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Coupler curves of four-bar linkages plotted in graphical programs

This work is devoted to the plotting of coupler curves in the environment of graphical programs. As there is a large variety of shapes, for the purpose of this study, the authors selected those curves that feature a cusp form. In the research, two software programs were used, i.e., AutoCAD and Rhinoceros with the Grasshopper plug-in. Two types of curves were defined: a fixed and a moving centrode, in which the points of the moving centrode define the coupler curves whose cusps are located on the fixed centrode. In conclusion, two design tools were compared and the curves in question were discussed in detail.

1. Introduction

Coupler curves delineated by coupler points of four-bar linkages are a dilemma that has been challenging researchers for a long time. It is believed that it was in 1875 when analytical studies concerning the properties of these curves began [1]. Back then, Samuel Roberts published the first accounts of the algebraic properties of the planar four-bar curve. Since then, researchers have been investigating this subject using a variety of methods and techniques. Two main directions of study can be distinguished:

- the plotting of coupler curves and their variants for all types of double crank four-bar linkages, crank-rocker linkages, double rocker linkages and slider-crank linkages [1–6],
- the synthesis of mechanisms when coupler curves or their fragments are given [7–14].

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The problem of coupler curves appears in numerous handbooks on the subject of the fundamentals of the construction of mechanisms and machinery [15–17].

In the teaching of machine kinematics, various methods and software tools are used. These include, among others: Adams, Geogebra, GIM and Cinderella (a freeware tool provided by Springer) [2].

The research presented in this work is in line with the current of investigations concerning the coupler plane, in which new capabilities of digital technologies are being tested. Two graphical programs were selected:

- AutoCAD, which has featured parametric constraints since its 2010 version (geometric and dimensional);
- Rhinoceros with the Grasshopper plug-in, which graphically illustrates mathematical and geometric relationships written in the form of algorithms.

The motivation for taking up this subject was the fact that constructions featuring coupler curves are geometrically correlated. Thus, graphical programs seem to be appropriate tools to plot them. The capabilities of programs were tested without the knowledge of the programming language.

Earlier studies of coupler planes utilised two programs: mathematical Mathcad and graphical AutoCAD [18]. Analytical correlations were exploited for the purpose of obtaining the results in a form of the coordinates of points belonging to each individual curve. The methodology is presented using the example of the fixed centrode as a curve plotted by instantaneous centres of rotation of the four-bar linkage (Fig. 1).

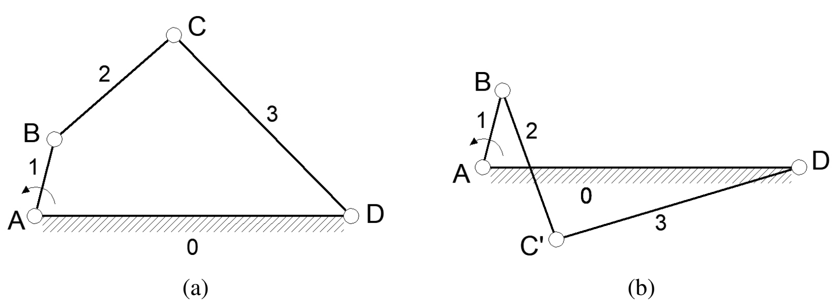


Fig. 1. Four-bar linkage

The following data were assumed: $x_A = 0$, $y_A = 0$, $x_D = 200$, $y_D = 0$, $l_{AB} = 50$, $l_{BC} = 100$, $l_{CD} = 160$. The rotation angle of link 1 around kinematic pair A was marked by α_1. The location of point B was determined from equations:

$$x_B(\alpha_1) = x_A + l_{AB}\cos(\alpha_1), \tag{1a}$$

$$y_B(\alpha_1) = y_A + l_{AB}\sin(\alpha_1). \tag{1b}$$

The distance between point B and D was specified:

$$L_{BD}(\alpha_1) = \sqrt{(x_D - x_B(\alpha_1))^2 + (y_D - y_B(\alpha_1))^2}. \tag{2}$$

