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On Derivation of Discrete Time Fourier Transform from Its Continuous Counterpart

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***Abstract*—This paper is devoted to some problems that appear in derivations of the discrete time Fourier transform from a formula for its continuous time counterpart for transformation from the time into the frequency domain as well as to those regarding transformation in the inverse direction. In particular, the latter ones remained so far an unresolved problem. It is solved for the first time here. Many detailed explanations accompanying the solution found are presented. Finally, it is also worth noting that our derivations do not exploit any of such sophisticated mathematical tools as the so-called Dirac delta and Dirac comb.**

***Keywords*—Sampling of signals, relation between discrete and continuous time Fourier transforms**

I. Introduction

PROBABLY this paper would never have been written if its author was not confronted with the need to explain the so-called discrete time Fourier transform (DTFT) to his students. Note that a standard way of introducing the DTFT in programs of digital signal processing courses at the universities is by presenting solely its definition. That is by providing only two expressions: one for the transformation from the discrete time domain into the continuous frequency domain, and the second regarding the inverse direction. See for instance some example lecture slides available in the Internet [1], [2].

Also, the DTFT is presented similarly in the textbooks on digital signal processing, as for example in such notable ones as [3]-[7]. Some authors only mention that the DTFT is a counterpart of the continuous time Fourier transform (CTFT) that was conceived for discrete time signals. Nothing is rather said about possibility of derivation of the formulas for the DTFT from those for the CTFT provided for signals of the continuous time, what would be interesting for students. Why? Because they learn first analog signals and analog systems, and the CTFT associated with them. So, they expect afterwards, when it comes time for discussion of discrete-time signals and discrete systems, and of an appropriate mathematical tool similar to the CTFT, a smooth transition between these themes. However, it turns out to be not so simple. Mostly, as we shall see in the next sections, because of some unexpected problems appearing in derivation of the inverse DTFT from its "continuous counterpart". That is in its derivation from the formula defining the CTFT. And, this is the reason why the lecturers and authors of textbooks on signal processing do not follow this way. They prefer to go the other way, which is however very unnatural. Why? Because it requires reformulating the sampled signal to arrive at an infinite sequence of non-physical signal objects. More precisely, this approach assumes modeling of the sampled signal by a modulated Dirac comb [3]-[7]. That is by an infinite sequence of the so-called Dirac impulses, which, as it is well known, are not usual functions [8], [9], [10]. They represent signal objects called distributions that have no physical counterparts.

Obviously, one can tell the students that there is no other way of introducing the DTFT than the one sketched above. However, the most inquisitive among them will not be then satisfied when receiving such an answer. They will suspect that the material on the DTFT in this form is incomplete, and therefore it must exploit some mathematical tricks, which are hard to understand. And, in fact, we must admit that they will be then right.

As the author of this paper experienced some penetrating questions raised by students, which regarded pitfalls in formulating the DTFT, as mentioned above, he decided to try to find a more consistent way of introducing the DTFT. A way in which the DTFT derives from the CTFT. And, it has proved possible; this paper reports on the results achieved.

The remainder of this paper is organized as follows. In the next section, we present derivation of the DTFT from the CTFT for transformation from the time into the frequency domain. The derivation of the inverse DTFT formula from a relevant expression for the inverse CTFT is a subject of the third section. The next section is devoted to checking the correctness of the main formula for the inverse DTFT derived in this paper by substituting in it a new expression found for the DTFT. Applicability of the new formulas derived here for calculation of the DTFT and its inverse in the case of consideration of under-sampled signals is analyzed thoroughly in the fifth section. The paper ends with a concluding remark.

II. Derivation of DTFT from CTFT for Transformation from Time into Frequency Domain

In this section, we will derive a defining formula for the DTFT from the corresponding formula for the CTFT for the case of transformation from the time into the frequency domain. So, to this end, let us consider a signal $x(t)$ of a continuous time variable $t \in \mathrm{R}$, where R denotes the set of real numbers. Further, assume that this signal is sampled uniformly with a sampling period *T*, providing an infinite set of signal samples. Here, we denote this set of samples by $\{..., x(-T), x(0), x(T), ...\} = \{x(nT)\}$, $n = ..., -1, 0, 1, ...$.

