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Non integer order, state space model of heat transfer process using Atangana-Baleanu operator

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Abstract. In the paper a new, state space, non integer order model of an one-dimensional heat transfer process is proposed. The model uses a new operator with Mittag-Leffler kernel, proposed by Atangana and Beleanu. The non integer order spatial derivative is expressed by Riesz operator. Analytical formula of the step response is given, the convergence of the model is discussed too. Theoretical results are verified by experiments.

Key words: non integer order systems, heat transfer equation, infinite dimensional systems, fractional order state equation, Riesz operator, Caputo operator, Atangana-Baleanu operator.

1. Introduction

The modeling of processes and phenomena hard to analyse with the use of other tools is one of main areas of application non integer order calculus. Non integer models for many physical phenomena were presented by various Authors, for example [3, 4, 6, 7, 22, 27]. Analysis of anomalous diffusion problem with the use of fractional order approach and semigroup theory was presented for example by [23]. An observability problem for fractional order systems was presented for example by [11]. Minimal energy control for FO descriptor systems was analysed for example by [24].

Heat transfer processes can also be modeled with the use of non integer order approach. This problem has been investigated for example by [1, 5, 14, 15]. The use of Caputo-Fabrizio operator in modeling of heat transfer processes was discussed by [25], the use operators with non singular kernel to modeling of thermal processes was deeply analysed in paper [2].

This paper is intented to propose and analyse a new, state-space model for heat transfer process in one dimensional plant. The considered model derives directly from time-continuous model given by [19] and [20] after replacing the Caputo (C) operator by Atangana-Baleanu (AB) operator. This operator was proposed by [2], it was presented also by [12], its use to modeling of heat transfer was considered for example by [25]. An interesting collection of recent results discussing the use of AB operator in modeling of different physical phenomena can be found in [8]. The analytical solution of the Christov diffusion equation is given in [26].

The paper is organized as follows: preliminaries describe the Atangana-Baleanu operator and its Laplace transform. Next the considered experimental heat plant and its time-continuous, fractional order, state space model using C operator is given. Furthermore the passing to AB model is presented and elementary properties of the proposed model: spectrum decomposition and convergence are analyzed. The analytical formula of the step response is also given and proved. Finally the proposed results are compared to the experimental results.

2. Preliminaries

A presentation of elementary ideas is started with a definition of a non integer-order, integro-differential operator. It was given for example by [4, 10, 13, 22]:

Definition 1. (The elementary non integer order operator) The non integer-order integro-differential operator is defined as follows:

$$
{}_aD_t^\alpha f(t) = \begin{cases} \dfrac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} & \alpha > 0 \\ f(t) & \alpha = 0 \\ \displaystyle\int_a^t f(\tau)(\mathrm{d}\tau)^\alpha & \alpha < 0 \end{cases}. \tag{1}
$$

where a and t denote time limits for operator calculation, $\alpha \in \mathbb{R}$ denotes the non integer order of the operation.

The fractional-order, integro-differential operator can be described by different definitions, given by Grünwald and Letnikov, Riemann and Liouville (RL) and Caputo (C). In this paper the C definition is applied (see for example [4, 10, 13, 22]):

Definition 2. (The Caputo definition of the FO operator)

$$
{}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(M-\alpha)} \int_0^\infty \frac{f^{(M)}(\tau)}{(t-\tau)^{\alpha+1-M}} \, \mathrm{d}\tau. \tag{2}
$$

where $M - 1 < \alpha < M$ denotes the non integer order of operation and $\Gamma(..)$ is the Gamma function.

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For the Caputo operator the Laplace transform can be defined (see for example [9]):

Definition 3. (The Laplace transform for Caputo operator)

$$\begin{aligned}\mathscr{L}({}_0^C D_t^\alpha f(t)) &= s^\alpha F(s), \quad \alpha < 0\\ \mathscr{L}({}_0^C D_t^\alpha f(t)) &= s^\alpha F(s) - \sum_{k=0}^{M-1} s^{\alpha-k-1}{}_0 D_t^k f(0),\\ &\alpha > 0, \quad M-1 < \alpha \leq M \in \mathbb{Z}.\end{aligned} \tag{3}$$

Consequently, the inverse Laplace transform for non integer order function is expressed as follows ([13]):

$$\mathscr{L}^{-1}[s^\alpha F(s)] = {}_0 D_t^\alpha f(t) + \sum_{k=0}^{M-1} \frac{t^{k-1}}{\Gamma(k-\alpha+1)} f^{(k)}(0^+) \qquad M-1 < \alpha < M, \quad M \in \mathbb{Z}. \tag{4}$$

A fractional-order linear state space system is described as:

$$\begin{aligned}{}_0 D_t^\alpha x(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t),\end{aligned} \tag{5}$$

where $\alpha \in (0,1)$ denotes the fractional order of the state equation, $x(t) \in \mathbb{R}^N$, $u(t) \in \mathbb{R}^L$, $y(t) \in \mathbb{R}^P$ are the state, control and output vectors respectively, A, B, C are the state, control and output matrices, respectively.

