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The parametric optimization of a system with two delays and a PI controller

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In the paper a Lyapunov matrices approach to the parametric optimization problem of a time-delay system with two commensurate delays and a PI-controller is presented. The value of integral quadratic performance index is equal to the value of the Lyapunov functional for the initial function of the time-delay system. The Lyapunov functional is determined by means of the Lyapunov matrix. In the paper is presented the example of a scalar system with two delays and a PI controller.

Key words: time-delay system, Lyapunov matrix, Lyapunov functional

1. Introduction

The Lyapunov functionals are used to test the stability of systems, in calculation of the robustness bounds for uncertain time delay systems, in computation of the exponential estimates for the solutions of time delay systems. The Lyapunov quadratic functionals are also used to calculation of a value of a quadratic performance index, for example the integral of the squared error, in the parametric optimization problem for time delay systems. One constructs a functional for a system with time delay with a given time derivative whose is equal to the negatively definite quadratic form of a system state. The value of that functional at the initial state of time delay system is equal to the value of the integral of squared error.

There are two methods of determination of the Lyapunov functionals. The first method was proposed by Repin [13]. This method was developed by Duda in many works. The last paper due to this method is [2]. The second method of determination of the Lyapunov functional by means of Lyapunov matrices is very

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popular, see for example [6–12, 14, 15]. Duda used this method in parametric optimization problem for a system with one delay [1], for a neutral system [3], for a system with both lumped and distributed time delay [5] and for a system with two delays [4]. In the paper [4] was considered the parametric optimization problem for a scalar system with two delays and a P controller.

In the paper is presented a Lyapunov matrices approach to the parametric optimization problem of a scalar time-delay system with two delays and a PI controller. The paper extends the results presented in [4] to the parametric optimization problem for a scalar system with two delays and PI controller.

The paper is organized as follows. In Section 2 is presented the formula for the value of the performance index and the properties of the Lyapunov matrix. In Section 3 is presented the procedure of determination of the Lyapunov matrix for a system with two delays. The parametric optimization problem for a scalar system with two delays and a PI controller is formulated and solved in Section 4. In Section 5 is given an example. Conclusions ends the paper.

2. Preliminaries

Let us consider a retarded type time-delay system

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = \sum\limits_{j=0}^{m} A_j x_t(-r_j), \\ x_{t_0}(\theta) = \varphi(\theta) \end{cases} \tag{1}$$

for $t \geqslant t_0$, $\theta \in [-r, 0]$, where $x(t) \in \mathbb{R}^n$, $A_j \in \mathbb{R}^{n\times n}$, $0 = r_0 < r_1 < \ldots < r_m = r$, x_t is a shifted restriction of x and is defined by the formula

$$x_t(\theta) = x(t+\theta). \tag{2}$$

Functions $x_t, \varphi \in PC([-r, 0], \mathbb{R}^n)$ – the space of piece-wise continuous vector valued functions defined on the segment $[-r, 0]$.

In parametric optimization problem will be used the index of quality

$$J = \int\limits_{t_0}^{\infty} x^T(t) W x(t)\,\mathrm{d}t, \tag{3}$$

where $W \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix.

The value of the performance index (3) is given by the formula, see for instance [1]

$$J = \varphi^T(0)U(0)\varphi(0) + 2\varphi^T(0)\sum_{j=1}^{m}\int_{-r_j}^{0} U^T(\theta + r_j)A_j\varphi(\theta)\,\mathrm{d}\theta +$$

$$+\sum_{j=1}^{m}\sum_{k=1}^{m}\int_{-r_j}^{0}\int_{-r_k}^{0}\varphi^T(\theta)A_j^T U(r_j - r_k + \theta - \eta)A_k\varphi(\eta)\,\mathrm{d}\eta\,\mathrm{d}\theta, \tag{4}$$

where U is a matrix valued function and is called the Lyapunov matrix.

Theorem 1 [14]. *The Lyapunov matrix $U(\xi)$ has the following properties:*

Dynamic property

$$\frac{\mathrm{d}}{\mathrm{d}\xi}U(\xi) = \sum_{j=0}^{m} U(\xi - r_j)A_j \tag{5}$$

for $\xi \geqslant 0$.

Symmetry property

$$U(-\xi) = U^T(\xi) \tag{6}$$

for $\xi \geqslant 0$.

Algebraic property

$$\sum_{j=0}^{m}\left[U(-r_j)A_j + A_j^T U(r_j)\right] = -W. \tag{7}$$

3. A Lyapunov matrix for a system with two commensurate delays

Let us consider a system

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = A_0x(t) + A_1x(t-h) + A_2x(t-2h), \\ x(\theta) = \varphi(\theta) \end{cases} \tag{8}$$

for $t \geqslant 0$ and $\theta \in [-2h, 0]$, where $A_0, A_1, A_2 \in \mathbb{R}^{n\times n}$ and $\varphi \in PC([-2h, 0], \mathbb{R}^n)$, $0 < h \in \mathbb{R}$.

A set of equations (5), (6), (7) for system (8) takes a form

$$\frac{\mathrm{d}}{\mathrm{d}\xi}U(\xi) = U(\xi)A_0 + U(\xi - h)A_1 + U(\xi - 2h)A_2, \tag{9}$$

$$U(-\xi) = U^T(\xi), \tag{10}$$

$$\begin{aligned} U(0)A_0 + U(-h)A_1 + U(-2h)A_2 + A_0^T U(0) + \\ + A_1^T U(h) + A_2^T U(2h) = -W \end{aligned} \tag{11}$$

for $\xi \in [0, 2h]$.

