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Applicability of the Capstan Equation to Guitar Strings

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The effects of friction were observed in electric guitar strings passing over an electric guitar saddle. The effects of changing the ratio of the diameter of the winding to the diameter of the core of the string, the angle through which the string is bent, and the length on either side of the saddle were measured. Relative tensions were deduced by plucking and measuring the frequencies of vibration of the two portions of string. Coefficients of friction consistent with the capstan equation were calculated and were found to be lower than 0.26 for wound strings (nickel plated steel windings on steel cores) and lower than 0.17 for unwound (tin plated steel) strings. The largest values of friction were associated with strings of narrower windings and wider cores and this may be due to the uneven nature of the contact between the string and saddle for wound strings or due the surface of the windings deforming more, encouraging fresh (and therefore higher friction) metal to metal contact. It is advised to apply lubrication under the saddle to string contact point after first bringing the string up to pitch rather than before in order to prevent this fresh metal to metal contact.

Keywords: guitar; string; friction; saddle; capstan; equation; bridge.

1. Introduction

Guitar strings operate under tension such that they approximately obey the linear wave equation with a fixed end at the saddle and another fixed end at the nut or on the fret on which they have been stopped. While in some instruments (for example the Floyd Rose locking system (Rose, 1979)) the ends of the string are fixed by clamping, it is much more common for these fixed points to be created by the string curving around a supporting structure of relatively large mass while supporting tension on both sides. The change in direction of the tension either side of the supporting structure (such as the saddle, nut or fret) gives a large resultant force pressing the string to the support and ensuring that the mass per unit length along the string becomes discontinuously large at the point of contact with the support, reflecting vibrations on the string.

Tuning errors in fretted string instruments can arise due to various problems including intonation issues (Varieschi, Gower, 2010), temperature changes and changes in the rest length of the string as it stretches or relaxes with changes in tension. When players perform a gesture to modulate the tension by turning a tuning peg, dragging the string along or across a fret or moving a mechanism (for example the Fender Stratocaster synchronised tremolo system (Fender, 1956)) the tension in the sounding length of the string before and after the playing gesture sometimes differs causing noticeable errors in the tuning of the instrument. This is widely recognised as being due to friction stopping the string coming back to rest at its original position, leading to tension changes either side of the supports. Cleaning then lubricating the surfaces helps minimise such problems but tuning problems related to friction at supports is a problem worthy of scientific analysis.

It is reasonable to consider the applicability of the capstan equation to the tension differences either side of the string. This is achieved here by mounting Fender American Standard guitar tuning machines (or tuners) in various positions on a wooden board either side of saddle of the type fitted to a current American Standard Stratocaster (nickel-plated bent steel as in vintage instruments). Plugging the coefficient of static friction for the relevant surfaces into the capstan equation allows comparisons to be drawn. Since strings slip dynamically over supports as their tension is modulated and because of the bending stiffness of the strings, the capstan equation is not perfect as a model of guitar string behaviour. The coefficient of static friction which would best predict the tension differences between sections of string in practical circumstances are then calculated and any trends noted in the data when strings of different construction are tested in different tuner locations.

For wound strings, the importance of the ratio of the total mass of the string to the mass of the winding has been demonstrated to control the sensitivity of strings to player control and this applies to tremolo arm use and for the conventional pitch bends (Kemp, 2017). It is therefore worth demonstrating whether friction depends on this ratio as might be expected given that varying the winding diameter will alter the nature of the contact area between the saddle and string.

2. Theory

Neglecting string stiffness, the capstan equation describes the ratio of tensions ($T_A > T_B$) on either side of a flexible string wrapped around a cylindrical point of contact:

$$\frac{T_A}{T_B} = e^{\mu\theta}, \qquad (1)$$

where μ is the coefficient of friction between the string and the cylinder, and θ is the angle through which the string is bent (Stuart, 1961). The capstan equation assumes that the string is on the limit of static friction, that the string has negligible stiffness, and that the string is non-elastic.

For the fundamental frequency of a string, the wavelength is equal to two times the effective string length, l:

$$f = \frac{c}{2l}. \qquad (2)$$

From the derivation of the one-dimensional wave equation (French, 1971) we know:

$$c = \sqrt{\frac{T}{\rho_L}}, \qquad (3)$$

where the linear density is $\rho_L = \frac{m}{l}$, with m being the mass of the portion of the string of length l that sounds. Substituting this into Eq. (2) gives

$$f = \frac{1}{2l}\sqrt{\frac{T}{\rho_L}} = \frac{1}{2l}\sqrt{\frac{T}{\left(\frac{m}{l}\right)}}, \qquad (4)$$

which leads to

$$\frac{T_1}{T_2} = \frac{m_1 l_1}{m_2 l_2}\left(\frac{f_1}{f_2}\right)^2, \qquad (5)$$

where the subscripts denote the section of string to either side of the saddle, taken to be 1 for the left side and 2 for the right side throughout the experiment.

