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Measuring Process via Sampling of Signals, and Functions with Attributes

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Abstract—**In this paper, it has been shown that any measuring process can be modeled as a process of sampling of signals. Also, a notion of a special kind of functions, called here functions with attributes, has been introduced. The starting point here, in the first of the above themes, is an observation that in fact we are not able to measure and record truly continuously in time any physical quantity. The measuring process can be viewed as going stepwise that is in steps from one instant to another, similarly as a sampling of signals proceeds. Therefore, it can be modeled as the latter one. We discuss this in more detail here. And, the notion of functions with attributes, we introduced here, follows in a natural way from the interpretation of both the measuring process as well as the sampling of signals that we present in this paper. It turns out to be useful.**

Keywords—**Measuring process, sampling of signals, functions with attributes, Dirichlet function**

I. Introduction

THE author of this paper came up with an idea of a special kind of functions when analyzing some "hidden" topological properties of the reconstruction formula and the sampling theorem [1]-[6]. Here, these functions are called functions with attributes. They turn out to be useful in many problems and interpretations we are faced with in the area of signal processing. We will illustrate this using some examples.

Let us start with some general remarks and observations, which will lead to the notion of functions with attributes. To this end, imagine that we perform some measurements continuously, achieving as a result a signal of a continuous time. At least we think it is so. That is that we receive a continuous signal.

In what follows now, we will however argue that this is not exactly true. Why? Because of properties of a physical equipment we utilize for carrying out measurements. It is not able to react immediately. And this obviously regards all kinds and types of the measuring equipment, independently whether it will be optical, mechanical, electronic, or a combined one. For example, when measuring a voltage that changes with time, we see that our voltmeter, connected to a curve plotter or to a data archiving system, needs a finite time to register each of the successively incoming values of a voltage. Obviously, this finite time will be mostly so short that invisible to eyes or "not seen" on a plotted curve. In other words, when we perform any measurements, we move in fact from one point on the timeline (time axis) to another. That is in a not strictly continuous way. And, note that this way resembles sampling of continuous-time signals. Furthermore, the sampling can be performed uniformly or non-uniformly.

The observation described above is illustrated in Fig. 1. In this figure, $v(t)$, $v_m(t)$, and $v_m[kT]$ mean, respectively, a "true" voltage signal that is a continuous function of time, its measured form, and a picture of the latter shown in a form as it would be a sampled signal.

Comparison of the curves presented in Figures 1(b) and 1(c) shows that the process of measuring any physical quantity, which aims in receiving its time course, can be formally identified with the process of sampling a signal of a continuous time.

Now, before introducing functions with attributes, let us, once again, draw attention to the model of measuring process that we propose here. It is not a continuous one in the sense that it delivers continuously values of a measured quantity. In our model, we assume that the values are delivered from instant to instant, and the time between these instants can be very, very small (for example, say, 1 fs), but it always remains finite. We denote it here by *T*. Note further that this time can be deduced, for example, from the sensitivity parameter of a concrete measuring device. Moreover, observe that it can depend upon the signal shape and possibly change then its value from instant to instant. However, we think that in most cases it will be quite reasonable to assume its value to be constant. In the latter case, this will mean an equivalent uniform signal sampling as illustrated in Fig. 1(c). In contrast to this, an equivalent signal sampling will be non-uniform in the first case. Finally, let support our model by an illustration shown in Fig. 2.

Consider first Fig. 2(a) showing a period of a continuous time from t_1 to t_2, in which some measurements should be carried out. Topologically, this period represents a bounded subset of the set of real numbers R. However, it contains an infinite number of elements (instants) and its cardinality is $\mathfrak{c}$ (continuum).

In this paper, we take the view that there is no physical possibility to perform a measurement at each of these infinite number of instants, including a simultaneous delivering the corresponding data to a user (or archiving these data). This so because any measurement, in the sense as stated above, needs some finite time, say a "processing time". And it causes that an infinite number of measurements cannot be carried out in a finite time period. So, therefore, the only reasonable description of measurements in a finite period, which we are able to imagine, is the one presented in Fig. 2(b). In this figure, *T* denotes the "processing time" mentioned above. Moreover, measurements are performed on a finite number of instants, denoted by *kT* in Figure 2(b), where *k* means an integer that assumes the values: -3, -2, -1, 0, 1, 2, 3, 4, and 5.

