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Switching controller synthesis for discrete-time switched linear systems with average dwell time

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This paper addresses weighted L_2 gain performance switching controller design of discrete-time switched linear systems with average dwell time (ADT) scheme. Two kinds of methods, so called linearizing change-of-variables based method and controller variable elimination method, are considered for the output-feedback control with a supervisor enforcing a reset rule at each switching instant are considered respectively. Furthermore, some comparison between these two methods are also given.

Key words: discrete-time switched linear systems, L_2 performance, average dwell time, controller state reset, linear matrix inequalities

1. Introduction

There is an increasing interest from the scientific community in the study of linear switching system which comprises a collection of subsystems described by linear dynamics (differential/difference equations), together with a switching rule that specifies the switching among the subsystems; see the survey papers [1–3], the books [4–6]. Because of its strong engineering backgrounds, such systems can be used to construct a controller for a wide range of physical and engineering plants in practice. When the switching law is assumed to be arbitrary, one way to investigate analysis and synthesis problems of stability or control performance is

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to find a common Lyapunov function (CLF) for all the switching models [7–11]. Compared with results based on multiple Lyapunov functions (MLF) [12–14], such CLF-based conditions might become too conservative when a particular switching logic is concerned. Therefore, in the context of switched linear systems with controlled switching, it is reasonable to constitute the switching logic in the design and find a suitable one such that stabilization as well as performance improvement can be achieved. In particular, the ADT switching logic has been an efficient tool in switched system analysis and control synthesis, and also relevant to many practical applications in [15–17].

As to the controller synthesis of switched linear systems, many results have been obtained in the literature; see [12, 17–22] and the references cited in therein. The authors in [12] have firstly investigated the problems of stability and L_2-gain analysis, and state feedback synthesis for a class of discrete-time switched systems with ADT scheme. With the min-type, piecewise quadratic Lyapunov function, the authors in [18] have presented sufficient matrix inequality conditions expressed in terms of Riccati–Metzler inequalities, which can be solved by any LMI solver [19] coupled to a line search. Recently, under a relaxed min-switching logic perspective, the authors in [20] have used a modified Lyapunov–Metzler inequality to provide sufficient conditions of stability, L_2 gain performance analysis, and output feedback control synthesis for switched linear systems. Different from state-feedback control of switched linear systems, which can be solved with the multiple Lyapunov function (MLF) by LMI optimal technology, the output-feedback control synthesis for switched linear systems with the ADT scheme is non-trivial. The main difficulty stems from the fact that the associated boundary condition that constraints the jump between two adjacent Lyapunov-like functions always leads to non-convex synthesis conditions, which can be expressed by nonlinear matrix inequalities with coupled matrix variables. To deal with the problem, a two-step design method has excluded the boundary condition from the controller synthesis, but may result in an unacceptable disturbance attenuation level and/or an unreasonably large dwell time [21]. Different from the method by [21], the work in [22] has incorporated the boundary condition into the state feedback controller synthesis, and the associated BMI problem has been circumvented by using a controller state reset technique. More recently, with the reset control technique [23-26] and congruent transformation [27], the authors in [17] have presented a hybrid control scheme for the output-feedback control of continuous-time switched linear systems with ADT scheme. With a reset rule, both full-order and reduced-order controllers with guaranteed stability and optimal weighted H_∞ performance can be solved directly by LMI optimizations. However, till now, the output-feedback dynamic control scheme of discrete-time switched linear systems with ADT scheme has not yet been fully addressed with the reset control technique.

The main idea of this paper is to present switching state feedback and output-feedback control of discrete-time switched linear systems with the ADT scheme. Two kinds of methods, so called linearizing change-of-variables based method and controller variable elimination method, are considered for the output-feedback control with a supervisor enforcing a reset rule at each switching instant are considered respectively. First, by the congruent transformation and linearizing change-of-variables based method, solvability conditions based on linear matrix inequalities (LMIs) are presented for the existence of such switching control schemes. Second, by the Finsler's Lemma and controller variable elimination method, different solvability conditions of LMIs are also presented for the existence of such switching control schemes. Furthermore, since the solvability conditions decouple the relation between the controller variable matrices and multiple Lyapunov matrices, compared with linearizing change-of-variables based method, the number of matrix variables of solvability conditions is significantly reduced, which helps numerical efficiency in the design procedure. With the increasing of the number of subsystems of switching system, the second method has better numerical efficiency in the design procedure. Finally, a simple numerical example is given to show the validity of the developed results.

Throughout this paper, $\Re$ stands for the set of real numbers and $\Re_+$ for the positive real numbers. $\Re_{m\times n}$ is the set of real $m \times n$ matrices. The transpose of a real matrix M is denoted by M^T. The identity matrix of dimension $d \times d$ is denoted by I_d. $S^{n\times n}$ and $S_+^{n\times n}$ are used to denote the set of real symmetric $n \times n$ matrices and positive definite matrices, respectively. For $x \in \Re^n$, its norm is defined as $\|x\| := (x^T x)^{1/2}$. The space of square integrable functions is denoted by L_2. For two integers $k_1 < k_2$, we denote $\mathrm{I}[k_1,\ k_2] = \{k_1,\ k_1{+}1,\ \ldots,\ k_2\}$.