The fractional order derivative Atangana-Baleanu operator is obtained via replacing the exponential kernel in the Caputo-Fabrizio (CF) operator by the Mittag-Leffler kernel. It is defined using the C or RL definition of fractional order derivative. Using these definitions we obtain the Atangana-Baleanu-Caputo (ABC) or Atangana-Baleanu-Riemann (ABR) operator respectively [2]:

Definition 4. (The Atangana-Baleanu-Caputo (ABC) operator)

$$^{ABC}{}_a D_t^\alpha (f(t)) = M_\alpha \int_a^t f'(x) E_\alpha \left(-\alpha \frac{(t-x)^\alpha}{1-\alpha} \right) \mathrm{d}x. \tag{6}$$

where $E_\alpha(..)$ is the one parameter Mittag-Leffler function, M_α is the normalization function equal:

$$M_\alpha = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}. \tag{7}$$

In (7) $\Gamma(..)$ is the Gamma function.

Definition 5. (The Atangana-Baleanu-Riemann (ABR) operator)

$$^{ABR}{}_a D_t^\alpha (f(t)) = M_\alpha \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t f(x) E_\alpha \left(-\alpha \frac{(t-x)^\alpha}{1-\alpha} \right) \mathrm{d}x. \tag{8}$$

where $E_\alpha(..)$ is the one parameter Mittag-Leffler function, M_α is the normalization function expressed by (7), $\Gamma(..)$ is the Gamma function.

The Laplace transforms for the ABC and ABR derivatives are as follows:

Definition 6. (The Laplace transform of the ABC operator)

$$\mathscr{L}\{^{ABC}{}_a D_t^\alpha (f(t))\}(s) = \frac{M_\alpha}{1-\alpha} \frac{s^\alpha \{f(t)\}(s) - s^{\alpha-1} f(0)}{s^\alpha + \dfrac{\alpha}{1-\alpha}}. \tag{9}$$

Definition 7. (The Laplace transform of the ABR operator)

$$\mathscr{L}\{^{ABR}{}_a D_t^\alpha (f(t))\}(s) = \frac{M_\alpha}{1-\alpha} \frac{s^\alpha \{f(t)\}(s)}{s^\alpha + \dfrac{\alpha}{1-\alpha}}. \tag{10}$$

For the homogenous initial condition: $f(0) = 0$ both Laplace transforms are equal:

$$\mathscr{L}\{^{ABR}{}_a D_t^\alpha (f(t))\}(s) = \mathscr{L}\{^{ABC}{}_a D_t^\alpha (f(t))\}(s). \tag{11}$$

The non integer order spatial derivative was given by Riesz and it has the following form (see for example [28]):

Definition 8. (The Riesz definition of FO spatial derivative)

$$\frac{\partial^\beta \Theta(x,t)}{\partial x^\beta} = -r_\beta \left({}_0 D_x^\beta \Theta(x,t) + {}_x D_1^\beta \Theta(x,t) \right), \tag{12}$$

where:

$$r_\beta = \frac{1}{2\cos\left(\dfrac{\pi\beta}{2}\right)}. \tag{13}$$

In (12) ${}_0 D_x^\beta$ and ${}_x D_1^\beta$ denote left- and right-side Riemann-Liouville derivatives, defined as underneath:

$${}_0 D_x^\beta \Theta(x,t) = \frac{1}{\Gamma(2-\beta)} \frac{\partial}{\partial x} \int_0^x \frac{\Theta(\xi,t)\,\mathrm{d}\xi}{(x-\xi)^{\beta-1}}, \tag{14}$$

$${}_x D_1^\beta \Theta(x,t) = \frac{1}{\Gamma(2-\beta)} \frac{\partial}{\partial x} \int_x^1 \frac{\Theta(\xi,t)\,\mathrm{d}\xi}{(\xi-x)^{\beta-1}}. \tag{15}$$

In (14) and (15) $\Gamma(..)$ denotes the Gamma function, $\beta > 1$ is the non integer derivative order with respect to length.