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That is, in our example, a finite number of nine measurements

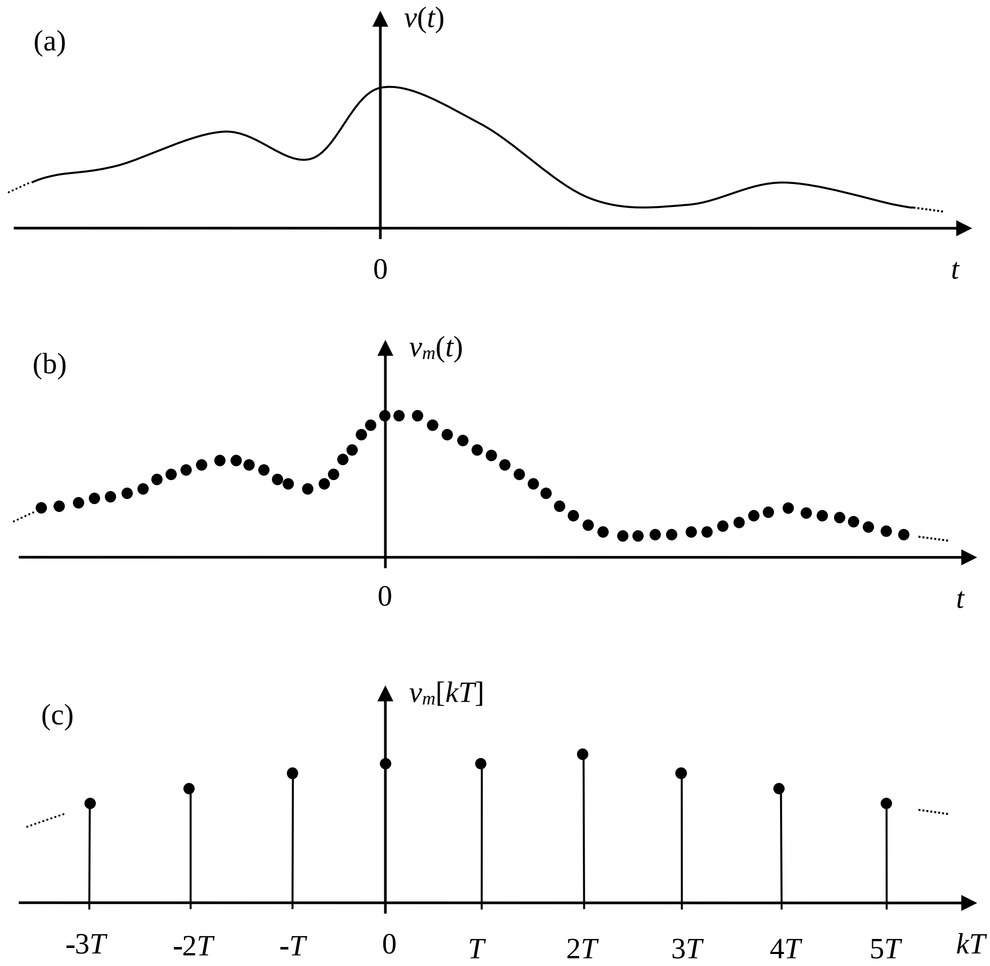

Fig. 1. (a) A true picture (but not measured) of a signal being a continuous function of time. (b) A picture of the signal from point (a) obtained in measurements. It consists of an infinite series of very closely spaced points (spaced uniformly or non-uniformly), which were obtained in measurements, and which approximate the function presented in point (a) above. (c) A fragment of a highly magnified picture of the one presented in point (b) with the marked discrete-time points *kT*, where $k = \ldots, -1, 0, 1, \ldots$ and *T* means a distance on the time axis between the measured points.