Assuming that the tension on the left portion of string (of length l_1) is higher than on the right, then the coefficient of friction consistent with a tension imbalance between two sides of the string can be found by substituting Eq. (5) into the capstan equation:

$$\mu = \frac{1}{\theta}\ln\left(\frac{m_1 l_1}{m_2 l_2}\left(\frac{f_1^{(L)}}{f_2^{(L)}}\right)^2\right), \qquad (6)$$

where the superscript (L) denotes that the tuning peg on the left has been turned to increase the tension of the string. If the peg on the right has been turned, on the other hand, then the right-hand string (of length l_2) will have a higher tension and the coefficient of static friction consistent with the capstan equation will be:

$$\mu = \frac{1}{\theta}\ln\left(\frac{m_2 l_2}{m_1 l_1}\left(\frac{f_2^{(R)}}{f_1^{(R)}}\right)^2\right). \qquad (7)$$

To improve the accuracy of results, the ratio $\frac{m_1 l_1}{m_2 l_2}$ can be expressed in terms of frequency ratios. For a given string and angle we may assume that the static coefficient of friction measured when the left tuner is turned is equal to that when the right tuner is turned, so the capstan equation predicts:

$$\frac{m_1 l_1}{m_2 l_2} = \left(\frac{f_2^{(L)}}{f_1^{(L)}}\right)\left(\frac{f_2^{(R)}}{f_1^{(R)}}\right), \qquad (8)$$

where, again, the superscripts L and R denote turning the left and right peg, respectively. Using the average values for $f_2^{(L)}/f_1^{(L)}$ and $f_2^{(R)}/f_1^{(R)}$ is considerably more accurate than using the ratio of lengths measured with a ruler. Using Eq. (6) and Eq. (8) we can show that:

$$\mu = \frac{1}{\theta}\ln\left(\frac{f_2^{(R)}}{f_1^{(R)}}\frac{f_1^{(L)}}{f_2^{(L)}}\right), \qquad (9)$$

and this equation was used to give the values of μ in the figures below.

3. Experiment

A one-string guitar was constructed from a piece of wood, with four tuning pegs, allowing for the measurement of four different bend angles, as shown in Fig. 1. The relevant distances and angles are shown in Table 1. The larger angles are similar to those observed for strings passing over the saddle in a Fender Stratocaster bridge while the lowest angles are more typical of those seen in guitars with tailpieces such as the Gibson Les Paul (where the precise angles depend on how the guitars are set up).

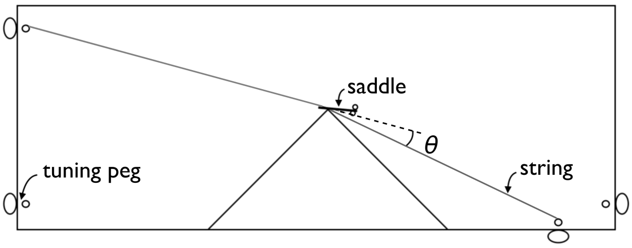

Fig. 1. One-string guitar with four tuning pegs. Screws were used to secure the saddle in place, along with an additional piece of wood underneath to prevent the saddle moving when a string under high tension is passed across.

Table 1. Distances between pegs measured with a ruler, and associated bend angles given by Eq. (10).

Left peg	Right peg	l_1 ruler [cm]	l_2 ruler [cm]	Peg to peg [cm]	θ [rad]
top	top	28.3	29.3	57.5	0.12 ± 0.05
top	bottom	28.3	24.7	52.7	0.21 ± 0.03
bottom	top	29.0	29.3	56.1	0.55 ± 0.01
bottom	bottom	29.0	24.7	50.7	0.67 ± 0.01

The bend angle θ can be found by measuring each relevant length with a ruler, and using the law of cosines:

$$\theta = \pi - \arccos\left(\frac{b^2 + c^2 - a^2}{2bc}\right), \tag{10}$$

where a is distance between the left and right pegs, b is the distance between the left peg and the saddle, and c is the distance between the saddle and the right peg.