3. The non integer order, state space model using Caputo operator

The simplified scheme of the considered heat plant is shown in Fig. 1. It has a form of a thin copper rod heated with an electric heater of the length Δx_u located at one end of rod. An output temperature is measured using Pt-100 RTD sensors Δx long attached in points: 0.29, 0.50 and 0.73 of rod length. More details of the construction are given in the section "Experimental Results".

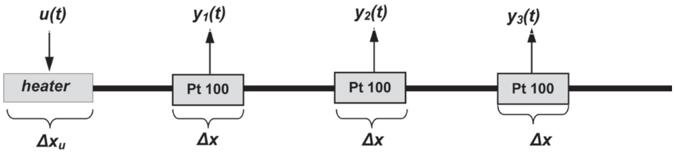

Fig. 1. The simplified scheme of the experimental system

The fundamental mathematical model describing the heat conduction in the plant is the partial differential equation of the parabolic type with the homogeneous Neumann boundary conditions at the ends, the homogeneous initial condition, the heat exchange along the length of rod and distributed control and observation. This equation with integer orders of both differentiations has been considered in papers [16–18].

The non integer order model with respect to both time and space coordinates, employing Caputo and Riesz operators was given in [20]. It takes the following form:

$$\begin{cases} {}^{C}D_t^{\alpha}Q(x,t) = a_w\dfrac{\partial^{\beta}Q(x,t)}{\partial x^{\beta}} - R_aQ(x,t) + b(x)u(t), \\ \dfrac{\partial Q(0,t)}{dx} = 0, \quad t \geq 0, \\ \dfrac{\partial Q(1,t)}{dx} = 0, \quad t \geq 0, \\ Q(x,0) = Q_0, \quad 0 \leq x \leq 1, \\ y(t) = k_0 \displaystyle\int_0^1 Q(x,t)c(x)\,\mathrm{d}x, \end{cases} \tag{16}$$

where α, $\beta > 0$ denote non integer orders of the system, a_w, R_a denote coefficients of heat conduction and heat exchange, k_0 is a steady-state gain of the model. Now we can express (16) as the following, inifinite dimensional state equation:

$$\begin{cases} {}^{C}D_t^{\alpha}Q(t) = AQ(t) + Bu(t), \\ Q(0) = Q_0, \\ y(t) = k_0CQ(t), \end{cases} \tag{17}$$

where:

$$\begin{cases} AQ = a_w\dfrac{\partial^{\beta}Q(x)}{\partial x^{\beta}} - R_aQ, \\ \mathscr{D}(A) = \left\{Q \in H^2(0,1) : Q'(0) = 0, Q'(1) = 0\right\}, \\ a_w, R_a > 0, \\ H^2(0,1) = \left\{u \in L^2(0,1) : u', u'' \in L^2(0,1)\right\}, \\ CQ(t) = \langle c, Q(t)\rangle, \quad Bu(t) = \langle bu(t)\rangle, \\ Q(t) = [q_1(t), q_2(t)..]^T. \end{cases} \tag{18}$$

In (18) $\mathscr{D}(A)$ denotes the field of the state operator A, $'$, $''$ denotes the first and second derivative with respect to length, $\langle..\rangle$ is the standard scalar product.

The following set of the eigenvectors for the state operator A creates the orthonormal basis of the state space:

$$h_n = \begin{cases} 1, & n = 0 \\ \sqrt{2}\cos(n\pi x), & n = 1, 2, \ldots . \end{cases} \tag{19}$$

Eigenvalues of the state operator are expressed as underneath:

$$\lambda_n = -a_w\pi^{\beta}n^{\beta} - R_a, \quad n = 0, 1, 2, \ldots \tag{20}$$

and consequently the state operator takes the form:

$$A = diag\{\lambda_0, \lambda_1, \lambda_2, \ldots\}. \tag{21}$$

Next, the spectrum σ of the state operator A is expressed as underneath:

$$\sigma(A) = \{\lambda_0, \lambda_1, \lambda_2, \ldots\}. \tag{22}$$

From (20) it follows at once that $\lambda_0 > \lambda_1 > \lambda_2 \ldots$.