In what follows, we assume that $x(t)$ means such a real signal, whose bandwidth is finite. That is it denotes a

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bandlimited signal having a Fourier transform $X(f)$ satisfying the following equation:

$$X(f) \equiv 0 \text{ for } |f| > f_m > 0, \tag{1}$$

where $f \in \mathrm{R}$ means a continuous frequency variable.

Moreover, note that assuming (1) means that the Fourier transform $X(f)$ of $x(t)$ is identically zero outside a closed frequency interval $<-f_m, f_m>$. In other words, the bandwidth of the signal $x(t)$ is equal to $B = f_m - 0 = f_m$, where f_m can be called the maximal frequency in its spectrum.

Now, we can express the condition under which the function $x(t)$ can be approximated exactly, or, in other words, it can be perfectly reconstructed from its samples at every point $t \in \mathrm{R}$. It has the following form:

$$T \le 1/(2f_m). \tag{2}$$

And, the reconstruction formula is

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\mathrm{sinc}(t/T - n) = x(t)\ , \tag{3}$$

where the function $\mathrm{sinc}(t)$ is defined as

$$\mathrm{sinc}(t) = \begin{cases} \dfrac{\sin(\pi t)}{\pi t} & \text{for } t \neq 0 \\ 1 \quad \text{for } t = 0 \end{cases}\ . \tag{4}$$

In (3), $\hat{x}(t)$ means a function being an approximation of $x(t)$ that exploits the set of signal samples $\{x(nT)\}$ defined above.

After all these preliminaries, let us now start with the task we mentioned at the beginning of this section. And, first, we will consider the case when we do not know whether the condition (2) is fulfilled or not. In this general case, we can write the following:

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\,dt \cong \\ &\cong \sum_{n=-\infty}^{\infty} x(n\Delta t)\exp(-j2\pi fn\Delta t)\Delta t \end{aligned}\ , \tag{5}$$

where the definition of the Riemann integral was applied. Moreover, the indices $n \in \mathrm{Z}$, where Z denotes the set of integers.

Assuming in (5) that the infinitesimal increase of time Δt is equal to the sampling period T and $f_s = 1/T$ means the sampling frequency, we get for the expression in the second line of (5)

$$\begin{aligned} &\sum_{n=-\infty}^{\infty} x(n\Delta t)\exp(-j2\pi fn\Delta t)\Delta t = T\sum_{n=-\infty}^{\infty} x(nT)\cdot \\ &\cdot\exp(-j2\pi fnT) = \sum_{n=-\infty}^{\infty} T\cdot x(nT)\exp\left(-j2\pi\frac{f}{f_s}n\right) = \\ &= \sum_{n=-\infty}^{\infty} \overline{x}(nT)\exp(-j2\pi Fn) = \sum_{n=-\infty}^{\infty} \overline{x}[n]\exp(-j2\pi Fn)\ , \end{aligned} \tag{6}$$

where $\overline{x}(nT) = T\cdot x(nT) = T\cdot x[n] = \overline{x}[n]$ denotes a non-strictly-point sample of a signal, while $F = f/f_s$ the normalized frequency.

Connecting finally (5) with (6), we can write

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\,dt \cong \\ &\cong \sum_{n=-\infty}^{\infty} \overline{x}[n]\exp(-j2\pi Fn) = X(F) \end{aligned}\ , \tag{7}$$

where $X(F)$ means the DTFT of the sampled signal. That is of the sequence $\{x(nT)\} = \{x[n]\}$, $n = \ldots,-1,0,1,\ldots$. Further, let us also express symbolically an approximate equivalence of $X(F)$ with $X(f)$, observed in (7), in the following way:

$$\text{CTFT} \cong \text{DTFT}\ . \tag{8}$$

Now, we will show that it is possible to strengthen the result given by (7) in the case when we know that the condition (2) is satisfied in carrying out the sampling. Note that we can then use the reconstruction formula (3). Applying it in the definition of the CTFT leads to

$$\begin{aligned} &X(f) = \\ &= \int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} x(nT)\,\mathrm{sinc}(t/T - n)\exp(-j2\pi ft)\,dt = \\ &= \sum_{n=-\infty}^{\infty} x(nT)\int_{-\infty}^{\infty}\mathrm{sinc}(t/T - n)\exp(-j2\pi ft)\,dt\ . \end{aligned} \tag{9}$$

