

Research Paper

Scattering of Sound Waves in Two Stepped Non-uniformly Lined Duct

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Scattering of sound waves in two stepped cylindrical duct which are coated with different acoustically absorbent materials is investigated by using Wiener-Hopf technique directly and by determining scattering matrices. First, by using Fourier transform technique we obtain a couple of modified Wiener-Hopf equations whose solutions involve four sets of infinitely many unknown expansion coefficients providing systems of linear algebraic equations. Then we determine scattering matrices of the problem and we state the total transmitted field by using generalized scattering matrix method. Numerical results are compared for different parameters.

Keywords: Wiener-Hopf; scattering matrices; step discontinuity; impedance discontinuity; duct.

1. Introduction

The propagation of sound through the waveguide has a significant place for researchers for many years. According to MORSE (1948) and PIERCE (1981) a duct for sound propagation also behaves like a transmission line (e.g. air conditioning duct, car muffler, etc.). But there is a fact that some unwanted voices are produced during propagation of sound. Therefore physical and geometrical properties of the ducts have an important role in studies.

Looking at the first studies done, it is seen that studies about infinite rigid cylindrical and rectangular waveguides (RAYLEIGH, 1945) and semi-infinite rigid ducts (LEVINE, SCHWINGER, 1948). In their study Levine and Schwinger found reflection coefficients analytically by Wiener-Hopf technique. Then their analysis has been reexamined by a few researchers (MORSE, FESCHBACH, 1953; NOBLE, 1969). All these studies are continuous in order to the geometrical and the physical properties. In addition to these studies, involving discontinuities in waveguide have been made and this is still a popular topic among the researchers. We can say first studies about the discontinuity problem in cylin-

dric ducts was solved by MILES (1944; 1945a; 1945b) for acoustic waves and by PAPADOPULOS (1957) for electromagnetic waves. In his study, Miles analysed the effect of plane discontinuity on a plane wave propagated in a cylindrical duct. KARAL (1953) examined the impedance introduced by an abrupt change of circular cross section of a tube. He formed an acoustical transmission system by joining together two infinite circular ducts of different cross section and investigated the acoustic inductance for this sudden discontinuities. ALFREDSON (1972) used many small discontinuities to define duct shape and developed an approximate technique for calculating the behaviour of sound.

PACE and MITTRA (1964; 1971) studied on electromagnetic waveguide discontinuity problems. They developed generalised scattering matrix technique to deal with step discontinuity problems. Nilsson and Brander applied this method to acoustic waveguides. They investigated the propagation of sound from cylindrical ducts in different geometries. In the first study they analysed the modes of an infinite duct, paying particular attention to possible instabilities through the Wiener-Hopf technique (NILSSON, BRANDER, 1980a). The second and third studies are devoted to the

analysis of the reflection and transmission properties of a single discontinuity. There are two different kind of discontinuities, the first solvable by standard Wiener-Hopf techniques (NILSSON, BRANDER, 1980b) and the second requiring a generalization of these techniques (NILSSON, BRANDER, 1980c). They found scattering matrices for sudden area changes in a cylindrical waveguide with mean flow and Bulk-reacting lining. Then they used these matrices in a similar waveguide with several interacting discontinuities (NILSSON, BRANDER, 1980d). VANLARICUM and MITTRA (1969) used a basic Wiener-Hopf geometry to present a modified residue-calculus technique for solving a class of boundary value problems which include waveguide discontinuities for electromagnetic waves.

KERGOMARD and GARCIA (1987) studied on planar discontinuities. They used the mode matching technique and examined simple discontinuities in acoustic waveguides. In their study the convergence criteria and the number of modes to be considered for various of the parameters were studied in more detail provided that the work was limited to low frequencies. KERGOMARD (1991) presented an alternative to the scattering matrix approach and calculated of discontinuities in waveguides by using mode matching method. HOMENTCOVSCHI and MILES (2010) investigated planar discontinuities in rectangular waveguides and gave a new method named a re-expansion method. In their work, they analysed the wave scattering by thin plates and steps in rectangular acoustic waveguides having planar discontinuities.