The input operator B has the following form:

$$B = [b_0, b_1, b_2, \ldots]^T, \tag{23}$$

where $b_n = \langle b, h_n\rangle$, $b(x)$ denotes the heater function:

$$b(x) = \begin{cases} 1, & x \in [0, x_0] \\ 0, & x \notin [0, x_0] \end{cases} \tag{24}$$

With respect to (19) and (24) each element b_n takes the following form:

$$b_n = \begin{cases} x_u, & n = 0, \\ \dfrac{\sqrt{2}\sin(n\pi x_u)}{n\pi}, & n = 1, 2, \ldots \end{cases} \tag{25}$$

The output operator C is expressed as follows:

$$C = \begin{bmatrix} C_{s1} \\ C_{s2} \\ C_{s3} \end{bmatrix}. \tag{26}$$

Rows of output operator C are as underneath:

$$C_{sj} = \left[c_{sj,0}, c_{sj,1}, c_{sj,2}, \ldots\right] \quad j = 1, 2, 3 \ldots, \tag{27}$$

where $c_{sj,n} = \langle c, h_n\rangle$, $c(x)$ denotes the sensor function:

$$c(x) = \begin{cases} 1, & x \in [x_1, x_2] \\ 0, & x \notin [x_1, x_2] \end{cases} \tag{28}$$

with respect to (19) and (28) each element c_{jn} takes the form:

$$c_{jn} = \begin{cases} x_{j2} - x_{j1}, & j = 1, 2, 3, \quad n = 0, \\ \dfrac{\sqrt{2}\left(\sin(n\pi x_{j2}) - \sin(n\pi x_{j1})\right)}{n\pi}, & \\ & j = 1, 2, 3, \quad n = 1, 2, \ldots \end{cases} \tag{29}$$

4. The non integer order, state space model using Atangana-Baleanu operator

Assume that initial condition in state equation equals to zero. Consequently the use of definition (6) or (8) to the equation (17) gives:

$$\begin{cases} {}^{ABC,ABR}D_t^{\alpha}Q(t) = AQ(t) + Bu(t), \\ Q(0) = 0, \\ y(t) = y_0 CQ(t). \end{cases} \tag{30}$$

Assume that A, B and C operators of the equation (30) are expressed by (21), (23) and (26) respectively. The analysis of elementary properties of the proposed model: step response and convergence are presented in the next subsections.

The spectrum decomposition for system described by (16) is discussed in detail in the paper [20]. The replacement the C operator in (16) by ABC/R operator (6), (8) does not change the form of state and input operators. This implies that the spectrum of the system keeps the form (22) and consequently the system can be decomposed to subsystems related to single eigenvalues λ_n.

4.1. The step response of the model. Assume that initial condition is equal zero: $Q_0 = [0]$. Using the Laplace transform (9) or (10) to (30) we obtain the general formula of the time response of the considered system to control $u(t)$:

$$y(t) = y_0 \sum_{n=0}^{\infty} C\mathscr{L}^{-1}\{Q_n(s)\}, \tag{31}$$

where y_0 is a steady-state gain of the model, necessary to fit the response to experimental data, B and C are given by (23) and (26) respectively, $Q_n(s)$ is as follows:

$$Q_n(s) = \frac{\alpha_1 s^{\alpha} + \alpha}{(M_\alpha - \alpha_1\lambda_n)s^{\alpha} + \alpha\lambda_n} b_n U(s), \quad n = 0,1,2..., \tag{32}$$

where M_α is expressed by (7), $\alpha_1 = 1 - \alpha$. The formula (31) allows to compute the step response of the model to the each control signal $u(t)$, for which a Laplace transform exists. Particularly, if the control is the Heaviside function, the analytical formula of step response can be given. It is described by the following remark:

Remark 1. (The step response of the system (30) with homogenous initial condition) Consider the system described by the equation (30) with homogenous initial condition $Q_0 = [0]$. The step response of this system is given as follows:

$$y(t) = y_0 \sum_{n=0}^{\infty} CQ_n(t), \tag{33}$$

where:

$$Q_n(t) = \left(r_n - \frac{b_n}{\lambda_n}\right) E_\alpha(-q_n t^{\alpha}) + \frac{b_n}{\lambda_n} 1(t), \quad n = 0,1,2... \tag{34}$$

$$\begin{aligned} r_n &= \frac{\alpha_1 b_n}{M_\alpha - \alpha_1\lambda_n}, \\ q_n &= \frac{\alpha\lambda_n}{M_\alpha - \alpha_1\lambda_n}. \end{aligned} \tag{35}$$

Proof. The Laplace transform (32) for $u(t) = 1(t)$ takes the form:

$$Q_n(s) = \frac{b_n(\alpha_1 s^{\alpha} + \alpha)}{s((M_\alpha - \alpha_1\lambda_n)s^{\alpha} + \alpha\lambda_n)}, \quad n = 0,1,2... \tag{36}$$