To proceed further, note that the Fourier transforms of the function $\mathrm{sinc}(t/T - n)$ shifted on the time axis are given by

$$\begin{aligned} &\int_{-\infty}^{\infty}\mathrm{sinc}(t/T - n)\exp(-j2\pi ft)\,dt = \\ &= T\cdot\mathrm{rect}(fT)\exp(-j2\pi fnT) \end{aligned}\ , \tag{10}$$

where the function $\mathrm{rect}(x)$ means the following:

$$\mathrm{rect}(x) = 1 \text{ for } |x| \le \frac{1}{2} \text{ and } 0 \text{ for } |x| > \frac{1}{2}\ . \tag{11}$$

Next, substituting (10) into (9), we get

$$
\begin{aligned}
X(f) &= \sum_{n=-\infty}^{\infty} x(nT)\, T \cdot \operatorname{rect}(fT) \exp(-j2\pi fnT) = \\
&= \sum_{n=-\infty}^{\infty} \bar{x}[n] \operatorname{rect}\left(\frac{f}{f_s}\right) \exp\left(-j2\pi \frac{f}{f_s} n\right) = \\
&= \sum_{n=-\infty}^{\infty} \bar{x}[n] \exp(-j2\pi Fn) = X(F) \\
&\quad \text{for } |F| \le 1/2 \quad \text{and} \\
X(f) &= 0 = X(F) \quad \text{for } |F| > 1/2 \;.
\end{aligned} \tag{12}
$$

Observe now that the result achieved in (12) resembles the one in (7). Maybe they are the same? To check this, consider once again (7) and specialize it for the latter case in which the following:

$$
f_m \le f_s/2 \quad \Leftrightarrow \quad F_m = \frac{f_m}{f_s} \le \frac{f_s/2}{f_s} = \frac{1}{2} \tag{13}
$$

holds. So, taking into account the above, we rewrite now (7) in such a way

$$
\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t) \exp(-j2\pi ft)\, dt \cong \\
&\cong \sum_{n=-\infty}^{\infty} \bar{x}[n] \exp(-j2\pi Fn) \quad \text{for } |F| \le |F_m| \le 1/2 \\
&\text{and} \\
X(f) &= 0 \quad \text{for } |F| > |F_m| > 1/2 \;.
\end{aligned} \tag{14}
$$

Comparison of (12) with (14) shows that the latter converges with the former. And, obviously, this fact is not casual. Moreover, the formulas in (12) and (14) are identical when $f_m = f_s$ (giving $|F_m| = 1/2$), and with the equality symbol instead of $\cong$ in (14).

Not fulfilling the condition (13) means the occurrence of the so-called aliasing effect [3], [4] in the operation of signal sampling. In this case, the sampled signal is distorted in a specific way, and a kind of distortion that then arises is referred to as an aliasing. And, in what follows, let us denote this distorted in such a way signal by $x_a(t)$. Further, note that an equivalent of (7) for this signal will have the following form:

$$
\begin{aligned}
X_a(f) &= \int_{-\infty}^{\infty} x_a(t) \exp(-j2\pi ft)\, dt \cong \\
&\cong \sum_{n=-\infty}^{\infty} \bar{x}[n] \exp(-j2\pi Fn) = X_a(F)
\end{aligned} \quad , \tag{15}
$$

where $X_a(f)$ and $X_a(F)$ mean the standard Fourier transform of $x_a(t)$ and the DTFT of its sampled version, respectively. Obviously, now, $x(t) \ne x_a(t)$ and thereby also $X(f) \ne X_a(f)$. However, the sequences: $\{x(nT)\} = \{x[n]\}$ for the under-sampled signal $x(t)$ and the corresponding sequence $\{x_a(nT)\} = \{x_a[n]\}$, representing the sampled signal $x_a(t)$, are identical. This fact is taken into account in (15).

It can be easily shown that the signal $x_a(t)$ is a low-pass signal having its maximal frequency in its spectrum equal to $1/(2T)$. That is we have $f_m = 1/(2T) = f_s/2$ here, giving $|F_m| = 1/2$. For more details, see [11] as well as section V in this paper.