2. Definition and problem statement

The state space realization of a discrete-time switched linear system P is expressed as

$$\sum_P := \begin{cases} x_p(k+1) = A_{p,\delta} x_p(k) + B_{u,\delta} u(k), \\ y(k) = C_{y,\delta} x_p(k) + D_{yu,\delta} u(k), \end{cases} \tag{1}$$

where the vectors $x_p(k) \in \Re^n$, $u(k) \in \Re^u$, and $y(k) \in \Re^q$ denote the state, the control input, the measured output, respectively. δ is a piecewise constant function of time, called a switching signal, which takes its values in the finite set $\mathrm{I}[1,\ N_p]$, $N_p > 1$ is the number of subsystems.

It is assumed that (A1) The triple $(A_{p,i},\ B_{u,i},\ C_{y,i})$ is stabilizable and detectable for all $i \in \mathrm{I}[1,\ N_p]$; (A2) $D_{yu,i} = 0$ for all $i \in \mathrm{I}[1,\ N_p]$. Note that the first assumption guarantees the existence of a stabilizing dynamic output-feedback controller for each subsystem, and the second one can be relaxed by loop transformation. We also assumed that δ is continuous from the right everywhere and obeys an ADT switching logic [15], whose definition is recalled as following.

Definition 1 *For a switching signal δ and any $t_2 > t_1 > t_0$, let $N_\delta(t_1, t_2)$ be the switching numbers of δ over the interval $[t_1, t_2)$. If $N_\delta(t_1, t_2) \leqslant N_0 + (t_2 - t_1)/\tau_a$ holds for $N_0 \geqslant 1$, $\tau_a > 0$, then τ_a and N_0 are called the average dwell time and the chatter bound, respectively.*

Problem 1 *Given a discrete-time switched linear system (1), under what conditions there exist state feedback and dynamic output-feedback with a supervisor enforcing a reset rule that make closed-loop switched linear system globally uniformly asymptotically stable (GUAS) and achieve a weighted L_2 gain performance under zero initial condition for every switching signal with the ADT scheme.*

The following lemma is useful to provide a systematic method to find a switching control scheme which guarantees globally uniformly asymptotically stability and a weighted L_2 gain performance for the switched linear system (1).

Lemma 1 (weighted L2 gain performance with level τ) *Consider a switched linear system given in (1). Given three tunable positive scalars $0 < \lambda_0 < 1$, $\lambda > 1$ and $\mu > 1$, if there exist symmetric positive definite matrices P_i such that*

$$\begin{pmatrix} A_i^T P_i A_i - \lambda_0^2 P_i & A_i^T P_i B_{u,i} & C_{y,i}^T \\ B_{u,i}^T P_i A_i & B_{u,i}^T P_i B_{u,i} - \tau I & D_{yu,i}^T \\ C_{y,i} & D_{yu,i} & -\tau I \end{pmatrix} < 0, \tag{2}$$

Using the solution P_i's of (2), we define the following piecewise Lyapunov function candidate $V(k) = V_{\delta(k)}(x) = x^T(k) P_{\delta(k)} x(k)$, where $P_{\delta(k)}$ is switched among the solution P_i's of (2) in accordance with the piecewise constant switching signal δ. Moreover, if the following inequalities are satisfied

$$V_j(x_p^+) \leqslant \mu V_i(x_i^+) \tag{3}$$

for all $i \in \mathrm{I}[1, \ N_p]$. Then, the switched linear system is globally uniformly asymptotically stable (GUAS) for every switching signal δ with the following ADT scheme: for any positive integer $j > 0$,

$$N_\delta(0, j) \leqslant \frac{j}{\tau_a^*}, \quad \tau_a^* = \frac{\ln(\mu)}{2\ln(\lambda)} \tag{4}$$

and achieves a weighted L_2 gain under zero initial condition, i.e.,

$$\sum_{j=0}^{+\infty} \lambda^{-2j} z^T[j]z[j] \leqslant \tau^2 \sum_{j=0}^{+\infty} w^T[j]w[j]. \tag{5}$$

Proof. The results are easily derived from Theorem 5 in [10].

Remark 1 *Using Schur complementary Lemma, according to [20, 29], LMIs (2) are equivalent with the following conditions:*

i) there exists symmetric positive matrices P_i *such that*

$$\begin{pmatrix} -P_i^{-1} & A_i & B_{u,i} & 0 \\ A_i^T & -\lambda_0^2 P_i & 0 & C_{y,i}^T \\ B_{u,i}^T & 0 & -\tau I & D_{yu,i}^T \\ 0 & C_{y,i} & D_{yu,i} & -\tau I \end{pmatrix} < 0; \tag{6}$$

ii) there exists symmetric positive matrices Q_i *such that*

$$\begin{pmatrix} -Q_i & A_i Q_i & B_{u,i} & 0 \\ Q_i A_i^T & -\lambda_0^2 Q_i & 0 & Q_i C_{y,i}^T \\ B_{u,i}^T & 0 & -\tau I & D_{yu,i}^T \\ 0 & C_{y,i} Q_i & D_{yu,i} & -\tau I \end{pmatrix} > 0. \tag{7}$$

3. Main results

In this section, a switching output feedback controller with the ADT scheme is considered to achieve the weighted L_2 gain performance (5) for a generalized switched linear system as

$$\sum_P := \begin{cases} x_p(k+1) = A_{p,\delta} x_p(k) + B_{w,\delta} w(k) + B_{u,\delta} u(k) \\ z(k) = C_{z,\delta} x_p(k) + D_{zw,\delta} w(k) + D_{zu,\delta} u(k) \\ y(k) = C_{y,\delta} x_p(k) + D_{yw,\delta} w(k). \end{cases} \tag{8}$$