CAMPOS (1984) investigated the problem of wave propagation from the waveguide of varying cross section by using numerous methods. HUDDE and LETENS (1985) considered circular lossless duct with an abrupt change of cross section. Their analysis includes a non rigid wall at the interface of both parts of the duct. They gave a matrix solution which represents purely the discontinuity untied from the influence of sound source and reflecting terminations in the duct. Sound wave propagation in varying cross section waveguides by multimodal decomposition was studied by PAGNEUX *et al.* (1996). GUPTA *et al.* (1995) analysed plane wave propagation in non-uniform ducts with mean flow by using a modified segmentation approach which divides the non-uniform duct into a series of finite number of short ducts. UTSUMI (1999) investigated transmission of sound through non-uniform circular ducts. His purpose on this study is to give a more efficient analytical method, which requires neither segmentation nor iterative calculation. WANG and SUN (2011) aimed in their study to develop a new segmentation approach in order to include the effect of lined ducts on the sound propagation and attenuation.

WARREN *et al.* (2002) investigated acoustic scattering in waveguides that are discontinuities in geometry and material property by using mod-matching method.

They considered the discontinuity between two ducts of different heights and at least one of the ducts is bounded by a membran. Similarly, the hybrid mod-matching technique has been used to analyse the propagation and scattering of acoustic waves in a flexible waveguide involving step discontinuity at an interface by AFZAL *et al.* (2014).

Researchers have been interested in ducts coated with acoustically absorbent lining because of its feature to reduce unwanted voices (RIENSTRA, 2007; SNAKOWSKA *et al.*, 2017; TIRYAKIOGLU, DEMIR, 2019; PEAKE, ABRAHAMS, 2020). In recent years, thinking of models that allow better absorption of sound has allowed them to work on non-uniform linings which vary circumferentially, axially, or both directions (MCAPLINE *et al.*, 2006; CAMPOS, OLIVEIRA, 2013). DEMIR (2017) used in his study non-uniform linings and defined the scattering matrices with help of the Wiener-Hopf technique. In his paper, he divided geometry of problem into two parts: expansion and contraction problem. At the end of the study, he used these scattering matrices to obtain the transmitted field in a lined waveguide with an inserted expansion chamber whose walls are treated by another acoustically absorbent material. In another work, DEMIR (2016) investigated the transmission of sound in a duct with sudden area expansion and extended inlet. The walls of the duct lying in overlap region lined with different acoustically absorbent materials. He solved this problem by using Wiener-Hopf technique.

The geometry of this problem is dealt with a two-step waveguide and each compartment of this waveguide is thought to be covered with a different sound absorber material. The presence of two step discontinuity and nonuniform admittance distribution makes the problem interesting when considering transmission of waves through waveguides. Due to the existing two step discontinuity, problem reduced to two coupled Wiener-Hopf equations differently from single step discontinuity. On the other hand, nonuniformities in the admittance result in different eigenvalues and eigenfunction expansions in the related regions of waveguide. These are naturally included in the Wiener-Hopf equations and affect the obtained solution. One of our aims in this paper is to find out the influence of nonuniform admittance distribution on sound transmission in waveguides with multiple step discontinuity. In this context, results of the analytical method applied are obtained numerically by taking different radii and different impedance values. Problem is investigated with Wiener-Hopf technique directly and with determination of scattering matrices. For direct solution, considering the mathematical model, the problem is divided into regions and the necessary boundary and continuity conditions are defined. Then, by using the Fourier transform for the scattered field and applying the boundary conditions in the transform domain,

the problem is reduced to a pair of Wiener-Hopf equations whose solutions consist of four sets of infinitely many unknown expansion coefficients providing four systems of linear algebraic equations. In the other solution, problem is discussed with two different geometries and scattering matrices are determined with the help of Wiener-Hopf equations. The total transmitted field is obtained by using generalised scattering matrix method. In this paper, scattering matrices technique is used to verify the results obtained by direct solution.