The relation (10) implies

$$U(-h) = U^T(h) \quad \text{and} \quad U(-2h) = U^T(2h)$$

so we can write equation (11) in a form

$$\begin{aligned} U(0)A_0 + U^T(h)A_1 + U^T(2h)A_2 + A_0^T U(0) + \\ + A_1^T U(h) + A_2^T U(2h) = -W. \end{aligned} \tag{12}$$

Formula (10) extends the function U defined on the segment $[0, 2h]$ to the segment $[-2h, 0]$.

Indeed for $\xi \in [0, 2h]$, $U(-\xi) = U^T(\xi)$. For $\tau = -\xi$, $U(\tau) = U^T(-\tau)$ and $\tau \in [-2h, 0]$.

We define the functions $U_1(\xi)$, $U_2(\xi)$, $Z_1(\xi)$, $Z_2(\xi)$ for $\xi \in [0, h]$

$$U_1(\xi) = U(\xi), \tag{13}$$

$$U_2(\xi) = U(h + \xi), \tag{14}$$

$$Z_1(\xi) = U(\xi - h) = U^T(-\xi + h), \tag{15}$$

$$Z_2(\xi) = U(\xi - 2h) = U^T(-\xi + 2h). \tag{16}$$

For $\xi \in [0, h]$ equation (9) can be written in a form

$$\frac{\mathrm{d}}{\mathrm{d}\xi} U_1(\xi) = U_1(\xi)A_0 + Z_1(\xi)A_1 + Z_2(\xi)A_2\,. \tag{17}$$

For $\xi + h = \varsigma \in [h, 2h]$

$$U(\varsigma) = U(\xi + h) = U_2(\xi),$$
$$U(\varsigma - h) = U(\xi) = U_1(\xi),$$
$$U(\varsigma - 2h) = U(\xi - h) = Z_1(\xi)$$

and equation (9) can be written in a form

$$\frac{\mathrm{d}}{\mathrm{d}\xi} U_2(\xi) = U_2(\xi)A_0 + U_1(\xi)A_1 + Z_1(\xi)A_2\,. \tag{18}$$

We compute the derivative of $Z_1(\xi)$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\xi}Z_1(\xi) &= \frac{\mathrm{d}}{\mathrm{d}\xi}U^T(-\xi+h) = \frac{\mathrm{d}}{\mathrm{d}\tau}U^T(\tau)\frac{\mathrm{d}\tau}{\mathrm{d}\xi} = -\frac{d}{d\tau}U^T(\tau) = \\ &= -A_0^TU^T(\tau) - A_1^TU^T(\tau-h) - A_2^TU^T(\tau-2h) = \\ &= -A_0^TU^T(-\xi+h) - A_1^TU^T(-\xi) - A_2^TU^T(-\xi-h) = \\ &= -A_0^TZ_1(\xi) - A_1^TU_1(\xi) - A_2^TU_2(\xi),\end{aligned} \tag{19}$$

where $\tau = -\xi + h$ and the derivative of $Z_2(\xi)$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\xi}Z_2(\xi) &= \frac{\mathrm{d}}{\mathrm{d}\xi}U^T(-\xi+2h) = \frac{\mathrm{d}}{\mathrm{d}\tau}U^T(\tau)\frac{\mathrm{d}\tau}{\mathrm{d}\xi} = -\frac{\mathrm{d}}{\mathrm{d}\tau}U^T(\tau) = \\ &= -A_0^TU^T(\tau) - A_1^TU^T(\tau-h) - A_2^TU^T(\tau-2h) = \\ &= -A_0^TU^T(-\xi+2h) - A_1^TU^T(-\xi+h) - A_2^TU^T(-\xi) = \\ &= -A_2^TU_1(\xi) - A_1^TZ_1(\xi) - A_0^TZ_2(\xi),\end{aligned} \tag{20}$$

where $\tau = -\xi + 2h$.

We have received a set of ordinary differential equations

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}\xi}U_1(\xi) = U_1(\xi)A_0 + Z_1(\xi)A_1 + Z_2(\xi)A_2, \\ \dfrac{\mathrm{d}}{\mathrm{d}\xi}U_2(\xi) = U_1(\xi)A_1 + U_2(\xi)A_0 + Z_1(\xi)A_2, \\ \dfrac{\mathrm{d}}{\mathrm{d}\xi}Z_1(\xi) = -A_1^TU_1(\xi) - A_2^TU_2(\xi) - A_0^TZ_1(\xi), \\ \dfrac{\mathrm{d}}{\mathrm{d}\xi}Z_2(\xi) = -A_2^TU_1(\xi) - A_1^TZ_1(\xi) - A_0^TZ_2(\xi)\end{cases} \tag{21}$$

for $\xi \in [0, h]$ with initial conditions

$$U_1(0), \quad U_2(0), \quad Z_1(0), \quad Z_2(0).$$

There hold relations

$$U(0) = U_1(0), \quad U(h) = U_2(0), \quad U(2h) = U_2(h)$$

and therefore equation (12) takes a form

$$U_1(0)A_0 + U_2^T(0)A_1 + U_2^T(h)A_2 + A_0^TU_1(0) + A_1^TU_2(0) + A_2^TU_2(h) = -W. \tag{22}$$