Now, we seek to design a controller ($A_{k,\delta} \in \Re^{n_k \times n_k}$) of fixed order n_k as

$$\begin{aligned} \begin{bmatrix} x_c(k+1) \\ u(k) \end{bmatrix} &= \begin{bmatrix} A_{k,\delta} & B_{k,\delta} \\ C_{k,\delta} & D_{k,\delta} \end{bmatrix} \begin{bmatrix} x_c(k) \\ y(k) \end{bmatrix}, \\ x_c^+(k) &= \Delta_{ij} x_c(k), \quad \text{when switching occurs,} \end{aligned} \tag{9}$$

where $x_c(k) \in \Re^{n_k}$ is the controller state. Just as introduced in [17], an important characteristic for (9) is that no plant state information is required for controller state reset.

Connecting the system (8) and controller (9), eliminating the variables u, y, we obtain the closed-loop switching system as

$$\begin{bmatrix} \widetilde{x}(k+1) \\ z(k) \end{bmatrix} = \begin{bmatrix} \bar{A}_\delta & \bar{B}_{w,\delta} \\ \bar{C}_{z,\delta} & \bar{D}_{zw,\delta} \end{bmatrix} \begin{bmatrix} \widetilde{x}(k) \\ w(k) \end{bmatrix}, \tag{10}$$

and $\widetilde{x}^+(k) = T_{ij}\widetilde{x}(k)$, where

$$\bar{A}_\delta = \begin{bmatrix} A_{p,\delta} + B_{k,\delta}D_{k,\delta}C_{y,\delta} & B_{k,\delta}C_{k,\delta} \\ C_{k,\delta}C_{y,\delta} & A_{k,\delta} \end{bmatrix}, \qquad \bar{B}_{w,\delta} = \begin{bmatrix} B_{w,\delta} + B_{k,\delta}D_{k,\delta}D_{yw,\delta} \\ B_{k,\delta}D_{yw,\delta} \end{bmatrix},$$

$$\bar{C}_{z,\delta} = \begin{bmatrix} C_{z,\delta} + D_{zu,\delta}D_{k,\delta}C_{y,\delta} & D_{zu,\delta}C_{k,\delta} \end{bmatrix}, \quad \bar{D}_{zw,\delta} = \begin{bmatrix} D_{zw,\delta} + D_{zu,\delta}D_{k,\delta}D_{yw,\delta} \end{bmatrix},$$

and

$$T_{ij} = \begin{bmatrix} I_n & 0 \\ 0 & \Delta_{ij} \end{bmatrix}. \tag{11}$$

Furthermore, when controller gain matrices containing all the unknown controller parameters are defined as

$$J_\delta = \begin{bmatrix} A_{k,\delta} & B_{k,\delta} \\ C_{k,\delta} & D_{k,\delta} \end{bmatrix}, \tag{12}$$

if we augment the open-loop switched system (8) with states corresponding to the controller (9), the augmented system will be obtained as

$$\begin{aligned} \left[\begin{array}{c} \widetilde{x}(k+1) \\ \hline z(k) \\ \hline \widetilde{y}(k) \end{array}\right] &= \left[\begin{array}{c} x_p(k+1) \\ x_c(k+1) \\ \hline z(k) \\ \hline x_c(k) \\ y(k) \end{array}\right] \\ &= \left[\begin{array}{cc|c|cc} A_{p,\delta} & 0 & B_{w,\delta} & 0 & B_{u,\delta} \\ 0 & 0 & 0 & I_{n_k} & 0 \\ \hline C_{z,\delta} & 0 & D_{zw,\delta} & 0 & D_{zu,\delta} \\ \hline 0 & I_{n_k} & 0 & 0 & 0 \\ C_{y,\delta} & 0 & D_{yw,\delta} & 0 & 0 \end{array}\right] \left[\begin{array}{c} x_p(k) \\ x_c(k) \\ \hline w(k) \\ \hline x_c(k+1) \\ u(k) \end{array}\right] \\ &= \begin{bmatrix} \widetilde{A}_{p,\delta} & \widetilde{B}_{w,\delta} & \widetilde{B}_{u,\delta} \\ \widetilde{C}_{z,\delta} & \widetilde{D}_{zw,\delta} & \widetilde{D}_{zu,\delta} \\ \widetilde{C}_{y,\delta} & \widetilde{D}_{yw,\delta} & 0 \end{bmatrix} \left[\begin{array}{c} \widetilde{x}(k) \\ \hline w(k) \\ \hline \widetilde{u}(k) \end{array}\right]. \end{aligned} \tag{13}$$

We have introduced the abbreviations as

$$\widetilde{x}(k) = \begin{bmatrix} x_p(k) \\ x_c(k) \end{bmatrix}, \qquad \widetilde{y}(k) = \begin{bmatrix} x_c(k) \\ y(k) \end{bmatrix} \qquad \text{and} \qquad \widetilde{u}(k) = \begin{bmatrix} x_c(k+1) \\ u(k) \end{bmatrix},$$

which allow us to write the control law as $\widetilde{u} = J_\delta \widetilde{y}$ with constraint condition $x_k^+ = \Delta_{ij} x_k$.