The ratio of string lengths was obtained by using Eq. (8) and for wound strings attached to the top left and top right pegs, for instance, this produced a value of $\frac{l_L}{l_R} = 1.0333 \pm 0.0002$, whilst using a ruler produced $\frac{l_L}{l_R} = 1.035 \pm 0.005$.

The process for testing each string is as follows:

1) The string is threaded through two tuning pegs and brought up to a high tension by turning both pegs.
2) The string is plucked multiple times.
3) The pegs are turned back until the string reaches the lowest tension at which a clear note is heard.
4) A recording is made of the left section of string being lightly plucked, whilst the right section is lightly damped by one finger.
5) A recording is made of the right section of string being lightly plucked, whilst the left section is lightly damped by one finger.
6) The left peg is moved through 1/4 of a turn.
7) Steps 4 to 6 are repeated until the string is difficult to bend.
8) The recordings are analysed and the data tabulated.
9) Steps 3 to 8 are repeated with the right peg turned instead of the left peg in step 6.

This process was repeated for each bend angle, in the order given in Table 1, and for each string. By starting with the smallest angle, effects due to permanent bends induced in the string were kept to a minimum. However, such effects were still present: due to the shorter length between the saddle and bottom right peg, a permanent bend was often present to the left of the saddle. This would have greater implications for strings with thicker cores because the permanent bend is stretched out relatively easily for narrower cores under sufficient tension. The calculation in Eq. (9) compensates for this effect by cancelling any impact of differences in mass times length on either side of the saddle.

After each turn of the tuning peg, the left section of string was always plucked before the right; the strings may have slightly loosened between the two plucks due to work hardening. This may have lead to larger frequency differences being calculated. Steps 1 and 2 were included to reduce the effects of work hardening.

The section of string not being measured was lightly damped in steps 4 and 5 to prevent multiple frequencies being recorded. When testing the effects of changing the ratio of lengths, it was noted that damping the much shorter section in different places had an audible effect on the pitch of the longer section. The recordings were made using the internal microphone of a MacBook Pro laptop running Praat software (Boersma, Weenink, 2018). The software produces a line showing the dominant frequency at each time. The frequency was then taken from the average of a section within the flat tail of this line.

The first turn of the peg after loosening was consistently observed to produce anomalous results; hence, the first turn was ignored in the analysis. The anomalous results during the first turn of the peg are most likely due to measurements beginning when the string was not quite sufficiently taut: this made the string more likely to slip into the board and hit the side of the saddle. This happened for multiple strings, and was mitigated partway through the experiment by replacing one of the tuning pegs with a higher peg, such that the string was held further from the board. Furthermore, the lowest frequency notes produced a recording with a lower signal-to-noise ratio, making the extraction of a single frequency more difficult.

For each string and each choice of θ, the experiments performed by sequentially tightening in quarter turns using the left tuner (superscript (L)) and by sequentially tightening in quarter turns using the right tuner (denoted by superscript (R)) resulted in a column of sounding frequencies tabulated in Microsoft Excel for the left string f_1 and right string f_2. A column of the ratio f_2/f_1 was then calculated for each experiment. The average values of the resulting columns obtained were then calculated using the AVERAGE function in Excel and these average values of $\left(\frac{f_2^{(L)}}{f_1^{(L)}}\right)$ and $\left(\frac{f_2^{(R)}}{f_1^{(R)}}\right)$ substituted into Eq. (9) to obtain the friction coefficient, μ, for graphing.

Error bars were produced by first calculating the standard error (also known as the standard deviation of the mean or the standard uncertainty) of the ratio of frequencies when the left peg is tightened, $SE\left(\frac{f_2^{(L)}}{f_1^{(L)}}\right)$, and the standard error of the ratio of frequencies when the right peg is tightened $SE\left(\frac{f_2^{(R)}}{f_1^{(R)}}\right)$. The standard error in each case was calculated in Excel by taking the standard deviation of the column of frequency ratios (calculated using the STDEVA function) and dividing this by the square root of the number of measurements of those frequency ratios (using the SQRT and COUNT functions) (Quirk, 2016). The standard error of μ was then calculated using the law of error propagation (Drosg, 2009) to give:

$$SE(\mu) = \frac{1}{\theta}\sqrt{\left(SE\left(\frac{f_2^{(L)}}{f_1^{(L)}}\right)\right)^2 + \left(SE\left(\frac{f_2^{(R)}}{f_1^{(R)}}\right)\right)^2}. \qquad (11)$$

4. The effect of core thickness versus winding thickness

Six custom made strings were commissioned to have a similar overall gauge (around 32 thousandths of an inch) and mass, suitable for use as an electric guitar string with a nominal pitch of $A_2 = 110$ Hz, but with different core and winding gauges. These were constructed with Newtone strings. The winding to core diameter ratios of these strings were $d_w/d_{core} = 6/19$, 7/18, 8/16, 9/14, 10/13, and 11/11. These strings have steel cores with circular cross-section, and are round wound with nickel-plated steel.