The rotation angle of segment BD around kinematic pair D was marked by α_2. Two angle values were determined for two different mounting options using the relationship:

$$\alpha_2(\alpha_1) = \arctan\left(\frac{y_D - y_B(\alpha_1)}{x_D - x_B(\alpha_1)}\right) + \arccos\left(\frac{l_{BD}(\alpha_1)^2 + l_{BC}^2 - l_{CD}^2}{2l_{BD}(\alpha_1)l_{BC}}\right), \tag{3}$$

$$\alpha_{2'}(\alpha_1) = \arctan\left(\frac{y_D - y_B(\alpha_1)}{x_D - x_B(\alpha_1)}\right) - \arccos\left(\frac{l_{BD}(\alpha_1)^2 + l_{BC}^2 - l_{CD}^2}{2l_{BD}(\alpha_1)l_{BC}}\right). \tag{4}$$

The location of point C was specified:

$$x_C(\alpha_1) = x_B(\alpha_1) + l_{BC}\cos\left(\alpha_2(\alpha_1)\right), \tag{5a}$$

$$y_C(\alpha_1) = y_B(\alpha_1) + l_{BC}\sin\left(\alpha_2(\alpha_1)\right); \tag{5b}$$

$$x_{C'}(\alpha_1) = x_B(\alpha_1) + l_{BC}\cos\left(\alpha_{2'}(\alpha_1)\right), \tag{6a}$$

$$y_{C'}(\alpha_1) = y_B(\alpha_1) + l_{BC}\sin\left(\alpha_{2'}(\alpha_1)\right). \tag{6b}$$

To determine the instantaneous centres of rotation being the points of intersection of two straight lines (first line passing through the kinematic pairs A and B, second line passing through the kinematic pairs C and D), the following markings were introduced:

$$m(\alpha_1) = y_B(\alpha_1) - y_A, \tag{7}$$

$$n(\alpha_1) = -\left(x_B(\alpha_1) - x_A\right), \tag{8}$$

$$k(\alpha_1) = \left(x_B(\alpha_1) - x_A\right) y_A - \left(y_B(\alpha_1) - y_A\right) x_A, \tag{9}$$

$$m_1(\alpha_1) = y_C(\alpha_1) - y_D, \tag{10}$$

$$n_1(\alpha_1) = -\left(x_C(\alpha_1) - x_D\right), \tag{11}$$

$$k_1(\alpha_1) = \left(x_C(\alpha_1) - x_D\right) y_D - \left(y_C(\alpha_1) - y_D\right) x_D, \tag{12}$$

$$m_2(\alpha_1) = y_{C'}(\alpha_1) - y_D, \tag{13}$$

$$n_2(\alpha_1) = -\left(x_{C'}(\alpha_1) - x_D\right), \tag{14}$$

$$k_2(\alpha_1) = \left(x_{C'}(\alpha_1) - x_D\right) y_D - \left(y_{C'}(\alpha_1) - y_D\right) x_D. \tag{15}$$

Instantaneous centres of location (marked by $E(x_E(\alpha_1);\ y_E(\alpha_1))$ for basic assembly and $E'(x_{E'}(\alpha_1);\ y_{E'}(\alpha_1))$ for cross-shaped assembly options) were determined from equations:

$$x_E(\alpha_1) = \frac{n(\alpha_1)k_1(\alpha_1) - k(\alpha_1)n_1(\alpha_1)}{m(\alpha_1)n_1(\alpha_1) - n(\alpha_1)m_1(\alpha_1)}, \tag{16a}$$

$$y_E(\alpha_1) = \frac{k(\alpha_1)m_1(\alpha_1) - m(\alpha_1)k_1(\alpha_1)}{m(\alpha_1)n_1(\alpha_1) - n(\alpha_1)m_1(\alpha_1)}. \tag{16b}$$

$$x_{\mathrm{E}'}(\alpha_1) = \frac{n(\alpha_1)k_2(\alpha_1) - k(\alpha_1)n_2(\alpha_1)}{m(\alpha_1)n_2(\alpha_1) - n(\alpha_1)m_2(\alpha_1)}, \tag{17a}$$

$$y_{\mathrm{E}'}(\alpha_1) = \frac{k(\alpha_1)m_2(\alpha_1) - m(\alpha_1)k_2(\alpha_1)}{m(\alpha_1)n_2(\alpha_1) - n(\alpha_1)m_2(\alpha_1)}. \tag{17b}$$

Point coordinates were transferred to AutoCAD, obtaining curves determined discretely.

Working using two different computer programs was a labour-intensive and time-consuming process, leading to the drawing of selected curves. Improving possibilities of programs were the reason for returning to the subject of coupler curves, to check the possibility of faster performance.