Thus, the closed-loop system matrix is also an affine function of the controller gain matrix J_δ as

$$\begin{bmatrix} \bar{A}_\delta & \bar{B}_{w,\delta} \\ \bar{C}_{z,\delta} & \bar{D}_{zw,\delta} \end{bmatrix} = \begin{bmatrix} \widetilde{A}_{p,\delta} & \widetilde{B}_{w,\delta} \\ \widetilde{C}_{z,\delta} & \widetilde{D}_{zw,\delta} \end{bmatrix} + \begin{bmatrix} \widetilde{B}_{u,\delta} \\ \widetilde{D}_{zu,\delta} \end{bmatrix} J_\delta \begin{bmatrix} \widetilde{C}_{y,\delta} & \widetilde{D}_{yw,\delta} \end{bmatrix}. \tag{14}$$

3.1. Linearizing change-of-variables based method

According to LMIs (6) and the state space expression of the closed-loop system (10), we have

$$\begin{pmatrix} -P_i & P_i\bar{A}_i & P_i\bar{B}_{w,i} & 0 \\ \bar{A}_i^T P_i & -\lambda_0^2 P_i & 0 & \bar{C}_{z,i}^T \\ \bar{B}_{w,i}^T P_i & 0 & -\tau I & \bar{D}_{wz,i}^T \\ 0 & \bar{C}_{z,i} & \bar{D}_{wz,i} & -\tau I \end{pmatrix} < 0, \tag{15}$$

When the Lyapunov-like function $V_i(\widetilde{x})$ is chosen as $V_i = \widetilde{x}^T P_i \widetilde{x}$, the inequality (3) becomes

$$\mu P_i - T_{ij}^T P_j T_{ij} \geqslant 0 \tag{16}$$

for all $i \in \mathrm{I}[1,\ N_p]$.

The above inequalities (15) will be transformed into linear matrix inequality for dynamic output-feedback synthesis problem with linearizing change-of-variables based method.

Let us partition the matrices P_i and P_i^{-1} as

$$P_i := \begin{bmatrix} Y_i & N_i \\ N_i^T & ? \end{bmatrix} \qquad \text{and} \qquad P_i^{-1} := \begin{bmatrix} X_i & M_i \\ M_i^T & ? \end{bmatrix}, \tag{17}$$

where matrices $X_i \in \mathfrak{R}^{n\times n}$ and $Y_i \in \mathfrak{R}^{n\times n}$, '?' denotes block in these matrices with no importance for the derivations to be presented in the sequel. From $P_i P_i^{-1} = I$ we infer $P_i \begin{pmatrix} X_i \\ M_i^T \end{pmatrix} = \begin{pmatrix} I \\ 0 \end{pmatrix}$ which leads to

$$P_i \vartheta = \vartheta_1 \qquad \text{with} \qquad \vartheta := \begin{bmatrix} X_i & I \\ M_i^T & 0 \end{bmatrix}, \qquad \vartheta_1 := \begin{bmatrix} I & Y_i \\ 0 & N_i^T \end{bmatrix}, \tag{18}$$

which has already been used in [4, 6]. Then the non-linear transformation is shown as

$$\hat{A}_i := N_i A_{k,i} M_i^T + N_i B_{k,i} C_{y,i} X_i + Y_i B_{u,i} C_{k,i} M_i^T + Y_i (A_i + B_{u,i} D_{k,i} C_{y,i}) X_i\,,$$

$$\hat{B}_i := N_i B_{k,i} + Y_i B_{u,i} D_{k,i}\,,$$

$$\hat{C}_i := C_{k,i} M_i^T + D_{k,i} C_{y,i} X_i\,,$$

$$\hat{D}_i = D_{k,i}\,.$$

Then, appropriate congruence transformations with matrix ϑ can be applied to (15) so that the resulting constraints only involves the following terms

$$\vartheta^T P_i \bar{A}_i \vartheta = \begin{bmatrix} A_i X_i + B_{u,i}\hat{C}_i & A_i + B_{u,i}\hat{D}_i C_{y,i} \\ \hat{A}_i & Y_i A_i + \hat{B}_i C_{y,i} \end{bmatrix},$$

$$\vartheta^T P_i \bar{B}_i = \begin{bmatrix} B_{w,i} + B_{u,i}\hat{D}_i D_{yw,i} \\ Y_i B_{w,i} + \hat{B}_i D_{yw,i} \end{bmatrix},$$

$$\bar{C}_i \vartheta = \begin{bmatrix} C_{z,i} X_i + D_{zu,i}\hat{C}_i & C_{z,i} + D_{zu,i}\hat{D}_i C_{y,i} \end{bmatrix},$$

$$\vartheta^T P_i \vartheta = \begin{bmatrix} X_i & I \\ I & Y_i \end{bmatrix}.$$

We see that the above terms are affine with respect to $\hat{A}_i$, $\hat{B}_i$, $\hat{C}_i$, $\hat{D}_i$, X_i and Y_i. Thus, the matrix inequalities (15) turn out to LMIs with respect to the variables $\hat{A}_i$, $\hat{B}_i$, $\hat{C}_i$, $\hat{D}_i$, X_i, Y_i and τ.

