}(l_1+l_2)\right]\right]\\
&+ e_m\left[-\left(b+\hat{b}\right)\left[m_3 - \alpha_{21}\left(m_1^2 + 2m_1m_2 + m_2^2\right) - \alpha_{22}(m_1+m_2)\right]\right]\\
&+ e_n\left[-(c+\hat{c})\left[n_3 - \alpha_{31}\left(n_1^2 + 2n_1n_2 + n_2^2\right) - \alpha_{32}(n_1+n_2)\right]\right]\\
&\quad \Rightarrow \dot{V} = -\left[(a+1)e_l^2 + \left(b+\hat{b}\right)e_m^2 + (c+\hat{c})e_n^2\right].
\end{aligned} \tag{15}$$

Clearly, the differentiation of the Lyapunov function in (15) is negative; moreover, the controller for CMFPS will asymptotically stabilize the systems. In addition, the errors of CMFPS asymptotically converge to the origin as $t \to \infty$. The results above suggest the synchronization of the states of the drive and the response system.

5. Simulation results

Here, we show the application of the method discussed above by choosing some numerical examples and providing the graphical results. We use some appropriate numerical method and solve the systems (7)–(9), given by the specified initial values:

$$\begin{aligned}
&[l_1(0),\ m_1(0),\ n_1(0)]^{\mathrm{T}} = [1,\ -6,\ 0.1]^{\mathrm{T}},\\
&[l_2(0),\ m_2(0),\ n_2(0)]^{\mathrm{T}} = [1,\ 3,\ 5]^{\mathrm{T}},\\
&[l_3(0),\ m_3(0),\ n_3(0)]^{\mathrm{T}} = [-0.5,\ 2,\ 3.5]^{\mathrm{T}}.
\end{aligned}$$

The specific parameters used in the numerical solution is specified in the captions of each figure. Figure 1 and its subgraphs show the chaotic attractor for the specified parameters. Figure 2 and its subgraphs show the Four-Scroll chaotic system for the given parameters. The graphical result for the chaotic drive system with the specified parameters is shown in Fig. 3. The synchronization errors, given by e_i, for $i = 1,2,3$ are presented in Fig. 4. When the scaling functions are chosen by

$$\begin{aligned}
\varphi_1 &= 2(l_1 + l_2) + 5,\\
\varphi_2 &= 3(m_1 + m_2) + 4.5,\\
\varphi_3 &= 1.5(n_1 + n_2) + 1,
\end{aligned}$$

then, the combined (MFPS) is achieved and performed in Fig. 4a. The FPS between the three systems are obtained in Figure 4b, when $\alpha_{i2} = 0$, for $i = 1,2,3$ and taking the following scaling functions:

$$\begin{aligned}
\varphi_1 &= (l_1 + l_2),\\
\varphi_2 &= 2(m_1 + m_2),\\
\varphi_3 &= 2.5(n_1 + n_2).
\end{aligned}$$

In particular, by choosing, $\varphi_1 = 1$, $\varphi_2 = 4$, and $\varphi_3 = 6$, the MPS appears according to Fig. 4c. Moreover, for the synchronization errors of systems (7)–(9) with $\varphi_i = 1$, we get complete synchronization in Fig. 4d while for $\varphi_i = -1$, then, we obtain anti-synchronization given in Fig. 4e.

The graphical results demonstrate that the chaotic systems achieve the (CMFPS) under the designed controller (10). Further, when $t \to \infty$, then both the drive and the response systems are synchronizing.