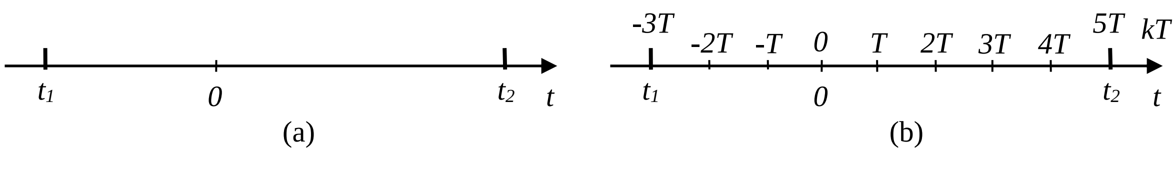

Fig. 2. (a) A period of a continuous time from t_1 to t_2, in which some measurements should be performed. (b) The same time period on the discrete-time axis (denoted by *kT*) that is "immersed" in the continuous timeline (denote by *t*).

are carried out in a finite time period lasting from t_1 to t_2.

Now, concluding all the above observations and remarks supported by Figures 1 and 2, we can really say with conviction that any measuring process can be treated equivalently as a sampling of a continuous-time signal. That is it can be treated similarly as a signal discretization. And because of this reason, the reconstruction formula and the sampling theorem [1]-[6] apply here; that is when analyzing measuring processes.

In fact, these two mathematical tools mentioned can really prove to be very powerful in interpreting and correctly manipulating measured data. For example, see that the signal values which were not measured in the case of Fig. 2(b) can be easily obtained through the use of the reconstruction formula. That is without carrying out any measurements. Their number is obviously infinite, as for the case of Fig. 2(a). So, in other words, we see clearly that the reconstruction formula and the accompanying sampling theorem play a role of a connecting element between the "images" presented in Figures 2(a) and 2(b). In fact, they probably lead to the simplest type of something we want to introduce in this paper and what we will call here functions with attributes. More details on them will be given in what follows.

II. DESCRIPTION OF CONTINUOUS TIME SIGNALS VIA ASSOCIATED FUNCTIONS WITH ATTRIBUTES

Let us begin this section with defining the purpose of introducing special objects we want to call functions with attributes, as mentioned above. Generally saying, they should connect to each other two or more images of a physical object we get in form of measured signals on it, and provide us all-in-one information about this object. For example, in the case considered in Introduction of exactly continuous-time signal (which is, as seen, an idealized one) and the related ones having discrete form (due to the nature of measurements), when all of them represent the same physical object, this will mean what we can express by saying "putting all these signals into a one box". Note that this is possible, as we argued in Introduction, because any measurement that provides signals can be viewed as (or be equivalent to) a sampling of a continuous-time signal. And the latter operation, when performed with sampling frequencies obeying the sampling theorem [1]-[6], allows just to relate to one another all the signals mentioned above. Then, via the reconstruction formula [1]-[6], they will mean the same. That is, in other words, they will constitute a signal object (let us call it in this way here) that will consist of an infinite number of equivalent signals; it is visualized in Fig. 3. Moreover, note that this figure provides also a graphical definition of the notion of a signal object.

First of all, see in Fig. 3(a) that the signal object presented there is a set consisting of an infinite number of elements which are signals. A primary one among these signals is the one presented on the top of the figure, being the signal of a continuous time, $x(t)$. All the other ones are its sampled versions achieved with the use of different sampling periods and/or different starting points of sampling (in the sense of location of the zeroth index *k*). For example, $x[k]$ and $x[k']$ in Fig. 3(a), where the sampling period *T* in square brackets is dropped, are sampled using the same sampling period, but possess two different locations of the zeroth sampling index *k*. Further, $x[k]$ and $x[k'']$, are sampled with the use of two different sampling periods *T* and *T*/2, respectively. The result of a non-uniform sampling is not illustrated in Fig. 3(a). Finally, it is assumed that all the sampled members of the set illustrated in Fig. 3(a) fulfil the sampling theorem [1]-[6] (as appropriate, its version for uniform or non-uniform sampling).

Fig. 3(b) stands for a useful abbreviated form of the signal object that was defined graphically in Fig. 3(a). Basic features of this object can be easily recognized on the figure. That is the time course of a continuous time function considered and points of its sampling. We will say here that possibility of sampling is its attribute, and any of its appropriately sampled versions is a result of "an attribute interaction on this function". And, as we know, the number of these interactions is infinite.