Theorem 1 *Given a discrete-time generalized switched linear system (1) and three tunable scalars $0 < \lambda_0 < 1$, $\lambda > 1$ and $\mu > 1$, if there exist $n \times n$ symmetric positive definite matrices X_i, Y_i, and rectangular matrices $\hat{A}_i$, $\hat{B}_i$, $\hat{C}_i$, $\hat{D}_i$, $\hat{\Delta}_{ij} \in \Re^{n\times n}$ for all $i \in \mathrm{I}[1,\ N_p]$ such that*

$$\begin{bmatrix} -X_i & -I & A_iX_i+B_{u,i}\hat{C}_i & A_i+B_{u,i}\hat{D}_iC_{y,i} & B_{w,i}+B_{u,i}\hat{D}_iD_{yw,i} & 0 \\ -I & -Y_i & \hat{A}_i & Y_iA_i+\hat{B}_iC_{y,i} & Y_iB_{w,i}+\hat{B}_iD_{yw,i} & 0 \\ (*)^T & (*)^T & -\lambda_0^2 X_i & -\lambda_0^2 I & 0 & X_iC_{z,i}^T+\hat{C}_i^TD_{zu,i}^T \\ (*)^T & (*)^T & (*)^T & -\lambda_0^2 Y_i & 0 & C_{z,i}^T+C_{y,i}^T\hat{D}_i^TD_{zu,i}^T \\ (*)^T & (*)^T & (*)^T & (*)^T & -\tau I & D_{zw,i}^T \\ (*)^T & (*)^T & (*)^T & (*)^T & (*)^T & -\tau I \end{bmatrix} < 0, \quad (19)$$

$$\begin{bmatrix} \mu X_i & \mu I_n & X_j & I_n \\ \mu I_n & \mu Y_i & \hat{\Delta}_{ij}^T & Y_i \\ X_j & \hat{\Delta}_{ij} & X_i & I_n \\ I_n & Y_i & I_n & Y_j \end{bmatrix} \geqslant 0; \qquad \begin{bmatrix} X_i & * \\ I_n & Y_i \end{bmatrix} > 0, \tag{20}$$

Once the variables $\hat{A}_i$, $\hat{B}_i$, $\hat{C}_i$, $\hat{D}_i$, X_i *and* Y_i *satisfying the constraint conditions above have been found, the output feedback controller matrices can be given as*

$$J_i = \begin{bmatrix} A_{k,i} & B_{k,i} \\ C_{k,i} & D_{k,i} \end{bmatrix} = \begin{bmatrix} N_i & Y_i B_{u,i} \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} \hat{A}_i - Y_i A_i X_i & \hat{B}_i \\ \hat{C}_i & \hat{D}_i \end{bmatrix} \begin{bmatrix} M_i^T & 0 \\ C_{y,i} X_i & I \end{bmatrix} \tag{21}$$

for all $i \in \mathrm{I}[1,\ N_p]$.

Proof. Just as introduced in Lemma 1, according to the derivation above with the congruent transformation and linearizing change-of-variables based method [29], the conditions (2) can be reformulated easily into conditions (19).

For the condition (3), when switching occurs, we have $V_j(\widetilde{x}^+) = \widetilde{x}^{+T} P_j \widetilde{x} = \widetilde{x}^T T_{ij}^T P_j T_{ij} \widetilde{x}$. Then, the boundary condition (3) can be converted to a matrix inequality, which can be further written by Schur complement Lemma as

$$\begin{bmatrix} \mu P_i & T_{ij}^T P_j \\ P_j T_{ij} & P_j \end{bmatrix} \geqslant 0.$$

Multiplying matrix $diag\{Z_i,\ Z_j\}$ to the right and its transpose from the left on both sides of above inequality, since $Z_j P_j T_{ij} Z_i^T = \begin{bmatrix} X_j & \hat{\Delta}_{ij} \\ I_n & Y_i \end{bmatrix}$, where $\hat{\Delta}_{ij} = X_j Y_i + X_{2,j} \Delta_{ij} Y_{2,i}^T$, the condition (19) can be deduced directly. □

3.2. Controller variable elimination method

By inserting (12) into (7), we can obtain the following matrix inequality formulation for the weighted L_2 gain performance problem. Inequality (7) is equivalent to the following formulation

$$\begin{pmatrix} -P_i^{-1} & \widetilde{A}_i & \widetilde{B}_{w,i} & 0 \\ \widetilde{A}_i^T & -\lambda_0^2 P_i & 0 & \widetilde{C}_{z,i}^T \\ \widetilde{B}_{w,i}^T & 0 & -\tau I & \widetilde{D}_{zw,i}^T \\ 0 & \widetilde{C}_{z,i} & \widetilde{D}_{zw,i} & -\tau I \end{pmatrix} + \begin{bmatrix} \widetilde{B}_{u,i} \\ 0 \\ 0 \\ \widetilde{D}_{zu,i} \end{bmatrix} J_i \begin{bmatrix} 0 & \widetilde{C}_{y,i} & \widetilde{D}_{yw,i} & 0 \end{bmatrix}$$
$$+ \left(\begin{bmatrix} \widetilde{B}_{u,i} \\ 0 \\ 0 \\ \widetilde{D}_{zu,i} \end{bmatrix} J_i \begin{bmatrix} 0 & \widetilde{C}_{y,i} & \widetilde{D}_{yw,i} & 0 \end{bmatrix} \right)^T < 0. \tag{22}$$

The following theorem gives us existence conditions of the output feedback controller with order n_k that make closed-loop system achieve the weighted L_2 gain performance by using MLFs under zero initial condition for every switching signal δ with the ADT scheme.