The comparative analysis of AutoCAD and Rhinoceros programs was performed for a crank-rocker four-bar linkage, for which the same given data were assumed: point coordinates at the base and the length of the individual links (Fig. 1). A fixed and moving centrode was established, in addition to coupler curves with cusp points.

2. Plotting curves using AutoCAD software

The method of plotting coupler curves with cusps was presented for the basic assembly option (Fig. 1a). The procedure for cross-shaped assembly was analogous.

The first construction was associated with defining the fixed centrode as a curve plotted by instantaneous centres of rotation of the mechanism. The points belonging to the curve are thus plotted at the intersection of straight lines including links 1 and 3 (in Fig. 2b it is the point marked with the letter G). The fixed centrode was constructed using the parametric capabilities of AutoCAD software – both geometric and dimensional (Fig. 2).

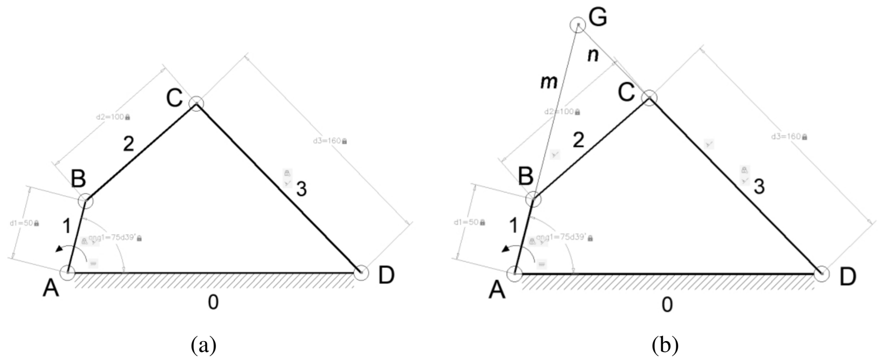


Fig. 2. Using parametric constraints in AutoCAD software to ensure: a) the proper kinematics of the four-bar linkage, b) determine instantaneous centre of rotation

The location of kinematic pairs A and D, which connect the mechanism with its frame, was determined using the *Fix* command. Using the *Coincident* command, links 2 and 3 were connected by kinematic pair B, while links 2 and 3 through pair C. Using the *Aligned* command (dimensional parametric), the lengths of individual link were set at $l_{AB} = 60$, $l_{BC} = 100$, $l_{CD} = 160$. Instantaneous centre of rotation G was plotted using sections *m* and *n*, which contained links 1 and 3. The following commands were used to properly connect the sections with the links: *Coincident*, connecting the endpoints of sections: *m* with link 1 using kinematic pair A, *m* and *n* at point G, *n* with link 3 using kinematic pair D, *Colinear*, ensuring the collinearity of sections and links. Driving link 1 was bound by the *ang1* command to the base, thereby determining the angle of the driving link displacement.

The geometric and dimensional constraints set in AutoCAD software, as well as the successively defined values of the *ang1* angle were used to plot the movements of the individual links and to determine the location of point G. A change of the angle of the driving link by 10 degrees each time (smaller values in selected positions) was assumed. The position of point G was highlighted by changing the successive angle values (Fig. 3). The fixed centrode in this version is a discretely given curve, composed of two branches c_{s1} and c_{s2}. The presented curve is an interpolation of points using the Spline command. The parallel course of the lines *m* and *n* means that the instantaneous centre of rotation is the infinite point. In this case, the mechanism takes two such positions. Thus, the instantaneous centres of rotations determine two branches of the fixed centrode (c_{s1}, c_{s2}).