The element $Q_n(s)$ can be presented as the following sum:

$$Q_n(s) = Q_n^1(s) + Q_n^2(s), \tag{37}$$

where:

$$Q_n^1(s) = \frac{b_n\alpha_1 s^{\alpha}}{s((M_\alpha - \alpha_1\lambda_n)s^{\alpha} + \alpha\lambda_n)} = r_n \frac{s^{\alpha}}{s(s^{\alpha} + q_n)}, \tag{38}$$

$$Q_n^2(s) = \frac{b_n\alpha}{s((M_\alpha - \alpha_1\lambda_n)s^{\alpha} + \alpha\lambda_n)} = p_n \frac{1}{s(s^{\alpha} + q_n)}, \tag{39}$$

$$p_n = \frac{\alpha b_n}{M_\alpha - \alpha_1\lambda_n}. \tag{40}$$

The inverse Laplace transforms of components $Q_n^{1,2}(s)$ are as follows (see [3], page 11, Eqs (1.34) and (1.35)):

$$Q_n^1(t) = r_n E_\alpha(-q_n t^{\alpha}), \tag{41}$$

$$Q_n^2(t) = \frac{p_n}{q_n}\left(1(t) - E_\alpha(-q_n t^{\alpha})\right). \tag{42}$$

Consequently $Q_n(t)$ takes the following form:

$$Q_n(t) = \left(r_n - \frac{p_n}{q_n}\right) E_\alpha(-q_n t^{\alpha}) + \frac{p_n}{q_n} 1(t). \tag{43}$$

Notice that:

$$\frac{p_n}{q_n} = \frac{b_n}{\lambda_n}. \tag{44}$$

The use of (44) in (43) gives directly (34) and the proof is completed. □

To additionally check the above result assume that $\alpha = 1.0$ (the integer order model with respect to time). This gives:

$$\begin{cases} \alpha_1 = 0, \\ M_\alpha = 1, \\ r_n = 0, \\ q_n = \lambda_n. \end{cases} \tag{45}$$

Applying (45) to (34) yields:

$$Q_n(t) = \frac{b_n}{\lambda_n}\left(1(t) - \exp(\lambda_n t)\right). \tag{46}$$

The n-th mode of step response (46) is the n-th mode of step response of integer order model of the considered system as it is given in [20], equation (26).

With respect to (31) and (34) the step response at the j-th output of the system is as follows:

$$y_j(t) = y_0 \sum_{n=0}^{\infty} C_{nj} Q_n(t), \quad j = 1,2,3. \tag{47}$$

The steady-state response of the n-th mode y_{jn}^{ss} can be obtained as a final value of (34) for $t \to \infty$. It is as follows:

$$y_{jn}^{ss} = \frac{c_{jn} b_n}{\lambda_n}. \tag{48}$$

The result (48) can be also obtained applying Final Value Theorem (FVT) to (36). With respect to (25), (29) and some elementary transformations we obtain the direct dependency between steady-state response of the n-th mode and parameters of the plant:

$$\begin{aligned} y_{jn}^{ss} = \frac{4}{\pi^2 n^2 \lambda_n} \sin\left(\frac{n\pi(x_{j2}-x_{j1})}{2}\right) \cdot \\ \cdot \cos\left(\frac{n\pi(x_{j2}+x_{j1})}{2}\right) \sin\left(\frac{n\pi x_u}{2}\right). \end{aligned} \tag{49}$$

The system described by (30)-(47) is inifite-dimensional. Its use in modeling of the considered experimental plant requires to use its finite dimensional approximation. This approximation can be obtained via truncating further modes of solution. It implies that the operators A, B and C can be interpreted as matrices and the solution (47) takes the form of the following finite sum:

$$y_j(t) = y_0 \sum_{n=0}^{N} C_{nj} Q_n(t), \quad j = 1,2,3. \tag{50}$$

The value of N is the crucial parameter of the finite dimensional model (50). Its analytical estimation is given in the next subsection.

4.2. Convergence. The convergence of the proposed model will be analyzed using approach presented in the paper [21]. It can be done by estimating the order N assuring a predefined value of Rate Of Convergence (ROC). In the considered case the ROC is defined as the increment of steady-state response y_{jn}^{ss} as a function of order N. This increment is equal to the absolute value of N-th mode of the steady-state response (48):

$$ROC_N = |y_{jN}^{ss}|. \tag{51}$$

The order N assuring the keeping predefined value Δ_N of ROC_N is described by the following proposition:

Proposition 1. (The order of model N assuring the predefined value of $ROC = \Delta_N$)
Consider the model of heat transfer process described by (30) with non integer order $0.0 < \alpha < 2.0$, the ROC of the model is defined by (51).