Theorem 2 *Given a discrete-time generalized switched linear system (8) and three tunable scalars $0 < \lambda_0 < 1$, $\lambda > 1$ and $\mu > 1$, if there exist $n \times n$ symmetric positive definite matrices X_i, Y_i, and matrices $\hat{\Delta}_{ij} \in \Re^{n\times n}$ such that*

$$\left(\begin{array}{c|c} N_{R,i} & 0 \\ \hline 0 & I \end{array}\right)^T \left(\begin{array}{cc|c} \lambda_0^{-2} A_i X_i A_i^T - X_i & \lambda_0^{-2} A_i X_i C_{z,i}^T & B_{w,i} \\ \lambda_0^{-2} C_{z,i} X_i A_i^T & -\tau I + \lambda_0^{-2} C_{z,i} X_i C_{z,i}^T & D_{zw,i} \\ \hline B_{w,i}^T & D_{zw,i}^T & -\tau I \end{array}\right) \left(\begin{array}{c|c} N_{R,i} & 0 \\ \hline 0 & I \end{array}\right) < 0, \tag{23}$$

$$\left(\begin{array}{c|c} N_{s,i} & 0 \\ \hline 0 & I \end{array}\right)^T \left(\begin{array}{cc|c} \lambda_0^{-2} A_i^T Y_i A_i - Y_i & \lambda_0^{-2} A_i^T Y_i B_{w,i} & C_{z,i}^T \\ \lambda_0^{-2} B_{w,i}^T Y_i A_i & -\tau I + \lambda_0^{-2} B_{w,i}^T Y_i B_{w,i} & D_{zw,i}^T \\ \hline C_{z,i} & D_{zw,i} & -\tau I \end{array}\right) \left(\begin{array}{c|c} N_{s,i} & 0 \\ \hline 0 & I \end{array}\right) < 0, \tag{24}$$

$$\begin{pmatrix} X_i & I \\ I & Y_i \end{pmatrix} \geqslant 0, \tag{25}$$

and

$$rank \begin{pmatrix} X_i & I \\ I & Y_i \end{pmatrix} \leqslant n + n_k \,. \tag{26}$$

$$\begin{pmatrix} \mu X_i & \mu I_n & X_j & I_n \\ \mu I_n & \mu Y_i & \hat{\Delta}_{ij}^T & Y_i \\ X_j & \hat{\Delta}_{ij} & X_i & I_n \\ I_n & Y_i & I_n & Y_j \end{pmatrix} \geqslant 0, \tag{27}$$

where $N_{R,i}$ and $N_{s,i}$ denote bases of the null spaces of $(B_{u,i}^T,\ D_{uz,i}^T)$ and $(C_{y,i},\ D_{wy,i})$ respectively. Then there exists a controller (9) with order n_k that make closed-loop switching linear system (10) with the condition (11) globally uniformly asymptotically stable (GUAS) and achieve the weighted L_2 gain performance under zero initial condition for every switching signal δ with the ADT $\tau_a \geqslant \tau_a^*$.

Proof. Consider the Lyapunov-like functions for the closed-loop system (10) with (11) as

$$V_i = \tilde{x}^T P_i \tilde{x} = \begin{bmatrix} x_p \\ x_c \end{bmatrix}^T \begin{bmatrix} X_i & X_{2,i} \\ X_{2,i}^T & X_{3,i} \end{bmatrix} \begin{bmatrix} x_p \\ x_c \end{bmatrix}.$$

Note that each V_i has one-to-one correspondence with the subsystems in (10).

We define

$$\begin{bmatrix} X_i & X_{2,i} \\ X_{2,i}^T & X_{3,i} \end{bmatrix}^{-1} = \begin{bmatrix} Y_i & Y_{2,i} \\ Y_{2,i}^T & Y_{3,i} \end{bmatrix} \qquad \text{and} \qquad Z_i = \begin{bmatrix} I & 0 \\ Y_i & Y_{2,i} \end{bmatrix}.$$

It is routine to verify that

$$0 \leqslant Z_i \cdot \begin{bmatrix} X_i & X_{2,i} \\ X_{2,i}^T & X_{3,i} \end{bmatrix} \cdot Z_i^T = \begin{bmatrix} X_i & I \\ I & Y_i \end{bmatrix}.$$

Also the Schur complement relationship

$$\begin{bmatrix} X_i & I \\ I & Y_i \end{bmatrix} = \begin{bmatrix} I & Y_i^{-1} \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} X_i - Y_i^{-1} & 0 \\ 0 & Y_i \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ Y_i^{-1} & I \end{bmatrix}$$

implies that

$$rank \begin{pmatrix} X_i & I \\ I & Y_i \end{pmatrix} = n + rank \left(X_i - Y_i^{-1} \right) = n + rank \left(X_i Y_i - I \right) \leqslant n + n_k \,,$$

where the last inequality follows from $I - X_i Y_i = X_{2,i} Y_{2,i}^T$ and $X_{2,i} \in \Re^{n \times n_k}$. $\begin{pmatrix} X_i & I \\ I & Y_i \end{pmatrix} \geqslant 0$ and $rank \begin{pmatrix} X_i & I \\ I & Y_i \end{pmatrix} \leqslant n + n_k$, in turn, $P_i > 0$ for all $i \in \mathrm{I}[1, \; N_p]$. □

Here, we also need to prove these conditions (2) and (3) of Lemma 1 for the system (10) with (11).