Following this, tests were carried out on five unwound tin-plated steel strings, with circular cross-sections and diameters of $\frac{9}{1000}''$, $\frac{10}{1000}''$, $\frac{11}{1000}''$, $\frac{13}{1000}''$, and $\frac{17}{1000}''$. A nickel plated bent steel saddle was used throughout the experiments.

Figure 2 shows the coefficient of friction, μ, calculated using Eq. (9) for the strings of different winding diameter to core diameter and for the four break angles. The measured coefficients of friction are in the range 0.06 and 0.26.

The larger values of the friction coefficient are observed for strings with thicker cores and narrower windings, but only for greater break angles. Factors involved may include the increased bending stiffness of strings due to wider cores having greater bending stiffness, the greater permanent bending of the core around the saddle for wider cores, and due to narrower windings on the saddle side of the string fusing under bending and compression introducing permanent shape changes which may also change the composition of the materials in the region of contact due to the deformation of the layer of nickel plating on the windings. Generally the effect of increased bending rigidity (associated with thicker cores) would tend to decrease the derived coefficient of friction rather than increase it due to the string bowing around the saddle and therefore being in contact over a reduced break angle, so this cannot be responsible for the observed effect (Stuart, 1961). The introduction of permanent bends should reduce the bowing of the string around the supports.

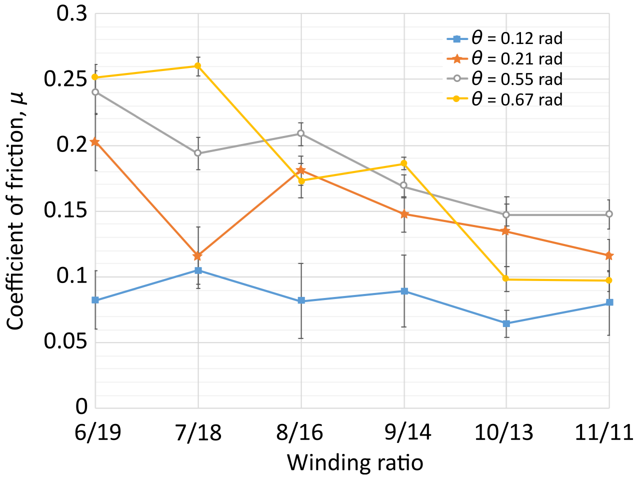

Fig. 2. The experimentally measured coefficient of friction consistent with the capstan equation plotted against ratio of winding diameter to core diameter for the string curving round the saddle through different angles, θ.

Changes of the composition of the materials in contact may be significant in explaining the apparent increase of friction for wider cores and narrower windings in this experiment. Narrower windings tend to deform more when bent over the saddle at larger break angles, leading to fresh nickel and possibly steel touching the saddle in place of the previous layer of oxide films and lubricating contaminants. Indeed, Straffelini (2015) avoids producing a generalised list of friction coefficients as "friction is a system property that strongly depend on the sliding conditions" (p. 38) and "Metal surfaces may easily undergo oxidation during sliding, with the formation of an oxide layer (or scale) that may reduce friction coefficient, thus acting as a lubricating skin" (p. 31). In practice adding lubrication after first installing a string to pitch would tend remove the observed effect.

Amontons' law states that the frictional force is equal to the normal reaction force multiplied by the coefficient of friction (Stuart, 1061). Any contribution to the observed increase in friction for small winding diameters due to area of contact arguments (for instance due to the stepped nature of the contact around the bent saddle) would violate Amontons' law. However, it has been observed that rough surfaces experience higher friction coefficients at low sliding velocities (less than 1 m/s, as would be appropriate here) (Lim *et al.*, 1989) so it is possible the discontinuous nature of the contact between wound strings and the saddle may increase friction to some extent.