The order N of model assuring the predefined value Δ_N of ROC meets the following inequality:

$$N \geq \frac{1}{\pi}\sqrt{\frac{\sqrt{R_a^2\Delta_N + 16a_w} - R_a\sqrt{\Delta_N}}{2a_w\sqrt{\Delta_N}}}. \tag{52}$$

Proof. With respect to (20) and (49) the condition $ROC \leq \Delta_N$ is equivalent to:

$$\Delta_N \geq \left|\frac{4}{\pi^2 N^2 (a_w \pi^\beta N^\beta + R_a)}\right| \cdot P, \tag{53}$$

where:

$$\begin{aligned} P = \left|\sin\left(\frac{N\pi(x_{j2}-x_{j1})}{2}\right)\cos\left(\frac{N\pi(x_{j2}+x_{j1})}{2}\right) \cdot \right. \\ \left. \cdot \sin\left(\frac{N\pi x_u}{2}\right)\right|. \end{aligned} \tag{54}$$

Notice that P expressed by (54) is not greater than one. It allows to assume that P equals to one. It will give us the upper estimation of N, but (53) takes to simplier form:

$$\left|\frac{4}{\pi^2 N^2 (a_w \pi^\beta N^\beta + R_a)}\right| \leq \Delta_N. \tag{55}$$

The absolute value in (55) can be ignored because the expression inside is always positive:

$$\frac{4}{\pi^2 N^2 (a_w \pi^\beta N^\beta + R_a)} \leq \Delta_N. \tag{56}$$

The left side of (56) will be called the non integer order limiter $L_{nio}(N)$:

$$L_{nio}(N) = \frac{4}{\pi^2 N^2 (a_w \pi^\beta N^\beta + R_a)}. \tag{57}$$

Next assume that $\beta = 2$ (we consider integer order model with respect to length). Then the non integer order limiter (57) takes its integer order form $L_{io}(N)$:

$$L_{io}(N) = \frac{4}{\pi^2 N^2 (a_w \pi^2 N^2 + R_a)}. \tag{58}$$

Consequently the inequality (56) turns to:

$$\Delta_N \pi^4 a_w N^4 + \Delta_N \pi^2 R_a N^2 - 4 \geq 0. \tag{59}$$

The solution of double quadratic inequality (59) gives directly the condition (52). This finishes the proof. □

The numerical verification of the above result is given in the next section.

5. Experimental results

Experiments were done with the use of the experimental system shown in Fig. 2. The length of rod is equal 260 [mm]. The control signal in the system is the standard current $0-20$ [mA] given from analog output of the PLC. This signal is amplified to the range $0-1.5$ [A] and it is the input for the heater. The temperature distribution along the rod is measured with the use of standard RTD sensors of Pt-100 type. In the considered case the size and location of sensors are following:

$$\begin{cases} x = 0.29: & x_1 = 0.26, & x_2 = 0.32 \\ x = 0.50: & x_1 = 0.47, & x_2 = 0.53 \\ x = 0.73: & x_1 = 0.70, & x_2 = 0.76 \end{cases}$$

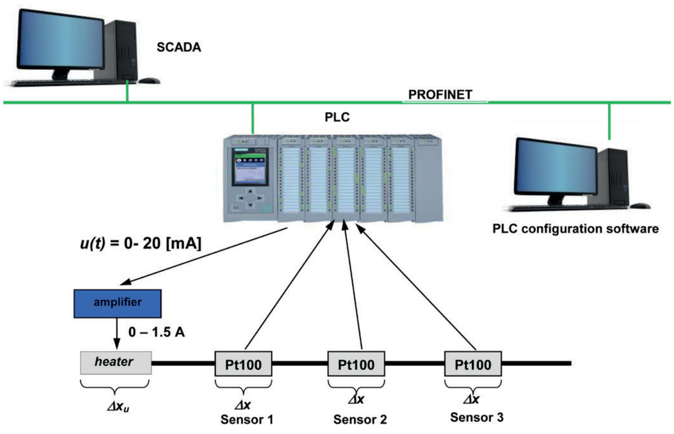

Fig. 2. The construction of the experimental system

Signals from the sensors are directly read by analog inputs of the PLC in Celsius degrees. Data from PLC are read and archivised by SCADA application. The whole system is connected via PROFINET. The temperature distribution with respect to time and length is shown in Fig. 3. The step response of the model was tested in time range from 0 to $T_f = 300$ [s] with sample time 1 [s], parameters were calculated via minimization of the MSE (Medium Square Error) cost function (60) using MATLAB *fminsearch* function.