The time dependence is assumed to be $\exp(-i\omega t)$ with ω being the angular frequency and suppressed throughout this paper.

2. Wiener-Hopf technique

2.1. Problem formulation

The geometry of this problem consists of an infinite cylindrical duct with two area expansions at $z = 0$ and $z = l$. Duct walls are assumed to be infinitely thin and they are defined by $\{\rho = a, z \in (-\infty, 0)\} \cup \{\rho \in (a, b), z = 0\} \cup \{\rho = b, z \in (0, l)\} \cup \{\rho \in (b, c), z = l\} \cup \{\rho = c, z \in (l, \infty)\}$, where (ρ, θ, z) denote the cylindrical polar coordinates. Also it is assumed that inner surface of duct is treated nonuniformly by acoustically absorbent linings which are denoted by η_1, η_2 , and η_3 (see Fig. 1). From the symmetry of the problem geometry and of the incident field, the total field everywhere is independent of θ . The incident sound wave propagating along the positive direction is defined by

$$u^i(\rho, z) = A_n J_0(\gamma_n \rho / a) e^{i\lambda_n z}, \quad (1)$$

$$ik a \eta_1 J_0(\gamma_n) + \gamma_n J_1(\gamma_n) = 0, \quad (2)$$

$$\lambda_n = \sqrt{k^2 - (\gamma_n / a)^2}. \quad (3)$$

Here $k = \omega / c$ denotes the wave number of the space and c is the speed of the sound. A_n is the amplitude of the incident wave. λ_n and γ_n are the modes of $z < 0$. For the sake of analytical convenience, we will assume that the surrounding medium is slightly lossy and k has a small positive imaginary part. The lossless case can be obtained by letting $\text{Im}k \rightarrow 0$ at the end of the

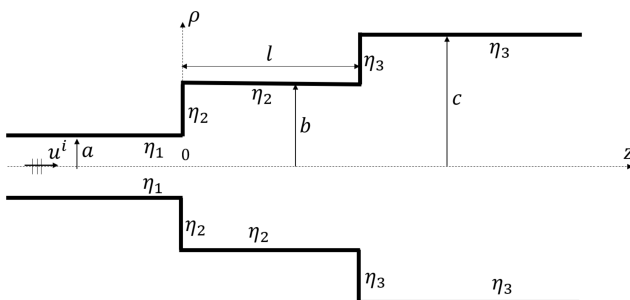


Fig. 1. Geometry of the problem.

analysis. The total field can be written in different regions as:

$$u^T(\rho, z) = \begin{cases} u_1(\rho, z) + u^i(\rho, z); & \rho < a, \quad -\infty < z < \infty, \\ u_2(\rho, z); & \rho \in (a, b), \quad z > 0, \\ u_3(\rho, z); & \rho \in (b, c), \quad z > l, \end{cases} \quad (4)$$

where u^i is the incident field as given by Eq. (1) and the fields $u_j(\rho, z)$, $j = 1, 2, 3$, which satisfy the Helmholtz equation,

$$\left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{\partial^2}{\partial z^2} + k^2 \right] u_j(\rho, z) = 0. \quad (5)$$

From the geometry of the problem, one can write the following boundary conditions

$$\left(ik\eta_1 - \frac{\partial}{\partial \rho} \right) u_1(a, z) = 0, \quad z < 0, \quad (6)$$

$$\left(ik\eta_2 - \frac{\partial}{\partial \rho} \right) u_2(b, z) = 0, \quad 0 < z < l, \quad (7)$$

$$\left(ik\eta_2 + \frac{\partial}{\partial z} \right) u_2(\rho, 0) = 0, \quad a < \rho < b, \quad (8)$$

$$\left(ik\eta_3 - \frac{\partial}{\partial \rho} \right) u_3(c, z) = 0, \quad l < z < \infty, \quad (9)$$

$$\left(ik\eta_3 + \frac{\partial}{\partial z} \right) u_3(\rho, l) = 0, \quad b < \rho < c, \quad (10)$$