Using the Kronecker product we can express equation (21) in a form

$$\begin{bmatrix} \dfrac{\mathrm{d}}{\mathrm{d}\xi} col\, U_1(\xi) \\ \dfrac{\mathrm{d}}{\mathrm{d}\xi} col\, U_2(\xi) \\ \dfrac{\mathrm{d}}{\mathrm{d}\xi} col\, Z_1(\xi) \\ \dfrac{\mathrm{d}}{\mathrm{d}\xi} col\, Z_2(\xi) \end{bmatrix} = \mathcal{H} \begin{bmatrix} col\, U_1(\xi) \\ col\, U_2(\xi) \\ col\, Z_1(\xi) \\ col\, Z_2(\xi) \end{bmatrix} \tag{23}$$

for $\xi \in [0, h]$ with initial conditions

$$col\, U_1(0), \quad col\, U_2(0), \quad col\, Z_1(0), \quad col\, Z_2(0),$$

where

$$\mathcal{H} = \begin{bmatrix} A_0^T \otimes I & 0 & A_1^T \otimes I & A_2^T \otimes I \\ A_1^T \otimes I & A_0^T \otimes I & A_2^T \otimes I & 0 \\ -I \otimes A_1^T & -I \otimes A_2^T & -I \otimes A_0^T & 0 \\ -I \otimes A_2^T & 0 & -I \otimes A_1^T & -I \otimes A_0^T \end{bmatrix}.$$

Solution of equation (23) is given in a form

$$\begin{bmatrix} col\, U_1(\xi) \\ col\, U_2(\xi) \\ col\, Z_1(\xi) \\ col\, Z_2(\xi) \end{bmatrix} = \Phi(\xi) \begin{bmatrix} col\, U_1(0) \\ col\, U_2(0) \\ col\, Z_1(0) \\ col\, Z_2(0) \end{bmatrix}, \tag{24}$$

where a matrix

$$\Phi(\xi) = \begin{bmatrix} \Phi_{11}(\xi) & \Phi_{12}(\xi) & \Phi_{13}(\xi) & \Phi_{14}(\xi) \\ \Phi_{21}(\xi) & \Phi_{22}(\xi) & \Phi_{23}(\xi) & \Phi_{24}(\xi) \\ \Phi_{31}(\xi) & \Phi_{32}(\xi) & \Phi_{33}(\xi) & \Phi_{34}(\xi) \\ \Phi_{41}(\xi) & \Phi_{42}(\xi) & \Phi_{43}(\xi) & \Phi_{44}(\xi) \end{bmatrix} \tag{25}$$

is a fundamental matrix of system (23).

We determine the initial conditions

$$col\, U_1(0), \quad col\, U_2(0), \quad col\, Z_1(0), \quad col\, Z_2(0).$$

From equation. (24) we obtain

$$\begin{aligned} col\, U_1(h) = col\, U_2(0) = \Phi_{11}(h) col\, U_1(0) + \\ + \Phi_{12}(h) col\, U_2(0) + \Phi_{13}(h) col\, Z_1(0) + \Phi_{14}(h) col\, Z_2(0), \end{aligned} \tag{26}$$

$$col\, Z_1(h) = col\, U_1(0) = \Phi_{31}(h) col\, U_1(0) + \\ + \Phi_{32}(h) col\, U_2(0) + \Phi_{33}(h) col\, Z_1(0) + \Phi_{34}(h) col\, Z_2(0), \quad (27)$$

$$col\, Z_2(h) = col\, Z_1(0) = \Phi_{41}(h) col\, U_1(0) + \\ + \Phi_{42}(h) col\, U_2(0) + \Phi_{43}(h) col\, Z_1(0) + \Phi_{44}(h) col\, Z_2(0), \quad (28)$$

$$col\, U_2(h) = \Phi_{21}(h) col\, U_1(0) + \Phi_{22}(h) col\, U_2(0) + \\ + \Phi_{23}(h) col\, Z_1(0) + \Phi_{24}(h) col\, Z_2(0). \quad (29)$$

We reshape equations (26), (27) and (28) and add equation (22). In this way we attain a set of algebraic equations which enables us to calculate the initial conditions of system (23).

$$\Phi_{11}(h) col\, U_1(0) + (\Phi_{12}(h) - 1) col\, U_2(0) + \\ + \Phi_{13}(h) col\, Z_1(0) + \Phi_{14}(h) col\, Z_2(0) = 0, \quad (30)$$

$$(\Phi_{31}(h) - 1) col\, U_1(0) + \Phi_{32}(h) col\, U_2(0) + \\ + \Phi_{33}(h) col\, Z_1(0) + \Phi_{34}(h) col\, Z_2(0) = 0, \quad (31)$$

$$\Phi_{41}(h) col\, U_1(0) + \Phi_{42}(h) col\, U_2(0) + \\ + (\Phi_{43}(h) - 1) col\, Z_1(0) + \Phi_{44}(h) col\, Z_2(0) = 0, \quad (32)$$

$$col\, U_2(h) = \Phi_{21}(h) col\, U_1(0) + \Phi_{22}(h) col\, U_2(0) + \\ + \Phi_{23}(h) col\, Z_1(0) + \Phi_{24}(h) col\, Z_2(0), \quad (33)$$

$$U_1(0) A_0 + U_2^T(0) A_1 + U_2^T(h) A_2 + A_0^T U_1(0) + \\ + A_1^T U_2(0) + A_2^T U_2(h) = -W. \quad (34)$$

We proved the following theorem.