Just as introduced in Theorem 1, according to Schur complementary and Finsler's lemmas [29], the existence of $n \times n$ symmetric positive definite matrices X_i, Y_i to conditions (23)–(26) is equivalent with the existence of symmetric positive definite matrices P_i to (22). That is, if there exist X_i, Y_i satisfying (23)–(26), there exist P_i satisfying (22). Furthermore, these matrices X_i, Y_i could be used to construct P_i satisfying (22), and then substituting these P_i into (22), then local controllers J_i for the subsystems in (10) can be obtained by using standard LMI optimal techniques. Consequently, the condition (2) of Lemma 1 could be satisfied.

For the condition (3), the derivation is the same as the introduction in Theorem 1. Then, we arrive at conditions (4).

Consequently, the solvability conditions of a switching controller (9) with order n_k could be obtained with (23)–(27).

Remark 2 *Theorem 2 gives us solvability conditions on the existence of a switching output feedback controller (9) with order n_k under which the weighted disturbance attenuation level can be attained with the ADT scheme. When the symmetric* *positive definite matrices X_i, Y_i satisfying (23)–(27) exist, just as introduced in [29], we can construct a positive matrix $P_i \in \Re^{(n+n_k)\times(n+n_k)}$ by finding a matrix $X_{2,i} \in \Re^{n\times n_k}$ such that $X_i - Y_i^{-1} = X_{2,i}X_{2,i}^T$. Then $P_i = \begin{bmatrix} X_i & X_{2,i} \\ X_{2,i}^T & I \end{bmatrix}$ is a proper Lyapunov matrix for the switching controller design problem. The order $n_{k,i}$ of corresponding controllers need be no larger than n, and in general can be chosen to be the rank of $X_i - Y_i^{-1}$. Substituting this P_i into (22), then controllers J_i for every local plant of the switched plant can be obtained directly by using standard LMI techniques.*

Theorem 2 states that a matrix P_i in $\Re^{(n+n_k)\times(n+n_k)}$ can be constructed from X_i, Y_i exactly when the LMIs (23)–(25) and (27) and rank conditions in (26) are satisfied. Rank conditions are not in general LMI, but notice that

$$rank\begin{pmatrix} X_i & I \\ I & Y_i \end{pmatrix} \leqslant 2n. \tag{28}$$

Therefore, if the order of controller $n_k \geqslant n$ is chosen, rank conditions become vacuous and only LMI conditions are left. That is, when a full order controller is considered, note that with any given pair of the dwell-time parameters $(\lambda,\ \mu)$, these conditions besides rank constraint (26) are LMIs. Thus, the determination of the minimal weighted L_2 gain τ, with respect to a given $(\lambda,\ \mu)$, can be formulated and solved as an LMI optimization problem:

$$\begin{array}{l} \min\ \tau \\ S.T.\ \ (23)\text{–}(25)\ \text{ and }\ (27) \end{array}. \tag{29}$$

3.3. Comparison between linearizing change-of-variables based method and controller variable elimination method

As to LTI plant, two kind of LMIs based conditions for the existence of H infinity controller synthesis are equivalent. However, for switched plant or LPV plant, numerical computation complexity of two methods is not consistent. Assume that the number of local subsystems is N, the number of LMI matrix variables is $8N$ for the first method, that is solved by a one-step procedure. By comparison, the second method has a two-step procedure, the number of LMI matrix variables is $3N$ in the first step; and the second step merely includes N LMI matrix variables. Even though the solution procedure for the second method has two steps, the number of LMI matrix variables in each step is much less than the first method. Consequently, the numerical efficiency in each step is better than the first method.

From the derivation above, when the number of subsystems is N_p, besides the matrices $\hat{\Delta}_{ij} \in \Re^{n\times n}$, linearizing change-of-variables based method has $6N_p$

LMI variables, which is much larger than the method presented here merely including $2N_p$.

In general, if the number of subsystems is quite large, with the increasing of the number of the LMI variables, the solutions of output feedback controller matrices for LMI conditions (21) may not be feasible. In comparison, since the method proposed here decouples the relation between the controller variable matrices and multiple Lyapunov matrices, the number of matrix variables is significantly reduced, which helps numerical efficiency in the design procedure.

4. Example

To show the effectiveness of the proposed method, a simple numerical example is illustrated with two vertices in [30] as

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -2+\gamma & -0.01 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x. \end{aligned} \tag{30}$$

Denote by A_1 and A_2 the values of the state matrix for $\gamma = -1$ and 1, respectively.

Assume two subsystems of the continuous-time switched plant above are discretized with sampling time 0.1, respectively, as

$$A_1 = \begin{bmatrix} 0.985 & 9.95e-2 \\ -0.298 & 0.984 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 4.98e-3 \\ 9.94e-2 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

and

$$A_2 = \begin{bmatrix} 0.995 & 9.98e-2 \\ -9.98e-2 & 0.994 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 4.99e-3 \\ 9.97e-2 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

As shown in Fig. 1, external disturbance rejection problem is considered here, where we choose the weighting function as $W_1(s) = \dfrac{50}{s+1}$, which is discreteized with sampling time 0.1 as $W_1(z) = \dfrac{4.758}{z-0.905}$. This weight has bandwidth 1 rad/s, so it might be used to get good tracking. The weighting L_2 performance optimal constrained problem is considered here as (5).