So, because of the above reason the analysis presented before for the case of sampling the signal $x(t)$ with the sampling frequency satisfying the condition (2) applies also here. That is in the case of sampling the signal $x_a(t)$. Therefore, the counterparts of (12) and (14) have now the following forms:

$$
\begin{aligned}
X_a(f) &= \sum_{n=-\infty}^{\infty} \bar{x}[n] \exp(-j2\pi Fn) \quad \text{for } |F| \le 1/2 \\
\text{and } \; X_a(f) &= 0 \quad \text{for } |F| > 1/2 \;,
\end{aligned} \tag{16}
$$

and

$$
\begin{aligned}
X_a(f) &\cong \sum_{n=-\infty}^{\infty} \bar{x}[n] \exp(-j2\pi Fn) \quad \text{for } |F| \le 1/2 \\
\text{and } \; X_a(f) &= 0 \quad \text{for } |F| > 1/2 \;,
\end{aligned} \tag{17}
$$

respectively. Moreover, note that the formulas in (16) and (17) are, in fact, identical here because the symbol $\cong$ in (17) can be replaced by the equality one.

Finally, at this point, we want to say that we come back yet to a more detailed discussion of (16) in section V.

III. Derivation of Inverse DTFT Formula from Relevant Expression for Inverse CTFT

This section will be devoted to the derivation of the inverse DTFT formula from a relevant expression for the CTFT. In other words, we will perform here a task similar to the one tackled with in the previous section, but now for the inverse direction of transformation. That is from the frequency into the time domain.

Let us start with the formula for the inverse CTFT; that is with

$$
x(t) = \int_{-\infty}^{\infty} X(f) \exp(j2\pi ft)\, df \quad . \tag{18}
$$

In the next step, let us introduce the following substitutions: $t = n\Delta t$, $\Delta t = T$, $f_s = 1/T$, $F = f/f_s$ in (18). This allows us to rewrite (18) as

$$
x(nT) = x[n] = f_s \int_{-\infty}^{\infty} X(F) \exp(j2\pi nF)\, dF \tag{19}
$$

or equivalently as

$$\bar{x}[n] = \int_{-\infty}^{\infty} X(F)\exp(j2\pi nF)\,dF \;, \tag{20}$$

where the usual convention of writing $X(f) = X(F)$ is used. Clearly, because the relation $f = f_s F$ holds.

Let us now divide the whole integration interval $(-\infty, \infty)$ of the integral in (20) into segments of the length equal to 1 and calculate this integral through integrating in each of these intervals and summing the results. So, for performing this task, we rewrite (20) as

$$\begin{aligned}\bar{x}[n] &= \int_{-\infty}^{\infty} X(F)\exp(j2\pi nF)\,dF = \ldots\ldots + \\ &+ \int_{-3/2}^{-1/2} X(F)\exp(j2\pi nF)\,dF + \\ &+ \int_{-1/2}^{1/2} X(F)\exp(j2\pi nF)\,dF + \\ &+ \int_{1/2}^{3/2} X(F)\exp(j2\pi nF)\,dF + \ldots\ldots\ldots \;.\end{aligned} \tag{21}$$

To proceed further, we introduce auxiliary variables $F_k = F_0 + k$, where $k \in \mathrm{Z}$, in places of occurrence of F in the successive integrals in (21) in such a way that F_k covers always the range of integration, which is taken at a given moment. That is, in our case, in the following segments indicated in (21): $\langle -3/2, -1/2 \rangle$, $\langle -1/2, 1/2 \rangle$, $\langle 1/2, 3/2 \rangle$, and so on. Further, note that the corresponding integer $k \in \mathrm{Z}$, mentioned above, can be interpreted as a "shifting" integer that shifts a basic integration interval $\langle -1/2, 1/2 \rangle$, associated with the corresponding variable F_0, to the intervals listed above. So, applying this in (21), we get

$$\begin{aligned}\bar{x}[n] &= \\ &= \ldots + \int_{-1/2}^{1/2} X(F_0 - 1)\exp\big(j2\pi n(F_0 - 1)\big)\,dF_0 + \\ &+ \int_{-1/2}^{1/2} X(F_0 + 0)\exp\big(j2\pi n(F_0 + 0)\big)\,dF_0 + \\ &+ \int_{-1/2}^{1/2} X(F_0 + 1)\exp\big(j2\pi n(F_0 + 1)\big)\,dF_0 + \ldots \;.\end{aligned} \tag{22}$$