Theorem 2 [4]. *The functions $U_1(\xi)$, $U_2(\xi)$, $Z_1(\xi)$, $Z_2(\xi)$ for $\xi \in [0, h]$ are obtained by solving the set of ordinary differential equations*

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}\xi} col\, U_1(\xi) \\ \frac{\mathrm{d}}{\mathrm{d}\xi} col\, U_2(\xi) \\ \frac{\mathrm{d}}{\mathrm{d}\xi} col\, Z_1(\xi) \\ \frac{\mathrm{d}}{\mathrm{d}\xi} col\, Z_2(\xi) \end{bmatrix} = \begin{bmatrix} A_0^T \otimes I & 0 & A_1^T \otimes I & A_2^T \otimes I \\ A_1^T \otimes I & A_0^T \otimes I & A_2^T \otimes I & 0 \\ -I \otimes A_1^T & -I \otimes A_2^T & -I \otimes A_0^T & 0 \\ -I \otimes A_2^T & 0 & -I \otimes A_1^T & -I \otimes A_0^T \end{bmatrix} \begin{bmatrix} col\, U_1(\xi) \\ col\, U_2(\xi) \\ col\, Z_1(\xi) \\ col\, Z_2(\xi) \end{bmatrix} \quad (35)$$

for which initial conditions col $U_1(0)$*, col* $U_2(0)$*, col* $Z_1(0)$*, col* $Z_2(0)$ *are obtained by solving the set of algebraic equations*

$$\begin{aligned}&\Phi_{11}(h)col\,U_1(0) + (\Phi_{12}(h) - 1)col\,U_2(0) + \\ &\quad + \Phi_{13}(h)col\,Z_1(0) + \Phi_{14}(h)col\,Z_2(0) = 0,\end{aligned} \tag{36}$$

$$\begin{aligned}&(\Phi_{31}(h) - 1)col\,U_1(0) + \Phi_{32}(h)col\,U_2(0) + \\ &\quad + \Phi_{33}(h)col\,Z_1(0) + \Phi_{34}(h)col\,Z_2(0) = 0,\end{aligned} \tag{37}$$

$$\begin{aligned}&\Phi_{41}(h)col\,U_1(0) + \Phi_{42}(h)col\,U_2(0) + \\ &\quad + (\Phi_{43}(h) - 1)col\,Z_1(0) + \Phi_{44}(h)col\,Z_2(0) = 0,\end{aligned} \tag{38}$$

$$\begin{aligned}&col\,U_2(h) = \Phi_{21}(h)col\,U_1(0) + \Phi_{22}(h)col\,U_2(0) + \\ &\quad + \Phi_{23}(h)col\,Z_1(0) + \Phi_{24}(h)col\,Z_2(0),\end{aligned} \tag{39}$$

$$\begin{aligned}&U_1(0)A_0 + U_2^T(0)A_1 + U_2^T(h)A_2 + A_0^T U_1(0) + \\ &\quad + A_1^T U_2(0) + A_2^T U_2(h) = -W,\end{aligned} \tag{40}$$

where

$$\Phi(\xi) = \begin{bmatrix} \Phi_{11}(\xi) & \Phi_{12}(\xi) & \Phi_{13}(\xi) & \Phi_{14}(\xi) \\ \Phi_{21}(\xi) & \Phi_{22}(\xi) & \Phi_{23}(\xi) & \Phi_{24}(\xi) \\ \Phi_{31}(\xi) & \Phi_{32}(\xi) & \Phi_{33}(\xi) & \Phi_{34}(\xi) \\ \Phi_{41}(\xi) & \Phi_{42}(\xi) & \Phi_{43}(\xi) & \Phi_{44}(\xi) \end{bmatrix} \tag{41}$$

is a fundamental matrix of system (35) and $\otimes$ *denotes the Kronecker product of matrices.*

4. Parametric optimization for a scalar system with two delays and a PI controller

Let us consider a scalar system with two delays and a PI-controller

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = ax(t) + bx(t-h) + cu(t-2h), \\ u(t) = -kx(t) - \dfrac{1}{T_i}\displaystyle\int_0^t x(\xi)\mathrm{d}\xi, \\ x(\theta) = \varphi_1(\theta), \end{cases} \tag{42}$$

$t \geqslant 0$, $x(t) \in \mathbb{R}$ is the state of system (42), $u(t) \in \mathbb{R}$ is the control, $\varphi_1(\theta)$ for $\theta \in [-2h, 0]$ is the initial function, $0 \leqslant h$, $2h$ are time delays, the parameter k is a gain and the parameter T_i is the integral time of a PI-controller.

One introduces the state variables $x_1(t)$ and $x_2(t)$ as follows

$$\begin{cases} x_1(t) = x(t), \\ x_2(t) = \dfrac{1}{T_i}\displaystyle\int\limits_0^t x(\xi)\,\mathrm{d}\xi. \end{cases} \tag{43}$$

The set of equations (42) can be written in a form

$$\begin{cases} \dfrac{\mathrm{d}x_1(t)}{\mathrm{d}t} = ax_1(t) + bx_1(t-h) + cu(t-2h), \\ \dfrac{\mathrm{d}x_2(t)}{\mathrm{d}t} = \dfrac{1}{T_i}x_1(t), \\ u(t) = -kx_1(t) - x_2(t), \\ x_1(\theta) = \varphi_1(\theta), \\ x_2(\theta) = \varphi_2(\theta), \end{cases} \tag{44}$$

where

$$\varphi_2(\theta) = \frac{1}{T_i}\int\limits_0^\theta \varphi_1(\xi)\,\mathrm{d}\xi \tag{45}$$

for $t \geqslant 0$ and $\theta \in [-2h, 0]$.

One can reshape equation (44) to a form

$$\begin{cases} \dfrac{\mathrm{d}x_1(t)}{\mathrm{d}t} = ax_1(t) + bx_1(t-h) - ckx_1(t-2h) - cx_2(t-2h), \\ \dfrac{\mathrm{d}x_2(t)}{\mathrm{d}t} = \dfrac{1}{T_i}x_1(t), \\ x_1(\theta) = \varphi_1(\theta), \\ x_2(\theta) = \varphi_2(\theta) \end{cases} \tag{46}$$

for $t \geqslant 0$ and $\theta \in [-2h, 0]$.

In parametric optimization problem we use the index of quality

$$J = \int\limits_0^\infty \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix} \begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \mathrm{d}t, \tag{47}$$

where $w_1, w_2 > 0$ and $x_1(t)$, $x_2(t)$ is a solution of Eq. (46) for initial functions φ_1 and φ_2.