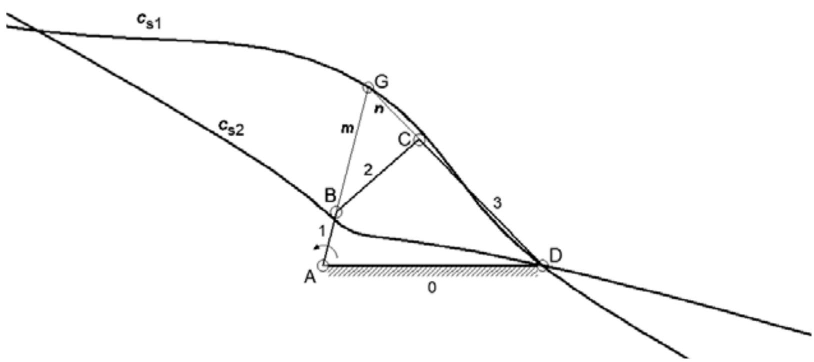
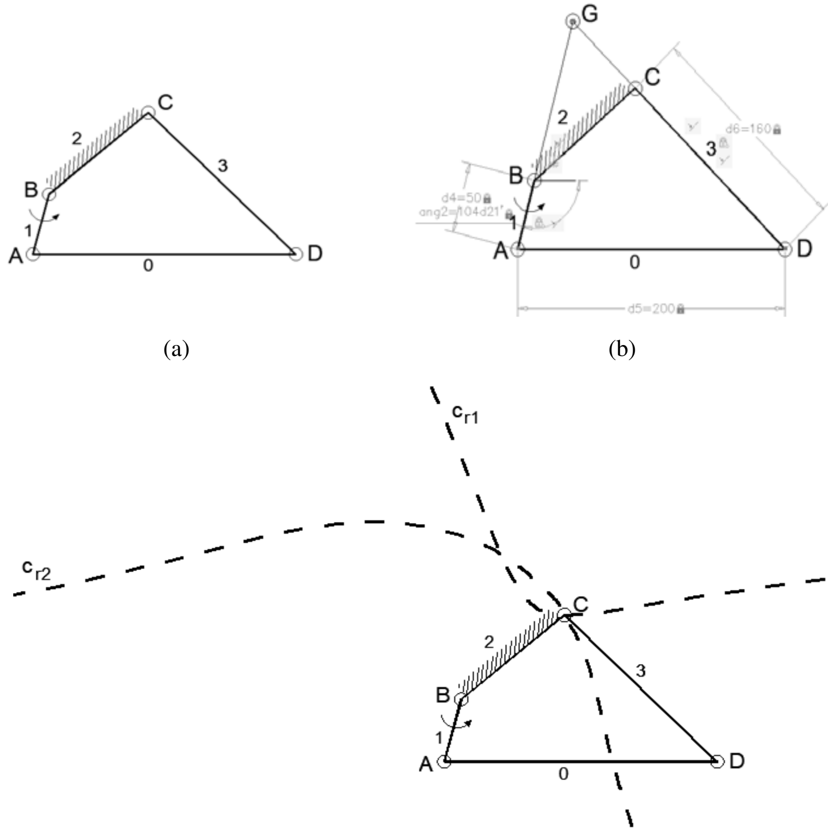


Fig. 3. Fixed centrode plotted for the four-bar linkage

The next stage in determining coupler curves with cusps was the construction of the moving centrodes of the four-bar linkage. They are curves plotted by instantaneous centres of rotation determined for the inversion of a given four-bar linkage. Thus, an inversion of the mechanism was performed – a linkage (link 2) was assumed to be the frame, making the location of link 0 changeable (Fig. 4a). The manner of proceeding when creating a moving centrode was carried out analogously as in the case of a fixed centrode – the appropriate geometric and dimensional constraints were set, with successive values of the *ang2* angle being set in order to plot the placement of instantaneous centres of rotation (Fig. 4b). The curve is composed of two branches: c_{r1}, c_{r2}.



(a) (b) (c)

Fig. 4. Plotting the moving centrode: a) mechanism inversion, b) setting parametric constraints, c) plotting the moving centrode

The moving centrode is plotted for a strictly defined position of the starting mechanism. For the four-bar linkage presented in Fig. 1, inversion was performed and a moving centrode was plotted for an *ang1* value of 75 degrees and 39 minutes. The movability of the centrode is associated with the fact that for a different *ang1* angle value another centrode is plotted. The moving centrode was relocated to the starting mechanism from Fig. 3 (Fig. 5). Point G is a common point and the point of contact of both centrodes. By assuming points on the moving centrode, the coupler curves are plotted, whose cusps are located on the fixed centrode. Point E_4 = G, located at the intersection of two branches of the moving centrode, which is the point of contact between the moving centrode (branch c_{r2}) and the fixed centrode (branch c_{s1}), plots a curve with two cusps. Point E_6, which has the same location as point C, has a movement path in the form of a circular arch with a centre at point D and a CD radius. Its length, spanning the distance between the branches of the fixed centrode, is at the same time the range of movement of rocker 3 of the four-bar linkage.