5. Comparison with typical friction coefficients

Typical values for the static coefficient of friction between nickel and nickel are 0.28 when greasy, and 0.7–1.1 when dry (Ramsdale, 2006). Measurements of the wound nickel-plated strings (in contact with nickel plated saddles) only produced values below 0.26. This is due to the capstan equation assuming that the strings are on the limit of static friction, which was often not the case in this experiment. A typical value of the coefficient of friction for lubricated sliding between nickel and nickel is 0.12 (Ramsdale, 2006) and hence the results suggest that when the tension increases on one side, the string slides across the saddle to almost equalise the tension imbalance, before coming to rest when the tension difference is consistent with the capstan equation where the coefficient of friction is approximately in line with published values for lubricated sliding contact. Lubrication had been applied to the saddle used more than two years prior to the experiments presented in this paper but was not applied immediately prior or during the experiments to keep the results consistent. The exact values for tensions differences are likely to depend strongly on the condition of the surfaces with dry, clean surfaces of fresh metal having higher values of friction than lubricated or lightly oxidised ones.

For the unwound strings, the contact is between the tin and nickel (due to the unwound strings being plated in tin and the steel saddles being plated with nickel). While standard values of friction between tin and nickel have not been found by the authors, the experimental data shows that for this system the friction coefficients are below 0.17 and average a little lower than that for the wound (nickel on nickel) measurements.

There is significant variation in the results, with the error bars (based on the standard error in measured frequency ratios on repeated tightenings of the tuning pegs) often not explaining the deviation of the data from clear trends. This adds support to the importance of small variations such as in condition of oxide films on the string at the point of contact with the saddle. Other sources of error include work hardening effects and the effects of bending and unbending of the string. The one negative coefficient of friction measured (in Fig. 3) corresponded to a measurement for the smallest break angle and may have resulted from the plucking process reversing the very small tension imbalance. Some of the randomness in the data, particularly for small break angles, may result similarly from the plucking process and/or vibration of the string modulating

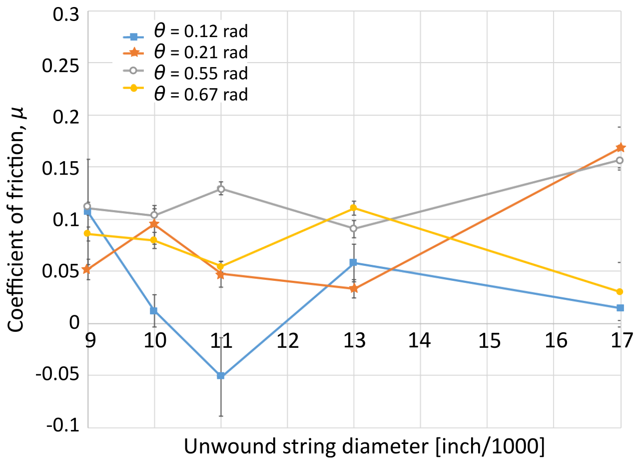

Fig. 3. The experimentally measured coefficient of friction consistent with the capstan equation plotted against the diameter of unwound strings for the string curving round the saddle through different angles, θ. The x axis values (using the thousandths of an inch scale that is standard for guitar strings) can be converted to millimetres through multiplying by 0.0254.

the normal force on the saddle sufficiently to cause some slippage. This process will tend to under predict the coefficient of friction to some extent because the tension difference will be lowered.

6. The effect of changing the length of string either side of the saddle

On a conventional guitar, the main section of each string is much longer than both sections from the nut and bridge to the ends of the string. Hence, three holes were drilled on the one-string guitar, along the line from the saddle to the bottom right tuning peg, such that a tuning peg could be inserted to each hole as shown in Fig. 4. This allowed for testing of multiple string length ratios for a fixed bend angle of 0.67 rad, i.e. from the bottom left peg. Tests were performed on the wound strings with ratios 6/19, 9/14, and 10/13. The additional holes were drilled at 4.0 cm, 8.1 cm, and 12.4 cm from the saddle.

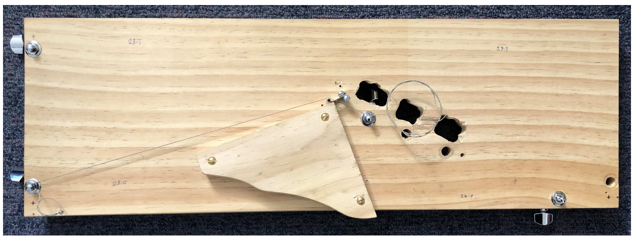

Fig. 4. Photograph of the one-string guitar modified to have seven possible tuning peg positions. Screws were used to secure the saddle in place, along with an additional piece of wood underneath to prevent the saddle moving when a string under high tension is passed across.