Now we can formulate the parametric optimization problem for system (46) which is equivalent to system (42).

Problem 2 *We search for such values of parameters k and T_i of the PI controller whose minimize the value of the performance index (47).*

The value of the performance index is given by formula (4). To obtain it we need the Lyapunov matrix $U(\xi)$.

System of equations (35) takes a form

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}\xi} col\, U_1(\xi) \\ \frac{\mathrm{d}}{\mathrm{d}\xi} col\, U_2(\xi) \\ \frac{\mathrm{d}}{\mathrm{d}\xi} col\, Z_1(\xi) \\ \frac{\mathrm{d}}{\mathrm{d}\xi} col\, Z_2(\xi) \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{bmatrix} \begin{bmatrix} col\, U_1(\xi) \\ col\, U_2(\xi) \\ col\, Z_1(\xi) \\ col\, Z_2(\xi) \end{bmatrix}, \tag{48}$$

where

$$U_1(\xi) = \begin{bmatrix} U_1^{11}(\xi) & U_1^{12}(\xi) \\ U_1^{21}(\xi) & U_1^{22}(\xi) \end{bmatrix}, \tag{49}$$

$$U_2(\xi) = \begin{bmatrix} U_2^{11}(\xi) & U_2^{12}(\xi) \\ U_2^{21}(\xi) & U_2^{22}(\xi) \end{bmatrix}, \tag{50}$$

$$Z_1(\xi) = \begin{bmatrix} Z_1^{11}(\xi) & Z_1^{12}(\xi) \\ Z_1^{21}(\xi) & Z_1^{22}(\xi) \end{bmatrix}, \tag{51}$$

$$Z_2(\xi) = \begin{bmatrix} Z_2^{11}(\xi) & Z_2^{12}(\xi) \\ Z_2^{21}(\xi) & Z_2^{22}(\xi) \end{bmatrix}, \tag{52}$$

$$G_{11} = G_{22} = \begin{bmatrix} a & 0 & \frac{1}{T_i} & 0 \\ 0 & a & 0 & \frac{1}{T_i} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{53}$$

$$G_{12} = G_{24} = G_{34} = G_{42} = 0_{(4,4)}, \tag{54}$$

$$G_{13} = G_{21} = \begin{bmatrix} b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{55}$$

$$G_{14} = G_{23} = \begin{bmatrix} -ck & 0 & 0 & 0 \\ 0 & -ck & 0 & 0 \\ -c & 0 & 0 & 0 \\ 0 & -c & 0 & 0 \end{bmatrix}, \tag{56}$$

$$G_{31} = G_{43} = \begin{bmatrix} -b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{57}$$

$$G_{32} = G_{41} = \begin{bmatrix} ck & 0 & 0 & 0 \\ c & 0 & 0 & 0 \\ 0 & 0 & ck & 0 \\ 0 & 0 & c & 0 \end{bmatrix}, \tag{58}$$

$$G_{33} = G_{44} = \begin{bmatrix} -a & -\frac{1}{T_i} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a & -\frac{1}{T_i} \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{59}$$

Initial conditions of system (48) one obtains by solving the algebraic equation

$$\begin{bmatrix} \Phi_{11}(h) & \Phi_{12}(h)-1 & \Phi_{13}(h) & \Phi_{14}(h) \\ \Phi_{31}(h)-1 & \Phi_{32}(h) & \Phi_{33}(h) & \Phi_{34}(h) \\ \Phi_{41}(h) & \Phi_{42}(h) & \Phi_{43}(h)-1 & \Phi_{44}(h) \\ Q_{41} & Q_{42} & Q_{43} & Q_{44} \end{bmatrix} \begin{bmatrix} col\, U_1(0) \\ col\, U_2(0) \\ col\, Z_1(0) \\ col\, Z_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -col\, W \end{bmatrix}, \tag{60}$$

where

$$col\, W = \begin{bmatrix} w_1 \\ 0 \\ 0 \\ w_2 \end{bmatrix}, \tag{61}$$

$$Q_{41} = P_1 + P_3\Phi_{21}(h), \qquad Q_{42} = P_2 + P_3\Phi_{22}(h),$$

$$Q_{43} = P_3\Phi_{23}(h), \qquad Q_{44} = P_3\Phi_{24}(h),$$

$$P_1 = \begin{bmatrix} 2a & \frac{1}{T_i} & \frac{1}{T_i} & 0 \\ 0 & 0 & a & \frac{1}{T_i} \\ 0 & a & 0 & \frac{1}{T_i} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{62}$$

$$P_2 = \begin{bmatrix} 2b & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{63}$$

$$P_3 = \begin{bmatrix} -2ck & 0 & 0 & 0 \\ -c & 0 & -ck & 0 \\ -c & 0 & -ck & 0 \\ 0 & 0 & -c & -c \end{bmatrix}, \tag{64}$$

$\Phi(\xi)$ is a fundamental matrix of solutions of equation (48) given by (41).