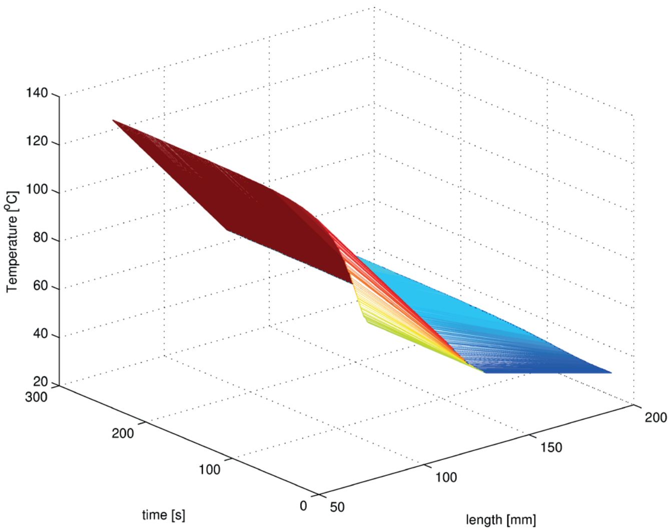

Fig. 3. The spatial-time temperature distribution in the plant

To accuracy estimation the typical MSE cost function was applied:

$$MSE = \frac{1}{3K_s} \sum_{j=1}^{3} \sum_{k=1}^{K_s} \left(y^{+}_{e_j}(k) - y^{+}_{j}(k) \right)^2 . \qquad (60)$$

In (60) K_s denotes the number of collected samples for one sensor, $y^{+}_{e_j}(k)$ and $y^{+}_{j}(k)$ are step responses of plant and model in k-th time moments.

The parameters a_w, R_a, α and β were estimated via minimization of the cost function (60) using MATLAB function *fminsearch*. Results are given in Table 1.

Table 1
Parameters of the AB model

N	α	β	a_w	R_a	MSE
22	0.9293	1.9990	0.0004	0.0714	0.0233

The step response of the proposed model using (50) is shown Fig. 4.