and continuity conditions:

$$u_1(a, z) + u^i(a, z) = u_2(a, z), \quad \rho = a, \quad z > 0, \quad (11)$$

$$\frac{\partial}{\partial \rho} [u_1(a, z) + u^i(a, z)] = \frac{\partial}{\partial \rho} u_2(a, z), \quad \rho = a, \quad z > 0, \quad (12)$$

$$u_2(b, z) = u_3(b, z), \quad \rho = b, \quad l < z < \infty, \quad (13)$$

$$\frac{\partial}{\partial \rho} u_2(b, z) = \frac{\partial}{\partial \rho} u_3(b, z), \quad \rho = b, \quad l < z < \infty. \quad (14)$$

In addition to these conditions, the radiation and edge conditions are as follows to ensure the uniqueness of the solution:

$$u_1(\rho, z) = R(\rho) e^{-i\lambda_1 z} + O(e^{-i\lambda_2 z}) \quad \text{for } z \rightarrow -\infty, \quad \rho < a, \quad (15)$$

$$u(\rho, z) = T(\rho) e^{i\tau_1 z} + O(e^{i\tau_2 z}) \quad \text{for } z \rightarrow \infty, \quad \rho < c, \quad (16)$$

and

$$u(a, z) = O\left(z^{\frac{2}{3}}\right), \quad z \rightarrow 0^+, \quad (17)$$

$$u(b, z) = O\left((z-l)^{\frac{2}{3}}\right), \quad z \rightarrow l^+, \quad (18)$$

$$\frac{\partial}{\partial \rho} u(a, z) = O\left(z^{-\frac{1}{3}}\right), \quad z \rightarrow 0^+, \quad (19)$$

$$\frac{\partial}{\partial \rho} u(b, z) = O\left((z-l)^{-\frac{1}{3}}\right), \quad z \rightarrow l^+, \quad (20)$$

where τ_n ($n = 1, 2, \dots$) are the modes of region III ($z \in (l, \infty)$).

2.2. Derivations of the modified Wiener-Hopf equations

Fourier transform of the Helmholtz equation in the region $\rho < a$ for $z \in (-\infty, \infty)$ is

$$\left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + (k^2 - \alpha^2) \right] F(\rho, \alpha) = 0. \quad (21)$$

Here $F(\rho, \alpha)$ is the Fourier transform of the field $u_1(\rho, z)$ defined as:

$$F(\rho, \alpha) = \int_{-\infty}^{\infty} u_1(\rho, z) e^{i\alpha z} dz = F_-(\rho, \alpha) + F_+(\rho, \alpha). \quad (22)$$

$F_-(\rho, \alpha)$ and $F_+(\rho, \alpha)$ are analytical functions in the lower and upper half plane of complex α plane (DEMIR, 2017), respectively. The solution of Eq. (21) due to the analytical properties of $F_{\pm}(\rho, \alpha)$ is as follows

$$F_-(\rho, \alpha) + F_+(\rho, \alpha) = \frac{J_0(K\rho)}{J(a, \alpha)} \Phi_1^+(a, \alpha). \quad (23)$$

Here $K(\alpha) = \sqrt{k^2 - \alpha^2}$ is the square root function defined in complex α plane and

$$\Phi_1^+(a, \alpha) = [ik\eta_1 F_+(a, \alpha) - \dot{F}_+(a, \alpha)], \quad (24)$$

where the dot stands for the derivation with respect to ρ and

$$J(a, \alpha) = ik\eta_1 J_0(Ka) + KJ_1(Ka), \quad (25)$$

J_0 and J_1 are the Bessel functions of integer order.