The Lyapunov functional for system (44) has a form, see formula (4)

$$\begin{aligned} v(\varphi) = {} & \varphi^T(0)U(0)\varphi(0) + 2\varphi^T(0)\int_{-h}^{0} U^T(\theta+h)A_1\varphi(\theta)\,\mathrm{d}\theta + \\ & + 2\varphi^T(0)\int_{-2h}^{0} U^T(\theta+2h)A_2\varphi(\theta)\,\mathrm{d}\theta + \\ & + \int_{-h}^{0}\int_{-h}^{0} \varphi^T(\theta)A_1^T U(\theta-\eta)A_1\varphi(\eta)\,\mathrm{d}\eta\,\mathrm{d}\theta + \\ & + \int_{-h}^{0}\int_{-2h}^{0} \varphi^T(\theta)A_1^T U(-h+\theta-\eta)A_2\varphi(\eta)\,\mathrm{d}\eta\,\mathrm{d}\theta + \\ & + \int_{-2h}^{0}\int_{-h}^{0} \varphi^T(\theta)A_2^T U(h+\theta-\eta)A_1\varphi(\eta)\,\mathrm{d}\eta\,\mathrm{d}\theta + \\ & + \int_{-2h}^{0}\int_{-2h}^{0} \varphi^T(\theta)A_2^T U(\theta-\eta)A_2\varphi(\eta)\,\mathrm{d}\eta\,\mathrm{d}\theta, \end{aligned} \tag{65}$$

where

$$\varphi(\theta) = \begin{bmatrix} \varphi_1(\theta) & \varphi_2(\theta) \end{bmatrix}^T,$$

$$A_1 = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}, \tag{66}$$

$$A_2 = \begin{bmatrix} -ck & -c \\ 0 & 0 \end{bmatrix}. \tag{67}$$

The value of the performance index (47) is equal to the value of the functional (65) for initial function φ

$$J = v(\varphi). \tag{68}$$

5. Example

In optimization process we will consider the following parameters of system (42) $a = -2$, $b = -1.5$ and $c = 0.4$ and the initial function φ_1 given by a formula

$$\varphi_1(\theta) = \begin{cases} x_0 & \text{for } \theta = 0, \\ 0 & \text{for } \theta \in [-2h, 0), \end{cases} \tag{69}$$

where $x_0 \in \mathbb{R}$ is an initial state of system (44).

Equation (45) implies

$$\varphi_2(\theta) = 0 \tag{70}$$

for $\theta \in [-2h, 0]$.

The value of functional (65) for φ given by formulas (69) and (70) is equal to

$$J = v(\varphi) = U_{11}(0)x_0^2\,. \tag{71}$$

Formula (71) implies that x_0 affects only the value J but not the optimal solution of the parametric optimization, because only $U_{11}(0)$ depends on k and T_i. Hence we take $x_0 = 1$.

We search for optimal parameters of a PI-controller whose minimize the index (71).

Formula (47) can be written in a form

$$J = \int_0^\infty \left(w_1 x_1^2(t) + w_2 x_2^2(t) \right) \mathrm{d}t. \tag{72}$$

We will consider two cases

1. $w_1 = 1, w_2 = 1$;
2. $w_1 = 1, w_2 = 0$.

In the second case performance index (72) has a form $J = \int_0^\infty x^2(t)dt$, because $x_1(t) = x(t)$. Such index of quality is often used for system (42).

Optimization results for $w_1 = 1$, $w_2 = 1$, obtained by means of Matlab function *fminsearch*, are given in Table 1.

Table 1: Optimization results for $w_1 = 1$, $w_2 = 1$

Delay	kopt	$1/T_i$ opt	Index value	critical gain	critical $1/T_i$
1.0	0.1520	0.0278	0.2965	5.0129	6.0899
1.5	0.7758	0.0654	0.3244	4.7419	4.0599
2.0	1.0706	0.0876	0.3325	4.6799	3.0099
2.5	1.2190	0.0954	0.3341	4.6799	2.3799
3.0	1.2934	0.0959	0.3340	4.6999	1.9699
3.5	1.3289	0.0934	0.3337	4.7299	1.6799
4.0	1.3451	0.0900	0.3335	4.7599	1.4599
4.5	1.3515	0.0864	0.3334	4.7899	1.2899
5.0	1.3540	0.0830	0.3334	4.8099	1.1499

Figure 1 shows the value of the index $J(k)$ for fixed $1/T_i = 0.0876$ and $h = 2$. You can see that there exists a critical value of the gain k_{crit}. The system (46) is

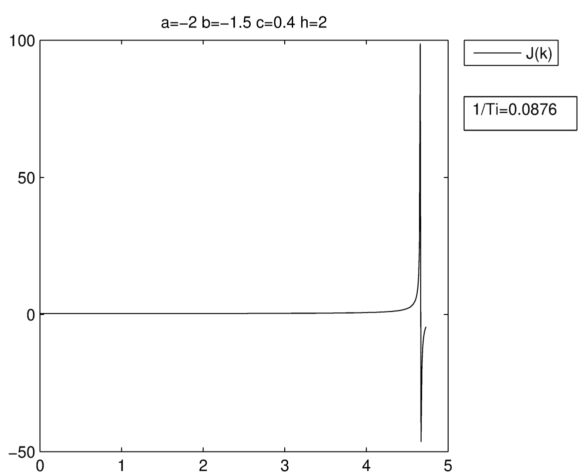

Figure 1: $J(k)$ for fixed $1/T_i = 0.0876$

stable for gains less than critical one and unstable for gains grater than critical. Critical gains were obtained for fixed $1/T_i$ equal to the optimal value.

Figure 2 shows the value of the index $J(1/T_i)$ for fixed $k = 1.0706$ and $h = 2$. There exists a critical value of the integral time $T_{i\,crit}$ too. This value determines the interval of stability. The critical values of $1/T_{i\,crit}$ were obtained for fixed k equal to the optimal value. Critical values k_{crit} and $1/T_{i\,crit}$ depend on the value of time delay. This dependence is presented in Table 1.