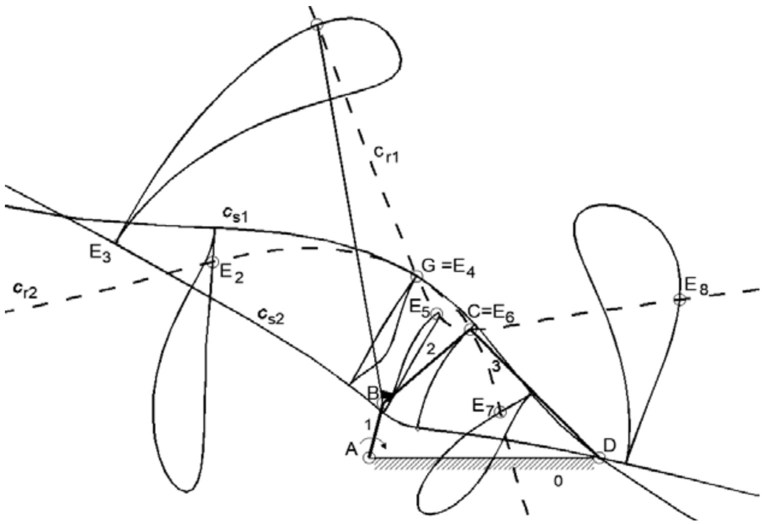
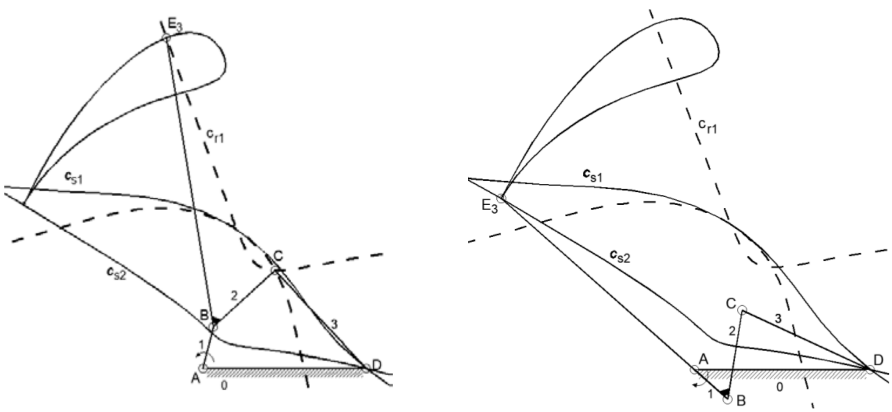


Fig. 5. Coupler curves with cusps

The adopted manner of connecting point E with the link (through section BE) led to the following observation. The cusp is obtained at the location in which link 1 and section BE take on the same direction (Fig. 6).



(a) (b)

Fig. 6. Coupler curve plotted by point E_3: a) position of the point on the moving centrode, b) of point E_3 on the cusp of the coupler curve for $ang1 = 318°69'$

3. Plotting curves in Rhinoceros and Grasshopper software

The process of creating an algorithm in the Grasshopper plug-in was started with generating four points: A, B, C and D, which defined the characteristic points of the four-bar linkage, and setting their mutual relations. It should be noted that, at this stage, the formulation of the algorithm takes place in an analogous manner to creating geometry in AutoCAD software. The difference is that, instead of executing specific AutoCAD commands, appropriate commands are entered into the algorithm, e.g., Move, Rotate, Circle, whose parameters are controlled by variables. In Grasshopper, variables can be defined using a slider, which makes it possible to fluidly change their value. Sliders are connected to blocks which represent individual functions, thus ensuring a flow of data.