In the next step, taking into account that $\exp(j2\pi nk) = 1$, $k \in \mathrm{Z}$, and dropping the index zero at the variable F_0 in (22), we arrive finally at

$$\begin{aligned}\bar{x}[n] &= \ldots. + \int_{-1/2}^{1/2} X(F - 1)\exp(j2\pi nF)\,dF + \\ &+ \int_{-1/2}^{1/2} X(F)\exp(j2\pi nF)\,dF + \\ &+ \int_{-1/2}^{1/2} X(F + 1)\exp(j2\pi nF)\,dF + \ldots. \;.\end{aligned} \tag{23}$$

Consider now for the moment that the function $X(F)$ occurring in (23) is not expressed as in (12), but as it is usually done in the literature, see, for example, [1]-[7], by the following formula:

$$X(f) = \sum_{n=-\infty}^{\infty} \bar{x}[n]\exp(-j2\pi Fn) \quad \text{for } F \in (-\infty, \infty). \tag{24}$$

In the literature, the expression (24) is treated as a definition of the DTFT. And, note that it represents, in this form, a periodic function with a period equal to 1. So, applying the periodicity of $X(F)$ in (23), we would then have arrived at

$$\begin{aligned}\bar{x}[n] &= \ldots + \int_{-1/2}^{1/2} X(F)\exp(j2\pi nF)\,dF + \\ &+ \int_{-1/2}^{1/2} X(F)\exp(j2\pi nF)\,dF + \\ &+ \int_{-1/2}^{1/2} X(F)\exp(j2\pi nF)\,dF + \ldots = \\ &= \text{infinity times} \int_{-1/2}^{1/2} X(F)\exp(j2\pi nF)\,dF = \\ &= \infty\end{aligned} \tag{25}$$

for all values of n, for which the following:

$$\int_{-1/2}^{1/2} X(F)\exp(j2\pi nF)\,dF \neq 0 \tag{26}$$

holds. This is, however, an unacceptable result. Therefore, we have no other choice but to conclude that the DTFT given by (24) constitutes an incorrect definition. Also, it seems to be clear that we can expect receiving a correct result for $\bar{x}[n]$ if only one component remains in (23). And, we arrive in such a situation, when we drop the periodicity property of the DTFT defined by (24), what is the case in the DTFT derived in (12).

Note that in accordance with (12), we have

$$\begin{aligned}&X(F+k) \text{ not identically zero for } k=0\\&\quad\text{in the range of } F\in\langle -1/2,1/2\rangle,\\&X(F+k) \text{ identically zero for every } k\neq 0\\&\quad\text{in the range of } F\in\langle -1/2,1/2\rangle.\end{aligned} \tag{27}$$

So, applying (27) in (23) results in

$$\bar{x}[n]=\int_{-1/2}^{1/2} X(F)\exp(j2\pi nF)\,dF. \tag{28}$$

That is we arrived here at the well-known formula [1]-[7] defining the inverse DTFT, or, in short, the IDTFT.

Obviously, we get the same result for $\bar{x}[n]$ from the formula (28) independently whether $X(F)$ is understood as the DTFT given by (12) or the DTFT expressed by (24). In case of the formula (24), this simply follows from the fact that only one segment of the periodic DTFT is taken then into account in (28), just the one lying in the range of $F\in\langle -1/2,1/2\rangle$.

Further, note that we can afford to a more consistent notation of the IDTFT with the "philosophy of the Fourier transform" if we use the DTFT as understood in (12) instead of the one given by (24). Simply, then, it will be allowed to write (28) in the following form:

$$\bar{x}[n]=\int_{-\infty}^{\infty} X_{(12)}(F)\exp(j2\pi nF)\,dF, \tag{29}$$

where $X_{(12)}(F)$ means $X(F)$ that should be calculated as shown in (12).

Here, we pay also attention to the fact that (28) represents a rule for calculation of the coefficients of the Fourier series (24). That is of the periodic function $X(F)$ considered in (24), which is expanded there in a Fourier series. Further, note also that the above observation is widely exploited in the literature [3]-[7]. However, in the context of the topic of our paper, it is quite irrelevant.