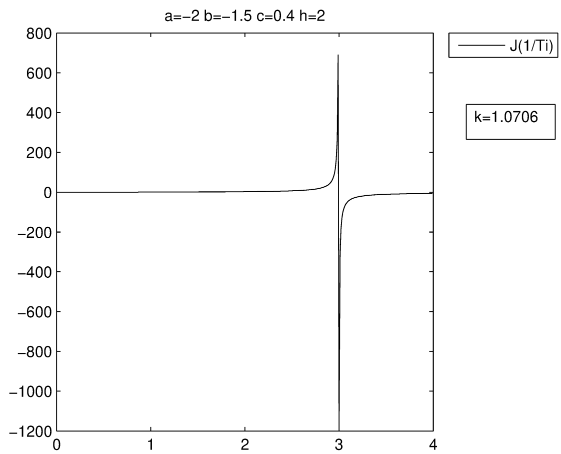

Figure 2: $J(1/T_i)$ for fixed $k = 1.0706$

Figure 3 shows the value of the index $J(k)$ for fixed $1/T_i = 0.0876$, $h = 2$ and for k less than critical gain. You can see that the function $J(k)$ is convex and has a minimum.

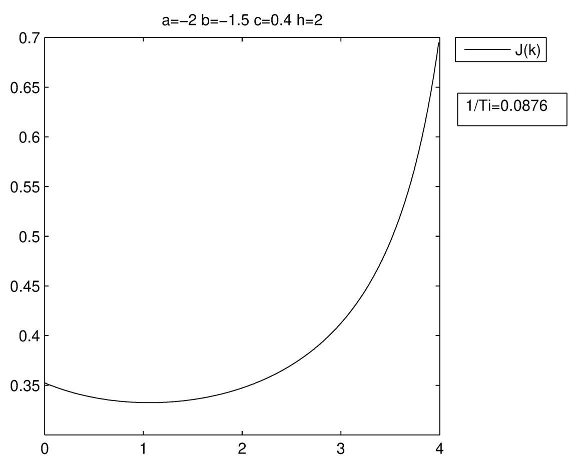

Figure 3: $J(k)$ for fixed $1/T_i = 0.0876$

Figure 4 shows the value of the index $J(1/T_i)$ for fixed $k = 1.0706$, $h = 2$ and for $1/T_i$ less than critical value. You can see that the function $J(1/T_i)$ is convex and has a minimum.

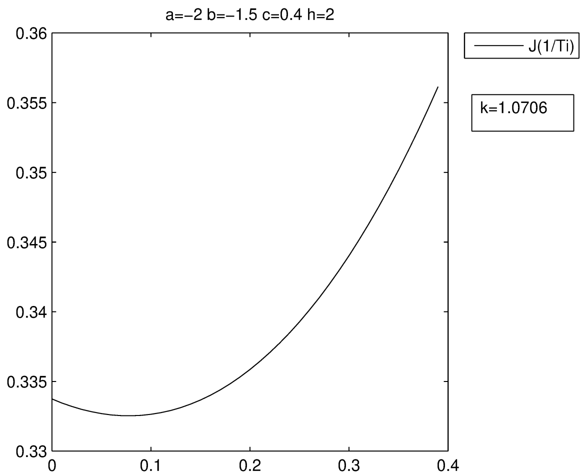

Figure 4: $J(1/T_i)$ for fixed $k = 1.0706$

Figure 5 shows the variable $x_1(t)$ for optimal parameters of the PI controller for $h = 1$, $k = 0.1520$, $1/T_i = 0.0278$ and for $w_1 = 1$, $w_2 = 1$.

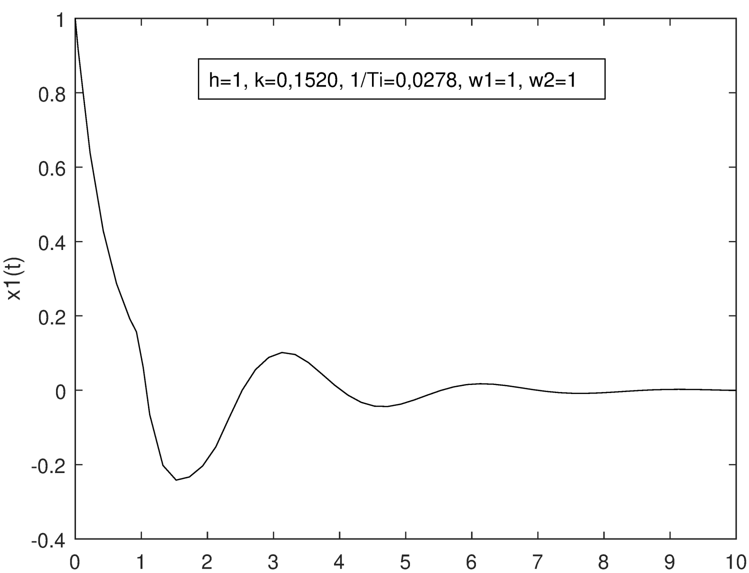

Figure 5: Variable $x_1(t)$ for optimal parameters of PI controller for $w_1 = 1$, $w_2 = 1$

Figure 6 shows the variable $x_2(t)$ for optimal parameters of the PI controller for $h = 1$, $k = 0.1520$, $1/T_i = 0.0278$ and for $w_1 = 1$, $w_2 = 1$.