Now, we will discuss two important properties of signal objects defined in a descriptive way above as well as graphically in Fig. 3. Namely, we will consider their boundedness (relating to sets of an infinite number of related signals – in the sense defined below) and their cardinality.

And, let us consider the boundedness property first. This boundedness we consider here refers to the set of possible values of the sampling period of a given function. (It is a closed bounded one.) And appropriately, we call those elements of a signal object, which correspond to the bound values of its sampling period, the bounding signal elements. Now, in more detail, see that really a continuous time function like that one which is visualized on the top of Fig. 3(a) is a bounding element (related with the corresponding bound value of the sampling period) of the set of elements of a given signal object. On the other side, its bounding elements are: 1. a version uniformly sampled with the so-called Nyquist rate [1]-[6] (that is with the minimal sampling frequency allowed by the sampling theorem) and, 2. a version non-uniformly sampled on the edge of satisfying an equivalent condition for this kind of sampling [7]-[12]. In other words, the first bounding element corresponds with choosing the sampling rate equal to infinity (meaning no sampling) and the second ones are connected with choosing the Nyquist rate or a related one. Moreover, note that then all the remaining sampled functions of a given continuous time function represent other signal objects, not this one described above. That is they belong to some other sets.

Consider now the size of a set constituting a signal object. Obviously, as already mentioned, it consist of an infinite number of elements. And, at first glance, it would seem that the set containing these elements is countable. That is its cardinality would be equal to the cardinality $\aleph_0$ [13], where the so-called aleph-zero symbol $\aleph_0$ means the cardinality of the set of natural numbers $\mathbf{N}$. In what follows, we will however show that this is not true. Namely, we will demonstrate that the cardinality of the above set is of $\mathfrak{c}$ or greater, where the symbol $\mathfrak{c}$ stands for the cardinality of the set of real numbers $\mathbf{R}$.

To show that the above claim is true, let us take into account a continuous time signal and sample it with a certain sampling period *T* that satisfies the sampling theorem. Next, observe that for this choice of *T* there exists an infinite number of possibilities of choosing a starting point of sampling on the closed line segment $<0,T>$. And, the cardinality of the latter line segment equals $\mathfrak{c}$. So, in consequence, this leads to obtaining a set of sampled signals that are visually different (because time-delayed). Further, consequently, the cardinality of that set we arrive at here is equal to $\mathfrak{c}$.

In the next step, see that in fact there exists an infinite number of possible correct (satisfying the sampling theorem) choices for choosing the value of *T*. Only condition is that they must lie in the closed line segment $<0,T_{\max}>$, where the zero value stands for performing no sampling and $T_{\max}$ for the maximal sampling period allowable by the sampling theorem. But, note that the cardinality of the latter line segment (interval) equals $\mathfrak{c}$. Furthermore, see also that for each of the values of period *T* lying in the interval $<0,T_{\max}>$ we can choose an infinite number of the starting points of sampling as described above. (Except of course of the case of $T = 0$, when no sampling is performed.)

Now, see that connecting these two procedures described above leads to a set which is an infinite union of sets of the cardinality $\mathcal{c}$; with this number of set union operations

And finally at this point, note that by using appropriate variants of the two procedures discussed above that is:

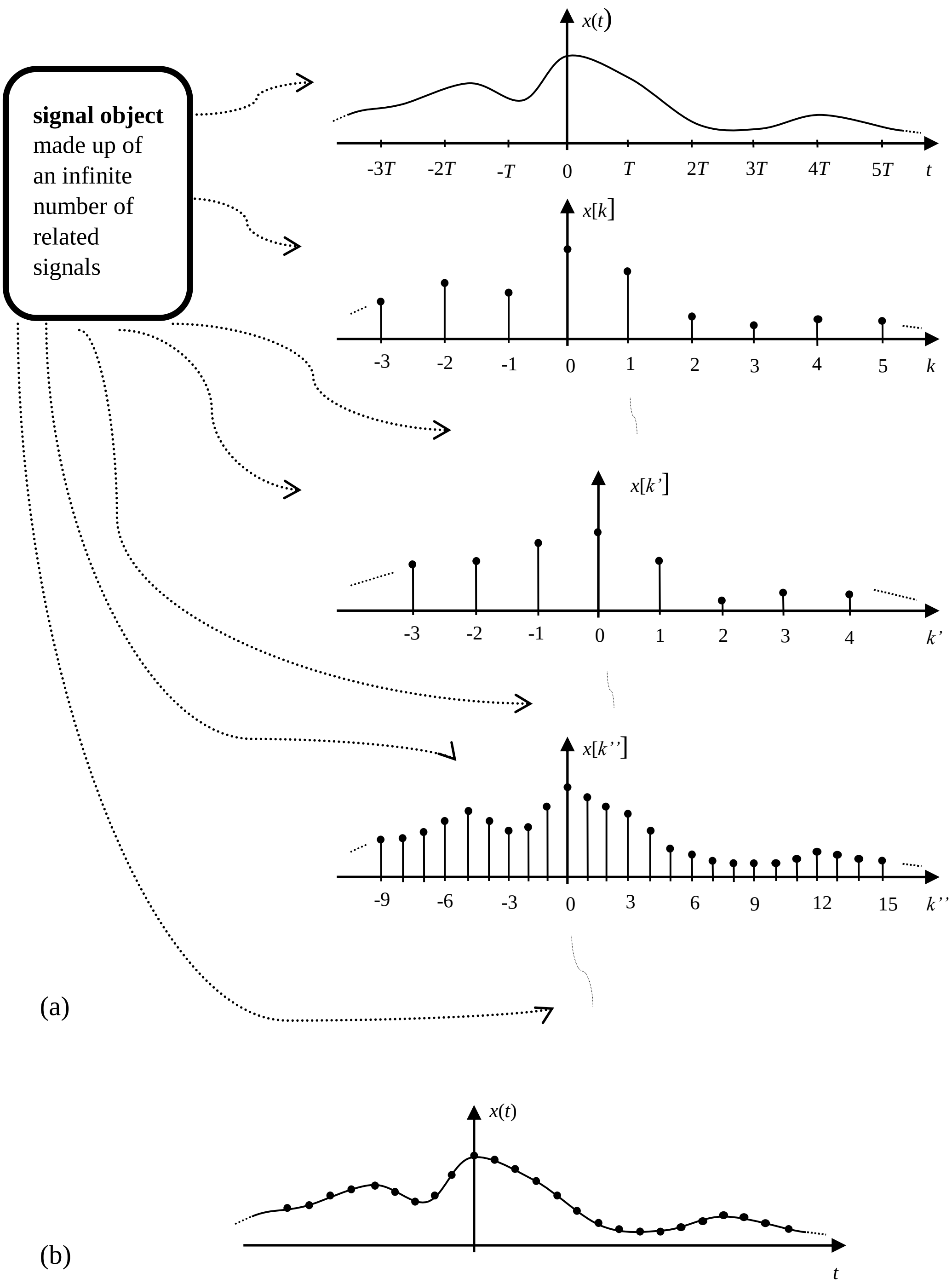

Fig. 3. (a) A signal object defined as an object that consists of an continuous-time signal and of an infinite number of its sampled versions. The signal sampled versions are assumed to be obtained with the use of different sampling periods and/or different starting points of sampling (in the sense of location of the zeroth index k). Further, all of them fulfil the sampling theorem. Moreover, the starting point of sampling can be chosen arbitrarily on the timeline, and the sampling operation itself can be uniform or non-uniform. (b) A pictorial representation of a signal object with sampling-type attribute or, in other words, of a signal possessing sampling-type attribute.

mentioned constituting a set of the cardinality $\mathcal{C}$. Obviously, the resulting set is of the cardinality $\mathcal{C}$ or greater.
1. choosing, now, admissible combinations of sampling periods and, 2. changing starting points of sampling in the case of non-uniform sampling we obtain next sets of elements belonging to a signal object considered. Obviously, these sets should be added to those discussed previously for a uniform sampling of a given signal. So, after that, the resulting set of elements of a signal object considered will be complete. And, the cardinality of it will be $\mathcal{C}$ or greater. That is maybe the size of the resulting set is exactly equal to $\mathcal{C}$ or it is another cardinal number. We do not consider however this question here.