It was found that changing the length of the section of string to the right of the saddle does not produce clear trends in the experimental derivation of the coefficient of friction, as shown in Fig. 5 with the coefficients varying by over a factor of two showing the sensitivity of the experiment to the surface condition of the string. The tendency of strings with narrowest windings to have a higher friction coefficient for large break angles is, nonetheless, maintained in this experiment.

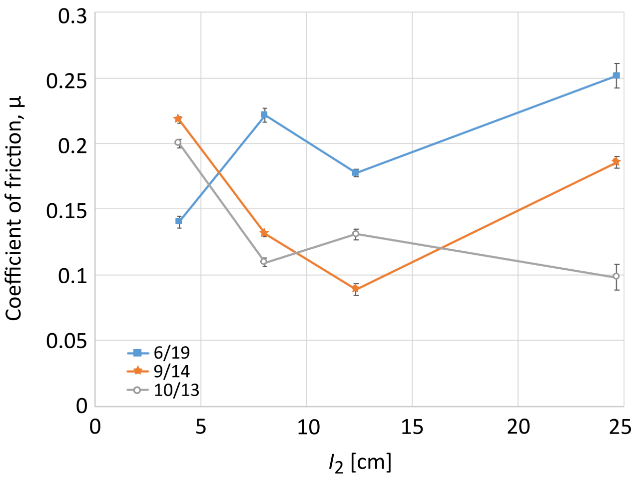

Fig. 5. The experimentally measured coefficient of friction consistent with the capstan equation plotted against the distance of string on the right hand side of the saddle, for the string curving round the saddle through an angle of $\theta = 0.67$ radians.

The combination of maximum observed friction and maximum angle gives a tension ratio of $\exp(\mu\theta) = 1.19$ to 3 significant figures. If the lengths of string either side of the saddle were the same then a maximum pitch difference supported either side would be approximately $1200 \log_2(\sqrt{1.19}) \approx 150$ cents, or one and a half semitones, implying huge pitch errors before and after bending the string for instance. In practice the length of string behind the saddle of a Fender Stratocaster bridge is in the range between 25 mm for newer models and 60 mm for vintage designs and this is much smaller than the scale length of around 648 mm. Slippage of the string across the saddle occurs whenever tension changes take place and the shorter side of string must undergo a much bigger stretch in relation to its length. This in turn means that the bulk of the tension change implied by the capstan equation due to slippage at the saddle will be experienced on the bridge side of the saddle, leaving the tension change and pitch change in the sounding length much smaller (but non-zero). A full treatment of this topic is beyond the scope of the current paper, but it is clear that minimising the distance of string behind the saddle minimises tuning problems since possible tension imbalances implied by the capstan equation can be absorbed by the string sliding across the saddle by smaller distances (changing the elongation of the main sounding length to a smaller degree).

7. Conclusions

Values for the coefficient of friction were calculated from the capstan equation based on measurements of the pitch of vibration for various constant bend angles. For wound strings, strings with narrow windings were found to have slightly larger coefficients of friction for large break angles. The results were consistent with the capstan equation where the coefficient of friction is less than 0.26 and likely to vary with the condition of the surfaces. Observed coefficients of friction were much less than most published values of static friction and were close to those associated with sliding contact. This suggests that when a tension imbalance is introduced (by turning a tuner or executing a conventional pitch bend) the string slides across supports (such as at the nut and saddle) to almost equalise the tension before coming to rest when the tension approximately matches the value given in the capstan equation for the coefficient of friction under sliding contact. It should be noted that in some practical cases surface shape roughness also encourages serious tension imbalances (such as between string windings and sharp edges on bridge plates and saddles) and interfaces between strings and sharp edges should minimized for tuning stability. Contact with sharp edges has been avoided in the experiments presented here.

Changing the ratio of lengths on either side of the saddle was shown to have no obvious effect on the derived coefficient of frequency. In practice having shorter length of string outside the main sounding length (for instance between the saddle and the ball end of the string) is beneficial to tuning stability as any changes in tension there will have lower displacements of string associated with it.

Permanent bends are introduced when a new string is installed, potentially changing the composition of the material on string surface and squeezing out any pre-existing surface films that may have otherwise lowered friction. When fitting a new string it is therefore advisable to install the string up to pitch, then move the string just enough in order to allow the application of lubrication between the newly formed bent surface of the string and the saddle. This process minimises fresh metal to metal contact and hence minimises friction related tuning problems. Applying lubrication was, however, out with the scope of the experimental tests within the current work and would be a useful topic for future study.

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