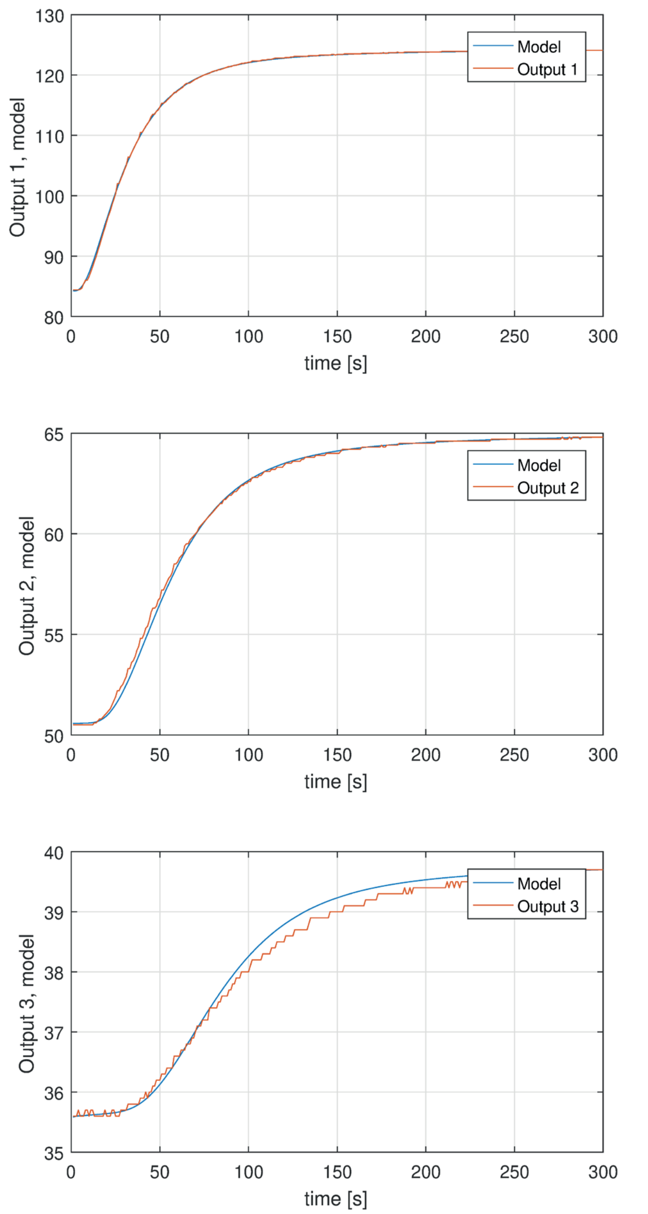

Fig. 4. Comparison the AB model to experiment for $N = 22$

The convergence was tested using the Proposition 1. The first predefined value of ROC was equal: $\Delta_N = 0.001$. Using condition (51) we obtain $N = 18$. The comparison limiters (57) and (58) to steady-state values of modes (48) is shown in Fig. 5.

Next the value $\Delta_N = 0.0001$ was analyzed. The use of condition (51) gives $N = 32$. All the limiters and steady state values of modes (48) are illustrated by Fig. 6.

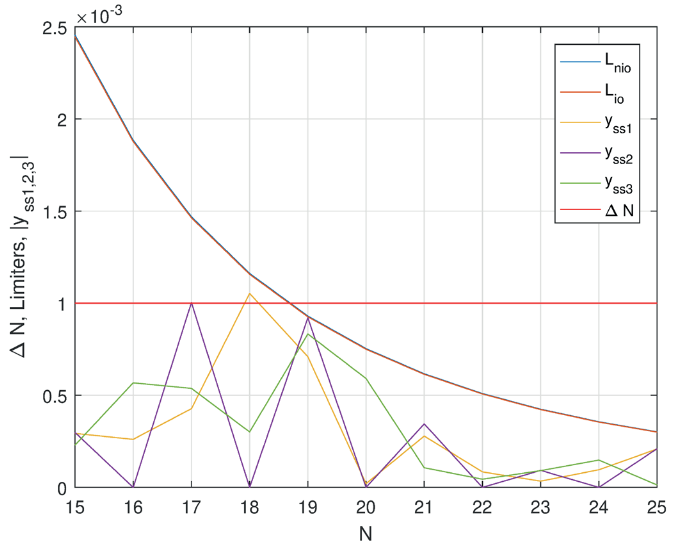

Fig. 5. Convergence estimation for $\Delta_N = 0.001$, and parameters of model given in Table 1

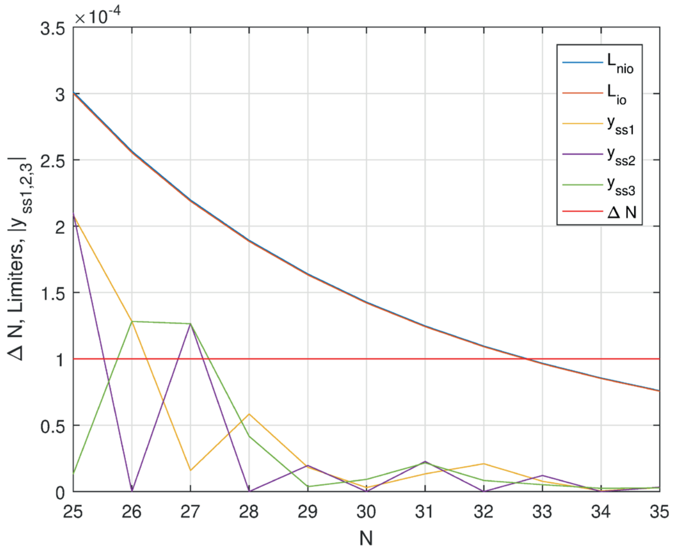

Fig. 6. Convergence estimation for $\Delta_N = 0.0001$, and parameters of model given in Table 1

Figures 5 and 6 show that the condition (52) gives upper estimation of N. The accuracy of estimation is better for bigger values of Δ_N and smaller size N of the model. For smaller values of Δ_N and higher values of N the estimation is more "cautious".

6. Final conclusions

The main final conclusion from the paper is that the AB operator can be used to construct the state space model of the one dimensional heat trasfer process. The diagonal form of the state operator allows to obtain the analytical form of the step response. The convergence of the model is also possible to analyze.

The further investigations of the proposed model will cover its detailed comparing to previously proposed C and CF models. The generalization of the presented results to fractional order, linear systems is also planned to do.

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