IV. Checking Correctness of (29) with Use of (12)

To complete our derivations presented in the previous section as well as to demonstrate correctness of a final formula (29), on the inverse path now, let us substitute (12) into (29). Then, we get

$$\begin{aligned}\bar{x}[n]=&\int_{-\infty}^{-1/2} 0\cdot\exp(j2\pi nF)\,dF+\\&+\int_{-1/2}^{1/2}\sum_{m=-\infty}^{\infty}\bar{x}[m]\exp(-j2\pi Fm)\exp(j2\pi nF)\,dF+\\&+\int_{1/2}^{\infty} 0\cdot\exp(j2\pi nF)\,dF,\end{aligned} \tag{30}$$

where $m\in\mathrm{Z}$. Obviously, the first and the third integrals in (30) equal zeros. Therefore, after the interchange of the symbols for summing and integrating in the second integral in (30), we obtain the following:

$$\bar{x}[n]=\sum_{m=-\infty}^{\infty}\bar{x}[m]\int_{-1/2}^{1/2}\exp(j2\pi F(n-m))\,dF. \tag{31}$$

Further, note that the value of the integral in (31) is given by

$$\begin{aligned}&\int_{-1/2}^{1/2}\exp(j2\pi F(n-m))\,dF=\frac{1}{j2\pi(n-m)}\cdot\\&\cdot\left[\exp(j2\pi(n-m))-\exp(-j2\pi(n-m))\right]\Big|_{-1/2}^{1/2}=\\&=0 \text{ for } n\neq m \text{ and}\\&\int_{-1/2}^{1/2}\exp(j2\pi F(n-m))\,dF=\int_{-1/2}^{1/2}1\cdot dF=\\&=F\Big|_{-1/2}^{1/2}=\frac{1}{2}-\left(-\frac{1}{2}\right)=1 \text{ for } n=m.\end{aligned} \tag{32}$$

So, applying (32) in (31) results in

$$\bar{x}[n]=\bar{x}[m=n]=\bar{x}[n]. \tag{33}$$

That is we obtained identity in (33), as expected. And, this ends our demonstration of correctness of (29) via (12).

V. DTFT and IDTFT for Under-Sampled Signals in More Detail

Now, we consider a question in a more detail whether the formulas (12) and (29) can be also used in the case of under-sampled signals. It is a legitimate question. Why? Because we used the reconstruction formula (3) in derivation of (12), while (3) is valid only when the conditions (1) and (2) are satisfied. Therefore, it is clear that we must check whether these conditions are fulfilled in the case of under-sampled signals. In what follows, we will show that the answer here is positive. This is so, mostly, because the constellation of samples of an under-sampled signal builds up, at the same time, a sequence of samples of a signal that is bandlimited with the maximal frequency in its spectrum, say, f_{ma} equal to $1/(2T)=f_s/2$.

Let us now denote by $x_a(t)$ the bandlimited signal mentioned above. That is a signal, which is a version of the signal $x(t)$ that was recovered from the samples of the latter one, obtained as a result of performing the operation of under-sampling. And, accordingly, denote by $X_a(f)$ the Fourier transform of $x_a(t)$. So, using this notation, we can rewrite (1) and (2) for the signal $x_a(t)$ as follows:

$$X_a(f)\equiv 0 \text{ for } |f|>f_{ma}=f_s/2=1/(2T) \tag{34}$$

and

$$T \le 1/(2 f_{ma}) = 1/(2 \cdot f_s/2) = T, \qquad (35)$$

respectively. And, both the conditions (34) and (35) are fulfilled for the signal $x_a(t)$. Therefore, we can apply the reconstruction formula (3) in this case. Here, it reads as follows:

$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT)\,\mathrm{sinc}(t/T - n) \qquad (36)$$

with

$$x_a(nT) = x(nT) \text{ for all } n \in \mathrm{Z}. \qquad (37)$$

So, consequently, the formulas (12) and (29) are then also valid. In other words, they are valid for the signal $x_a(t)$, too. However, as we will see a little bit later, their application in this case needs a correct interpretation.