In the second ($\rho \in (a, b)$, $z \in (0, l)$) and third ($\rho \in (b, c)$, $z \in (l, \infty)$) regions, when the procedure in (DEMIR, 2017) is followed and taken into account with the first region, couple of modified Wiener-Hopf equations (MWHE) are obtained as follows:

$$\begin{aligned} & -\frac{a}{2} F_-(a, \alpha) + V_1(\alpha) \Phi_1^+(a, \alpha) - \frac{e^{i\alpha l}}{\pi L(\alpha)} \Phi_2^+(b, \alpha) \\ & = \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{J(b, \alpha_m)}{J(a, \alpha_m)} \frac{(ik\eta_2 + i\alpha) f_m}{\alpha_m^2 - \alpha^2} \\ & \quad + \frac{a A_n J_0(\gamma_n)}{2i(\lambda_n + \alpha)}, \end{aligned} \quad (26)$$

$$\begin{aligned} & -\frac{a}{2} \frac{J(a, \alpha)}{J(b, \alpha)} F_-(a, \alpha) + V_2(\alpha) e^{i\alpha l} \Phi_2^+(b, \alpha) - \frac{b}{2} G_1(b, \alpha) \\ & = \frac{J(a, \alpha)}{J(b, \alpha)} \sum_{m=1}^{\infty} \frac{J(b, \alpha_m)}{\pi J(a, \alpha_m)} \frac{(ik\eta_2 + i\alpha) f_m}{(\alpha_m^2 - \alpha^2)} \\ & \quad - \sum_{m=1}^{\infty} \frac{(ik\eta_2 + i\alpha) f_m}{\pi(\alpha_m^2 - \alpha^2)} \\ & \quad + e^{i\alpha l} \sum_{m=1}^{\infty} \frac{J(c, \beta_m)}{\pi J(b, \beta_m)} \frac{(ik\eta_3 + i\alpha) h_m}{(\beta_m^2 - \alpha^2)} \\ & \quad + \frac{a}{2i} \frac{J(a, \alpha)}{J(b, \alpha)} \frac{A_n J_0(\gamma_n)}{(\lambda_n + \alpha)}, \end{aligned} \quad (27)$$

where

$$V_1(\alpha) = \frac{J(b, \alpha)}{\pi J(a, \alpha) L(\alpha)} = V_1^+(\alpha) V_1^-(\alpha), \quad (28)$$

$$V_2(\alpha) = \frac{J(c, \alpha)}{\pi J(b, \alpha) L(\alpha)} = V_2^+(\alpha) V_2^-(\alpha), \quad (29)$$

$$\Phi_2^+(b, \alpha) = [ik\eta_2 G_+(b, \alpha) - \dot{G}_+(b, \alpha)], \quad (30)$$

$$L(\alpha) = Y(b, \alpha) J(a, \alpha) - J(b, \alpha) Y(a, \alpha), \quad (31)$$

$$N(\alpha) = Y(c, \alpha) J(b, \alpha) - J(c, \alpha) Y(b, \alpha), \quad (32)$$

$$J(s, \alpha) = ik\eta_2 J_0(Ks) + KJ_1(Ks), \quad (33)$$

$$Y(s, \alpha) = ik\eta_1 Y_0(Ks) + KY_1(Ks), \quad s = a, b, c. \quad (34)$$

G_1 is an entire function and G_+ is a regular function in the upper half of the complex α plane as follows:

$$G_1(\rho, \alpha) = \int_0^l u_2(\rho, z) e^{i\alpha z} dz, \quad (35)$$

$$G_+(\rho, \alpha) = \int_l^{\infty} u_2(\rho, z) e^{i\alpha(z-l)} dz, \quad (36)$$

and α_m 's are the zeros of $L(\alpha)$ and β_m 's are the zeros of $N(\alpha)$. $V_{1,2}^+(\alpha)$ are regular and free of zeros on the region $\text{Im}(-k) < \text{Im} \alpha$ and $V_{1,2}^-(\alpha)$ are regular and free of zeros on the region $\text{Im} \alpha < \text{Im}(k)$. Following the similar method used in (PAGNEUX *et al.*, 1996) split functions can be obtained (DEMIR, 2017).