Let us now choose and formalize notation for functions (signals) with attributes. We propose here to denote them with the use of the following symbol: $x\left(t,\square_{t,x}\right)$, where t stands for a continuous time variable and $\square_{t,x}$ means an attribute interacting on the function $x(t)$. Note that in this notation the attribute $\square_{t,x}$ extends the function (signal) $x(t)$ to the function (signal) object $x\left(t,\square_{t,x}\right)$.

In other words, the subscript t by $\square_{t,x}$ indicates here the attribute interaction on the time variable, and the second one by $\square_{t,x}$, x, indicates that the time variable affected is an argument of the function $x(t)$. Moreover, note that in this paper we use the same symbol $\square_{t,x}$ for denoting a given attribute itself as well as for the result of its interaction on the time variable t, being the argument of the function $x(t)$. Its current meaning will follow from the context.

In this section, we consider, as already mentioned, the sampling-type attribute (or simply sampling on the time axis). And, note that its definition can be narrower or broader. For example, the sampling attribute can be restricted only to the uniform sampling or only to the non-uniform one. Furthermore, it can mean only the stochastic sampling. Or, it can stand, as discussed at the beginning of this section, for the uniform and non-uniform types of sampling taken together. Moreover, the sampling-type attribute definition can admit all the possible starting points of sampling (in the sense of location of the zeroth index k) or be restricted to only one or more fixed ones. And so on.

As already mentioned and shown before, all the elements of a given signal object (that is all the signals it contains) are equivalent to each other although they differ from each other visually. However, there exist two unique operations (operators, transformations) which allow to obtain from any of them, uniquely, an arbitrary element of a given signal object we wish. These two transformations are: 1. the sampling of a continuous time signal carried out in accordance with the Nyquist sampling theorem or a related one, and, 2. the mapping indicated by the so-called reconstruction formula. As well known, in the case of the uniform sampling, we can express them as follows below.

To this end, let us use the notation that is applied on the two top curves of Fig. 3(a). There, the values of bar heights $x[k]=x[kT]$, $k=\ldots,-1,0,1,\ldots$, are equal to the values $x(kT)$ of the first function (continuous one) at the points kT. Next assume that the following:

$$1/T = f_s \geq 2 f_m \qquad (1)$$

holds, where T means a sampling period, f_s the corresponding sampling frequency, and f_m stands for the maximal frequency present in the spectrum of the signal $x(t)$. Then, by virtue of the sampling theorem [1]-[6], the signals $x(t)$ and $x[k]=x[kT]$, $k=\ldots,-1,0,1,\ldots$, are equivalent to each other in the sense that they can be obtained from each other via the so-called reconstruction formula [1]-[6]

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} x(kT)\,\mathrm{sinc}(t/T-k) = \\ &= \sum_{k=-\infty}^{\infty} x[k]\,\mathrm{sinc}(t/T-k) \end{aligned} . \qquad (2)$$

In (2), the function $\mathrm{sinc}(t)$ is defined as

$$\mathrm{sinc}(t) = \begin{cases} \dfrac{\sin(\pi t)}{\pi t} & \text{for } t \neq 0 \\ 1 & \text{for } t = 0 \end{cases} . \qquad (3)$$