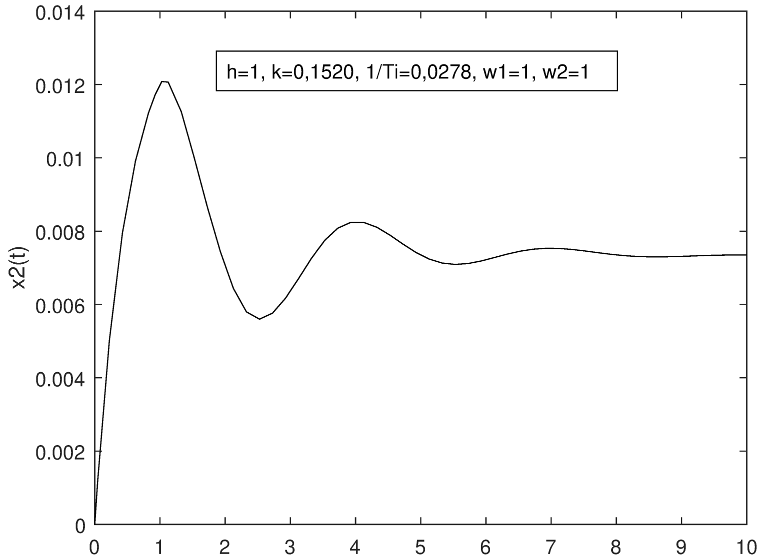

Figure 6: Variable $x_2(t)$ for optimal parameters of PI controller for $w_1 = 1$, $w_2 = 1$

In Table 2 are given optimization results for $w_1 = 1$, $w_2 = 0$. Values of critical gain k_{crit} and of critical integral time $T_{i\,crit}$ are the same as in Table 1. When we compare optimization results from Table 1 and Table 2 we notice differences in optimal values of integral time.

Table 2: Optimization results for $w_1 = 1$, $w_2 = 0$

Delay	kopt	$1/T_i$ opt	Index value
1.0	0.0563	1.5088	0.2880
1.5	0.7135	1.3821	0.3082
2.0	1.0332	1.1188	0.3154
2.5	1.1958	0.9062	0.3188
3.0	1.2780	0.7474	0.3211
3.5	1.3186	0.6290	0.3229
4.0	1.3381	0.5392	0.3244
4.5	1.3473	0.4699	0.3256
5.0	1.3517	0.4153	0.3267

Figure 7 shows the variable $x_1(t)$ for optimal parameters of the PI controller for $h = 1$, $k = 0.0563$, $1/T_i = 1.5088$ and for $w_1 = 1$, $w_2 = 0$.

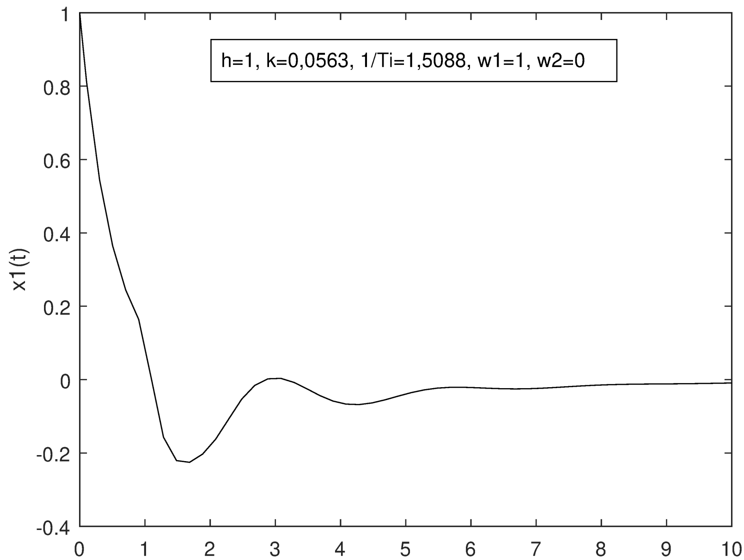

Figure 7: Variable $x_1(t)$ for optimal parameters of PI controller for $w_1 = 1$, $w_2 = 0$

Figure 8 shows the variable $x_2(t)$ for optimal parameters of the PI controller for $h = 1$, $k = 0.0563$, $1/T_i = 1.5088$ and for $w_1 = 1$, $w_2 = 0$.

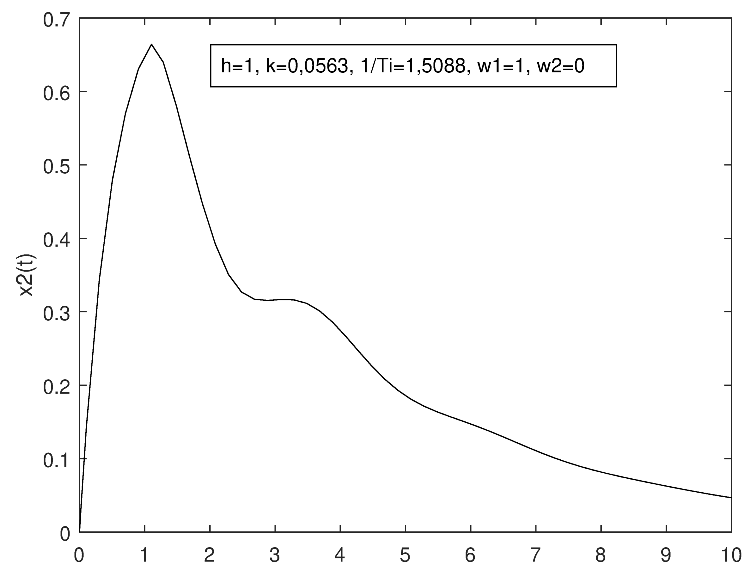

Figure 8: Variable $x_2(t)$ for optimal parameters of PI controller for $w_1 = 1$, $w_2 = 0$

6. Conclusions

In the paper a Lyapunov matrix approach to the parametric optimization problem of time-delay systems with two delays and a PI controller is presented. The value of integral quadratic performance index is equal to the value of the Lyapunov functional for initial function of time-delay system. The Lyapunov functional is determined by means of the Lyapunov matrix. The paper can be used in determination of the values of optimal gain and optimal integral time whose minimize ISE (Integral of Squared Error) for systems with two delays. Using formulas presented in Chapter 3 and a simple Matlab code it is possible to obtain the values of optimal gain and optimal integral time for required parameters of system (46).

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