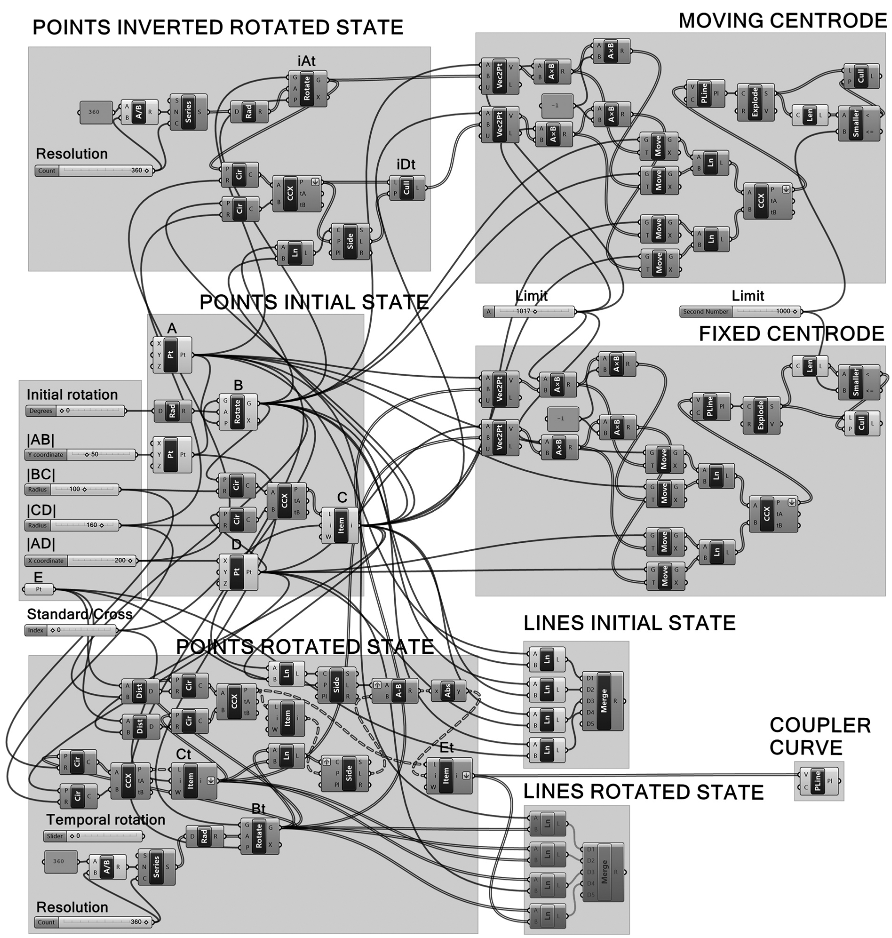


Fig. 7. Grasshopper algorithm

Point A, with coordinates 0,0,0 was defined as the first one. The coordinates of points B and D were defined with the use of the sliders, whose values determined the lengths of sections AB and AD (B coordinates: 0,|AB|,0 and D coordinates: |AD|,0,0). Point C was plotted using the intersection points of two circles with their centres at points B and D, respectively, and with radiuses corresponding to the lengths of sections BC and CD. These values were also defined using sliders. Due to the fact that two such intersection points exist, additional functions were introduced into the algorithm making it possible to select only one of them. Furthermore, the ability to define the initial state of the mechanism by rotating point B relative to A by a given angle using a slider was ensured as well.

Point E was defined directly within Rhinoceros, and introduced into the algorithm using the Point parameter. The location of the point is determined in a relation to the initial state of the mechanism (initial angle of rotation).

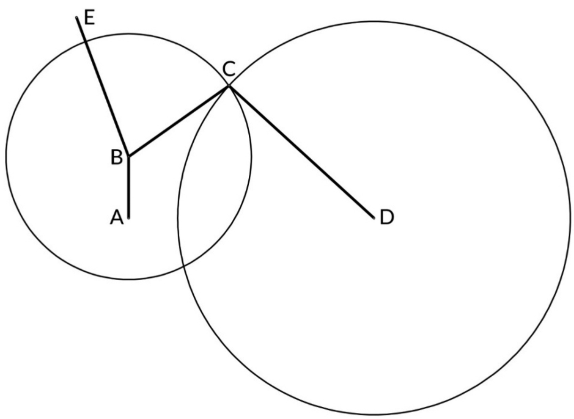


Fig. 8. Four-bar linkage generated by the algorithm with the circles determining the position of C point

In order to perform a simulation of the mechanism's operation, point B_t, was defined through the rotation of point B from its defined initial position relative to point A by an angle of rotation set using the slider. By analogy to point C, point C_t was determined, which constitutes an image of point C for a given rotation of point B. The momentary location of point E, marked as E_t, was defined by the intersection of two circles with their centres at points B_t and C_t and with radiuses corresponding to the distances between points B, E and C, E, respectively. The tracing of the mechanism's operation is possible by moving the slider that defines the rotation angle of point B and thus that of the remaining points of the four-bar linkage.

An essential characteristic of algorithms created using the Grasshopper plug-in is the ability to perform operations on big data. This property was used to plot the coupler curve formed by the positions of point E, marked as E_t. First, a list of the values of rotation angles, ranging from 0 to 359 with an interval of 1 was created, in the form of 0, 1, 2, . . . , 359. This list was linked with the *Rotate* tool, which defined the rotation of point B in place of a slider. This resulted in the generation of not just one, but 360 locations for point B_t, with analogous 360 positions for points C_t and E_t. The linking of successive locations of point R, marked as E_t, makes it possible to plot a coupler curve, which is also performed from the level of the algorithm.