Suppose that the switched plant (10) is required to operate over the time interval $t \in [0, 7]$sec with the following switching signal

$$\delta(k) = \begin{cases} 1, & 20l \leqslant k \leqslant (2l+1)\cdot 10, \quad \text{and } \; k \in [3,\ 50] \\ 2, & \text{otherwise} \end{cases},$$

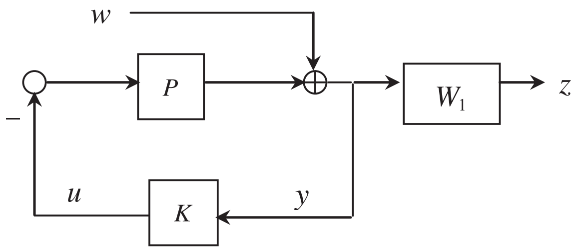


Figure 1: Block diagram for design

where $l = 0, 1, 2, 3$. The ADT of this switching signal can be calculated by $\tau_a^* = 7/6 = 1.17$, where 6 is the number of switches occurred during the time interval $[0, 7]$ sec. As such, we specify the dwell-time parameters $(\lambda, \mu) = (1.062, 4)$ for the controller synthesis, such that the ADT τ_a^* satisfies the requirement. Meanwhile the positive number λ_0 is chosen as $\lambda_0 = 0.998$.

Solving the optimization problem (19), (20) with LMI toolbox [29], a state feedback controller is designed in the form of (8) with the optimal value $\tau = 20.01$. And solving the optimization problem (23)–(27), we obtain a full-order output feedback controller in the form of (9) with the optimal value $\tau = 25.49$. As expected, the resulting value of τ appears to be 5.48 larger than that derived by using the state feedback control strategy. When the switching signal with the average dwell time τ_a^* is given as Fig. 2, time-domain simulation results of the state feedback and full-order output feedback control are obtained as Fig. 3 using the external disturbance signal: $w(k) = -1$, $k \geqslant 5$.

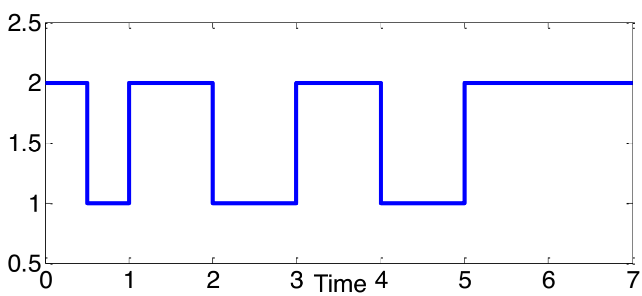
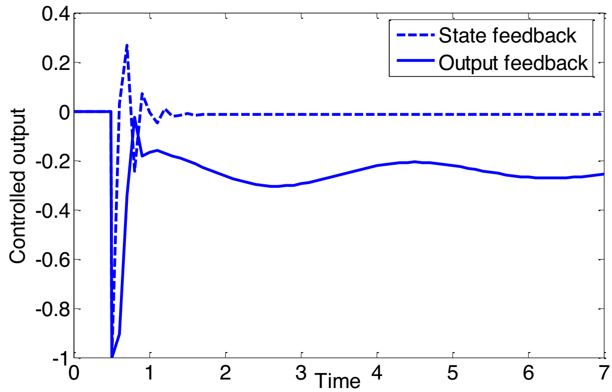
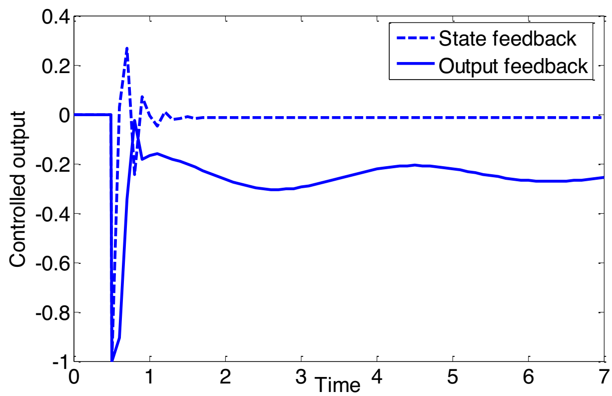


Figure 2: The switching signal

As seen in Fig. 3, from (a) the controlled output response of the state feedback has a quicker transient process and better static properties than that of the output feedback. Meanwhile, from (b) the control force of the state feedback is much larger than that of the output feedback.



(a) Controlled output



(b) Controller output

Figure 3: Simulation results using state feedback and output feedback

5. Conclusions

This paper has presented solvability conditions of switching control schemes for state-feedback control and output-feedback control of discrete-time switched linear systems with ADT scheme respectively. Based on these conditions, the design procedure is separated into two sequential steps. First, multiple Lyapunov functions for each closed-loop system have been obtained to guarantee globally uniformly asymptotically stability as well as the weighted L_2-gain performance. Second, by these multiple Lyapunov functions, each controller for each plant subsystem could be solved directly by LMI optimal technology. Furthermore, since these conditions decouple the relation between the controller matrices and multiple Lyapunov matrices, the number of matrix variables of LMI based solvability conditions is significantly reduced, which helps numerical efficiency.

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