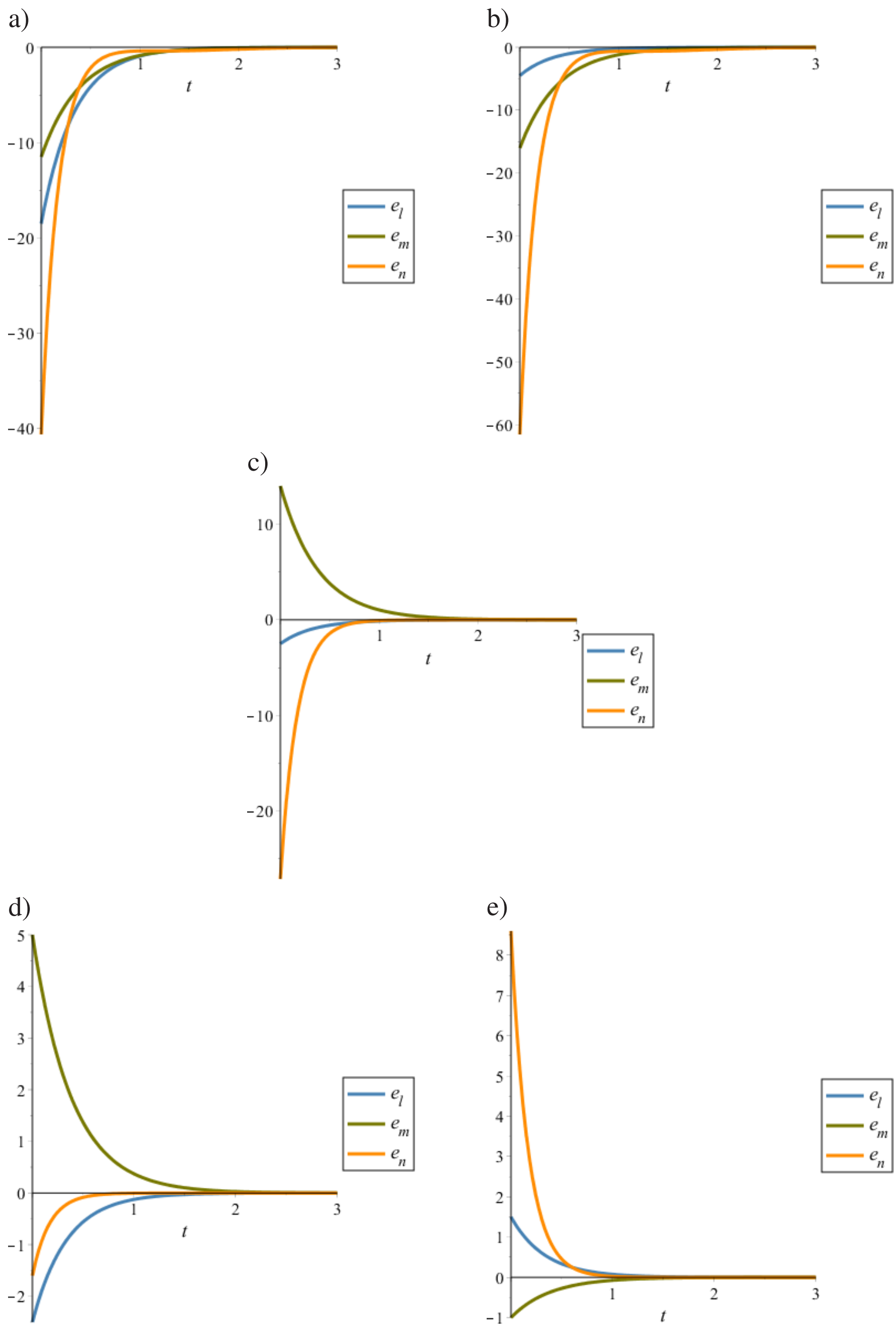

Figure 4: The plot shows the synchronization errors between systems (4)–(6): a) MFPS, b) FPS, c) MPS, d) complete synchronization and e) anti-synchronization

6. Conclusion

In the present paper, we successfully presented a novel method known as combined modified function projective synchronization. A Lyapunov function has been defined for the stability of an adaptive control and found to be asymptotically stable, which shows the synchronization of the drive and response systems. Further, we obtained the numerical results of the systems and showed the effectiveness of the suggested method. The graphical results suggests the applicability of the new method to chaotic systems. Thus, the new method considered in this paper could provide better results compared to the previous one.

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