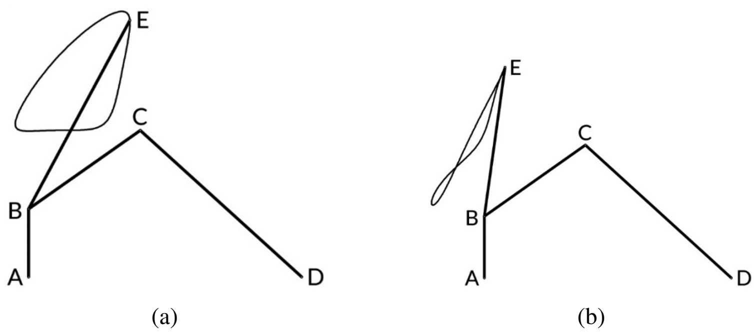


Fig. 9. a) b) Coupler curves generated by the algorithm for different locations of E point

Based on operations on big data and geometric principles described in the previous points, the fixed and moving centrodes were plotted for the initial position of the mechanism.

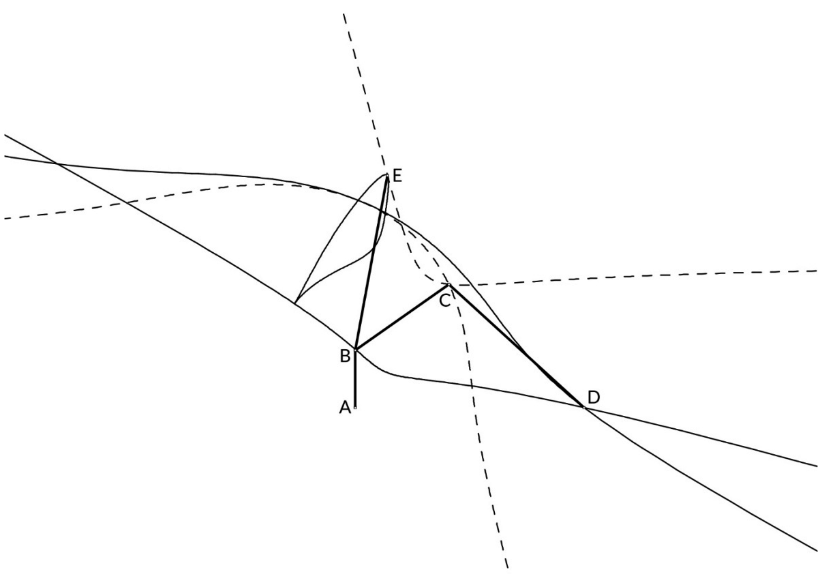


Fig. 10. Fixed centrode (solid line) and moving centrode (dashed line) generated by the algorithm; coupler curve whit the cusp located on the fixed centrode generated by the algorithm for the given point E located at the moving centrode

The fundamental benefit of the use of algorithms created using the Grasshopper plug-in is the fact that the algorithm described above makes it possible to analyse the shape of the coupler curve and centrodes not only for a single defined case, but also for mechanisms of any size, with any initial rotation angle and for any and all positions of point E. Changes to the initial rotation angle of the mechanism and the lengths of its links can be made by altering slider settings, while the position of point E by moving it in Rhinoceros itself.