Using the newly introduced notation for functions with attributes and the formulas (1)-(3), we can define, respectively, the sampling $H_s(\cdot)$ and reconstruction $H_r(\cdot)$ transformations (operators) in the following way:

$$\begin{aligned} &H_s\left(x(t)\right) = \square_{t,x}(T,k_0) = \left\{x(kT)=x[k]\right\}, \\ &k=\ldots,-1,0,1,\ldots, \end{aligned} . \qquad (4)$$

and

$$\begin{aligned} &H_r\left(\square_{t,x}(T,k_0) = \left\{x(kT)=x[k]\right\}\right) = \\ &= \sum_{k=-\infty}^{\infty} x[k]\,\mathrm{sinc}(t/T-k) = x(t) , \end{aligned} \qquad (5)$$

where $\left\{x(kT)=x[k]\right\}$, $k=\ldots,-1,0,1,\ldots$, means a set of samples of a given signal $x(t)$ for a given sampling period T and an assumed location of the zeroth index k_0 of k (not indicated in this notation). Convention assumed here is also that when the parameters T and k_0 are not given, for example, in $\square_{t,x}(T,k_0)$, this corresponds to the case of taking all the possible and admissible values of them. Always, the context will determine an actual usage. And finally at this point, we remark that after introducing the symbol $x\left(t,\square_{t,x}\right)$ a precise notation for the signal illustrated in Fig. 3(b) should be the one that uses the latter symbol (instead of $x(t)$).

III. The Dirichlet Function as an Instructive Example of Functions with Attributes

At the first glance, it can seem that the functions with attributes - so named by us here - are simply a kind of functions with parameters. Of course, this is partly true. However, the former ones mean much more and we will try to show this on an example of the so-called Dirichlet function [14], [15]. This example is very instructive.

The Dirichlet function is defined as

$$D(t)=\begin{cases}1 \text{ for } t\in\mathbf{Q}\\ 0 \text{ for } t\notin\mathbf{Q}\end{cases}, \tag{6}$$

where $\mathbf{Q}$ means the set of the rational numbers. This function is called the characteristic function of the rational numbers and can be also expressed analytically as [14]

$$D(t)=\lim_{m\to\infty}\lim_{n\to\infty}\cos^{2n}(m!\pi t)\ . \tag{7}$$

Consider now the function $x(t)=t$ and a function $x_D(t)=t\cdot D(t)$ that is derived from the former one. In what follows, we will show that these two functions build up a signal object in the sense defined in Section 2. Or, in other words, they form a function that we called a function with attributes in the previous sections. Let us denote it by $x\left(t,\square_{t,x}\right)$, similarly as before, where now this symbol refers to as only one function, $x(t)=t$. Its attribute will be defined as a condition following from the function $D(t)$, determining whether to multiply the time variable *t* by 1 or by 0. Or, more illustratively, as switching between the values of t when they are rational numbers, and zeros in case they are irrational ones. What, equivalently on the time axis, will correspond to stamping the successive rational and irrational numbers on this axis. (On this occasion, remember that in the case of signal sampling we have dealt with something similar: moving successively from one point to another on the time axis *t*, where the locations of these points followed from the relation $t = kT$, with $k=\ldots,-1,0,1,\ldots$.)

Concluding the above considerations, note that the following transformations (operators):

$$H_{sD}\left(x(t)=t\right)=\square_{t,x}=x_D(t)=t\cdot D(t)\ . \tag{8}$$

and

$$H_{rD}\left(\square_{t,x}\right)=\begin{cases}t \text{ if } \square_{t,x}=0\\ t \text{ if } \square_{t,x}\neq 0\end{cases}=t=x(t), \tag{9}$$

can be formulated, where $H_{sD}(\cdot)$ and $H_{rD}(\cdot)$ play the roles, respectively, of a sampling operator and of a signal recovery operator. And, because of this reason, we call them here, respectively, the Dirichlet-sampling and the Dirichlet-reconstruction operations.

So, finally, we can say that the function with the Dirichlet-type attribute $x\left(t,\square_{t,x}\right)$, defined by (8) and (9), builds up a unique signal (function) object. It consists of a pair of two functions, $x(t)=t$ and $x_D(t)=t\cdot D(t)$. Moreover, observe that the sets of arguments of these functions related with their non-zero values possess different cardinalities. For the first of them, this is $\mathfrak{c}$, but for the second $\aleph_0$ (note that something similar happened in the case of signal (function) objects discussed in Section II).

IV. Remark

The basic ideas behind an interpretation of the measuring process as a sampling of signals and behind a notion of functions with attributes have been developed here. Now, however, they will need more studies to prove their usefulness. The results of these investigations will be presented in the forthcoming papers.

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