2.3. Solution of the MWHEs

Performing standard factorization and decomposition procedures and then applying Liouville's theorem we get

$$\begin{aligned} & \frac{a}{2} \frac{F_-(a, \alpha)}{V_1^-(\alpha)} = \sum_{m=1}^{\infty} \frac{e^{i\delta_m l} J(a, \delta_m) V_1^+(\delta_m) \Phi_2^+(b, \delta_m)}{J'(b, \delta_m) (\delta_m - \alpha)} \\ & \quad + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{J(b, \alpha_m) (ik\eta_2 - i\alpha_m) f_m}{V_1^+(\alpha_m) J(a, \alpha_m) 2\alpha_m (\alpha_m + \alpha)} \\ & \quad - \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{J(b, \alpha_m) (ik\eta_2 + i\alpha) f_m}{V_1^-(\alpha) J(a, \alpha_m) (\alpha_m^2 - \alpha^2)} \\ & \quad - \frac{a}{2i} \frac{A_n J_0(\gamma_n)}{V_1^-(\alpha) (\lambda_n + \alpha)} \\ & \quad + \frac{a}{2i} \frac{A_n J_0(\gamma_n)}{V_1^+(\alpha) (\lambda_n + \alpha)}, \end{aligned} \quad (37)$$

$$\begin{aligned}
 V_1^+(\alpha)\Phi_1^+(a, \alpha) &= \frac{e^{i\alpha l} J(a, \alpha) V_1^+(\alpha) \Phi_2^+(b, \alpha)}{J(b, \alpha)} \\
 &+ \sum_{m=1}^{\infty} \frac{e^{i\delta_m l} J(a, \delta_m) V_1^+(\delta_m) \Phi_2^+(b, \delta_m)}{J'(b, \delta_m) (\delta_m - \alpha)} \\
 &+ \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{J(b, \alpha_m) (ik\eta_2 - i\alpha_m) f_m}{V_1^+(\alpha_m) J(a, \alpha_m) 2\alpha_m (\alpha_m + \alpha)} \\
 &+ \frac{a}{2i} \frac{A_n J_0(\gamma_n)}{V_1^+(\lambda_n) (\lambda_n + \alpha)}, \quad (38)
 \end{aligned}$$

and

$$\begin{aligned}
 V_2^+(\alpha)\Phi_2^+(b, \alpha) &= \frac{a}{2} \sum_{m=1}^{\infty} \frac{J(a, \delta_m) F_-(a, -\delta_m) e^{i\delta_m l}}{a^* (\delta_m + \alpha)} \\
 &+ \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \frac{e^{i\delta_p l} J(b, \alpha_m) J(a, \delta_p) (ik\eta_2 - i\delta_p) f_m}{\pi J(a, \alpha_m) b^* (\alpha_m^2 - \delta_p^2) (\delta_p + \alpha)} \\
 &+ \sum_{m=1}^{\infty} \frac{J(c, \beta_m) (ik\eta_3 - i\beta_m) h_m}{\pi J(b, \beta_m) V_2^+(\beta_m) 2\beta_m (\beta_m + \alpha)} \\
 &+ \frac{a}{2i} \sum_{m=1}^{\infty} \frac{J(a, \delta_m) e^{i\delta_m l} A_n J_0(\gamma_n)}{a^* (\lambda_n - \delta_m) (\delta_m + \alpha)}, \quad (39)
 \end{aligned}$$

where

$$\begin{aligned}
 a^* &= J'(b, -\delta_m) V_2^+(\delta_m), \\
 b^* &= J'(b, -\delta_p) V_2^+(\delta_p).
 \end{aligned}$$

2.4. Determination of the expansion coefficient

The expression of $\Phi_1^+(a, \alpha)$ in Eq. (38) involves the unknown coefficients f_m , g_m , $\Phi_2^+(b, \delta_m)$, and $F_-(a, -\delta_m)$. To determine these coefficients we substitute $\alpha = \alpha_1, \alpha_2, \dots, \alpha_N$ in Eq. (38), $\alpha = -\delta_1, -\delta_2, \dots, -\delta_N$ in Eq. (37), $\alpha = \beta_1, \beta_2, \dots, \beta_N$ in Eq. (39), and $\alpha = \delta_1, \delta_2, \dots, \delta_N$ in (39) and use correlation of f_m and g_m coefficients with