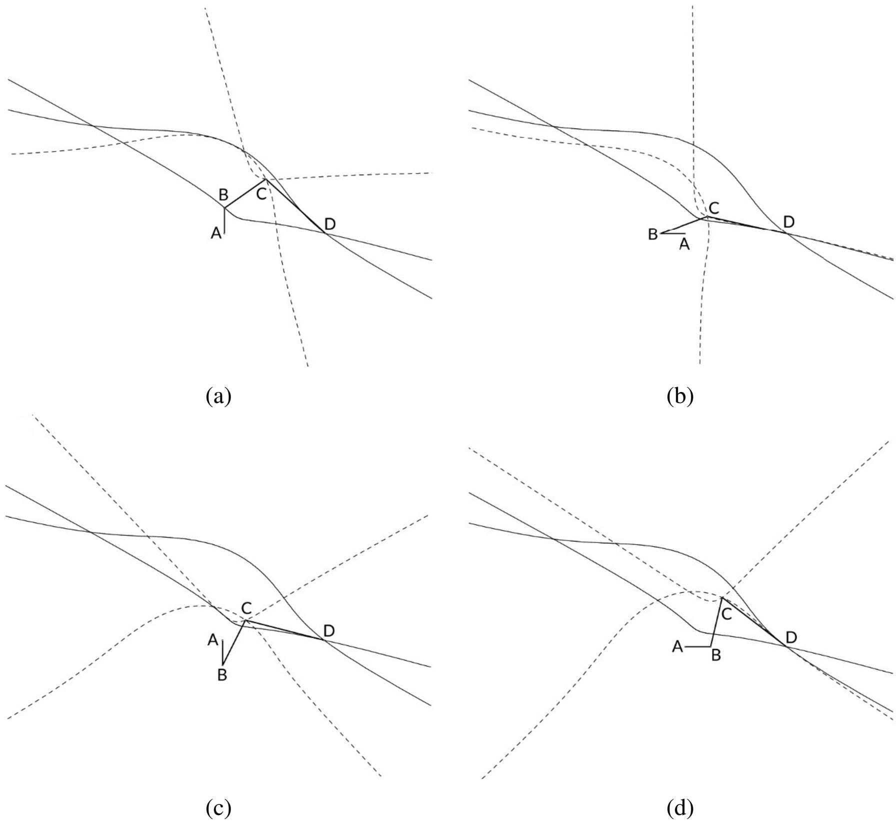


Fig. 11. Fixed (solid line) and moving (dashed line) centrodes generated by the algorithm for different settings of initial angle of rotation parameter: a) 0 degrees, b) 90 degrees, c) 180 degrees, d) 270 degrees

4. Conclusions

We have achieved the intended goal of the work, which was the plotting of coupler curves with a cusp in graphical software: AutoCAD and Rhinoceros. The research that had been performed became the source of numerous conclusions and observations.

It is visible both in AutoCAD and Rhinoceros software that cusps are located at the intersection of straight lines that contain links 1 and 3, as well as when link 1 and section BE take on the same direction. In Rhinoceros, when changing the initial angle of the mechanism's rotation, a rolling of the moving centrode along the fixed centrode can be observed. This is associated with the fact that both curves are always in contact at one shared point (in Fig. 5 it is point G) [18, 20].

Numerous capabilities of both tested programs, and differences between them, became apparent. In AutoCAD, which is a program well-known and highly popular around the world, with the use of parametric constraints we can perform a quick simulation of the movement of a four-bar linkage. We can then observe the movements of coupler points. However, the plotting of the curves themselves (the fixed and moving centrode, as well as the coupler curve) is associated with the tiresome marking of the positions of the points that define them. It is a time-consuming process and is thus recommended for plotting only selected curves.

Rhinoceros and Grasshopper programs find particular interest among architects [19], as well as in the case of research on geometric problems. They are, however, less-known in the community of researchers from other disciplines. However, the results that were obtained encourage us to familiarise ourselves with this design tool. It is very well-suited for the complete study of the coupler surface. The possibility of using big data and controlling inputs using sliders makes it possible to generate any number of curves. We can also change the dimensions of four-bar linkages, simultaneously define the course of centrodes and plot coupler curves in a fully automatic manner.

The mathematical programs Mathcad and Mathematica have the capacity of determining both a fixed and a moving centrode. With them, however, no mechanism can be built or its movement simulated. Hence, the results of numerical calculations must be transferred to programs such as Working Model or Inventor Professional, which are designed for building mechanisms and determining their kinematics and dynamics, but not for determining the centrodes themselves. Therefore, for the purpose of building mechanisms whose couplers draw curves with the cusp point, a mathematical program is necessary, whereas for simulating their movement a different program must be used. Rhinoceros and Grasshopper programs feature all the necessary properties to enable a simultaneous visualisation of all elements, i.e., of the mechanism itself, of both centrodes and of coupler curves that are drawn while the position of the driving links is changing. The knowledge of the programming language is not required to use these programs, unlike to the case of the Matlab program.

AutoCAD and its parametric constraints make it possible to quickly construct a four-bar linkage and simulate its movements. We can thus successfully use it to educate students and to visualise the repositioning of the mechanism's links. However, Rhinoceros and Grasshopper are better programs for the in-depth study of the coupler curve. Due to the geometric and mathematical relationships used in them, they can prove to be useful tools in many studies concerning mechanisms.

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