To explain this, it is advisable to denote the variable F as F_a and rewrite (12) and (29) incorporating this small notational change. So, we will have

$$X_a(f) = \sum_{n=-\infty}^{\infty} \bar{x}[n]\exp(-j2\pi F_a n) = X(F_a)$$
$$\text{for } |F_a| \le 1/2 \text{ and} \qquad (38)$$
$$X(f) = 0 = X(F_a) \quad \text{for} \quad |F_a| > 1/2,$$

and

$$\bar{x}[n] = \int_{-\infty}^{\infty} X_{(38)}(F_a)\exp(j2\pi n F_a)\,dF_a, \qquad (39)$$

respectively. In (39), $X_{(38)}(F_a)$ means $X(F_a)$ calculated according to (38). Further, F_a in (38) and (39) is given by

$$F_a = \frac{f}{1/T} = fT. \qquad (40)$$

Consider now the variable F occurring in a general formula (12) for the case when this formula could be used for the signal $x(t)$ that is when a sampling period used - denote it by T_r - would satisfy (2). Then, analogously to (40), the variable F would have been given by

$$F = \frac{f}{1/T_r} = fT_r. \qquad (41)$$

Further, see that the following relation:

$$T_r \le 1/(2 f_m) < 1/(2 f_{ma}) = 1/(2 \cdot 1/(2T)) = T \qquad (42)$$

holds.

So, see now that because occurrence of different proportionality coefficients T and T_r in (40) and (41), respectively, the variables F and F_a cannot be identified with each other.

In other words, applying (12) with F given by (41) results in $X(f)$ for the signal $x(t)$, however, with $F \to F_a$ expressed by (40) results in $X_a(f)$ for the signal $x_a(t)$.

In view of the above, see also that the sequences of samples $\{x(nT)\}$ and $\{x_a(nT)\}$ are never identical in the calculations of $X(f)$ and $X_a(f)$. The relation (37) indicates only that a sequence $\{x_a(nT)\}$ can build up a subsequence of $\{x(mT_r)\}$, where the sampling phases and the relation between the sampling periods T and T_r are appropriately chosen. For example, the following: $T_r = (1/2) \cdot T$ and $m = 2n$ leading to $\{x_a(nT)\} = \{x(2nT_r)\}$ could take place.

VI. Concluding Remark

It seems to us that it is very important to emphasize once again that the problem of derivation of the inverse DTFT from a formula defining its continuous-time counterpart has been transparently solved here. That is without using such sophisticated mathematical tools as the so-called Dirac delta and Dirac comb.

References

[1] Ch. A. Bouman, *Digital Image Processing I - Lecture 11 - DTFT, DSFT, Sampling, and Reconstruction*, https://cosmolearning.video-lectures/dtft-dsft-sampling-reconstruction/, accessed Dec. 2019.

[2] H. C. So, *Signals and Systems - Lecture 6 - Discrete-Time Fourier Transform*, www.ee.cityu.edu.hk/~hcso/ee3210.html, accessed Dec. 2019.

[3] J. H. McClellan, R. Schafer, and M. Yoder, *DSP First*, London: Pearson, 2015.

[4] M. Vetterli, J. Kovacevic, and V. K. Goyal, *Foundations of Signal Processing*, Cambridge: Cambridge University Press, 2014.

[5] R. Wang, *Introduction to Orthogonal Transforms with Applications in Data Processing and Analysis*, Cambridge: Cambridge University Press, 2010.

[6] V. K. Ingle and J. G. Proakis, *Digital Signal Processing Using Matlab*, Stamford, USA: Cengage Learning, 2012.

[7] A. V. Oppenhiem, S. Willsky, and S. H. Nawab, *Signals and Systems*, Upper Saddle River, USA: Prentice Hall, 1996.

[8] R. Strichartz, *A Guide to Distribution Theory and Fourier Transforms*, Boca Raton, USA: CRC Press, 1994.

[9] R. F. Hoskins, *Delta Functions: An Introduction to Generalised Functions*, Cambridge: Horwood Pub., 2009.

[10] R. Brigola, *Fourier-Analysis und Distributionen*, Hamburg: edition swk, 2013.

[11] A. Borys, “Under-sampling as a specific operation of filtering and signal shaping at the same time,” submitted to *Electronics Letters*, 4th January, 2020.