$$\Phi_1^+(a, \alpha_m) \quad \text{and} \quad \Phi_2^+(b, \beta_m),$$

then we get following infinite systems of linear algebraic equations:

$$\begin{aligned}
 &\frac{J(a, \alpha_r)}{\pi J(b, \alpha_r)} \vartheta_r^1 (ik\eta_2 + i\alpha_r) V_1^+(\alpha_r) f_r \\
 &= \sum_{m=1}^{\infty} \frac{e^{i\delta_m l} J(a, \delta_m) V_1^+(\delta_m) \Phi_2^+(b, \delta_m)}{J'(b, \delta_m) (\delta_m - \alpha_r)} \\
 &+ \sum_{m=1}^{\infty} \frac{J(b, \alpha_m) (ik\eta_2 - i\alpha_m) f_m}{\pi V_1^+(\alpha_m) J(a, \alpha_m) 2\alpha_m (\alpha_m + \alpha_r)} \\
 &+ \frac{a}{2i} \frac{A_n J_0(\gamma_n)}{V_1^+(\lambda_n) (\lambda_n + \alpha_r)}, \quad n, r = 1, 2, \dots \quad (40)
 \end{aligned}$$

$$\begin{aligned}
 V_2^+(\beta_r) \frac{J(b, \beta_r)}{\pi J(c, \beta_r)} \vartheta_r^2 (ik\eta_3 + i\beta_r) h_r \\
 = \frac{a}{2} \sum_{m=1}^{\infty} \frac{e^{i\delta_m l} J(a, \delta_m) F_-(a, -\delta_m)}{a^* (\delta_m + \beta_r)} \\
 + \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \frac{J(b, \alpha_m) J(a, \delta_p) (ik\eta_2 - i\delta_p) e^{i\delta_p l} f_m}{\pi J(a, \alpha_m) b^* (\alpha_m^2 - \delta_p^2) (\delta_p + \beta_r)} \\
 + \sum_{m=1}^{\infty} \frac{J(c, \beta_m) (ik\eta_3 - i\beta_m) h_m}{\pi J(b, \beta_m) V_2^+(\beta_m) 2\beta_m (\beta_m + \beta_r)} \\
 + \frac{a}{2i} \sum_{m=1}^{\infty} \frac{J(a, \delta_m) e^{i\delta_m l} A_n J_0(\gamma_n)}{a^* (\lambda_n - \delta_m) (\delta_m + \beta_r)}, \quad n, r = 1, 2, \dots, \quad (41)
 \end{aligned}$$

$$\begin{aligned}
 \frac{a}{2} \frac{F_-(a, -\delta_r)}{V_1^+(\delta_r)} &= \sum_{m=1}^{\infty} \frac{e^{i\delta_m l} J(a, \delta_m) V_1^+(\delta_m) \Phi_2^+(b, \delta_m)}{J'(b, \delta_m) (\delta_m + \delta_r)} \\
 &+ \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{J(b, \alpha_m) (ik\eta_2 - i\alpha_m) f_m}{V_1^+(\alpha_m) J(a, \alpha_m) 2\alpha_m (\alpha_m - \delta_r)} \\
 &- \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{J(b, \alpha_m) (ik\eta_2 - i\delta_r) f_m}{V_1^+(\delta_r) J(a, \alpha_m) (\alpha_m^2 - \delta_r^2)} \\
 &- \frac{a}{2i} \frac{A_n J_0(\gamma_n)}{V_1^+(\delta_r) (\lambda_n - \delta_r)} \\
 &+ \frac{a}{2i} \frac{A_n J_0(\gamma_n)}{V_1^+(\lambda_n) (\lambda_n - \delta_r)}, \quad n, r = 1, 2, \dots, \quad (42)
 \end{aligned}$$

$$\begin{aligned}
 V_2^+(\delta_r) \Phi_2^+(b, \delta_r) \\
 = \frac{a}{2} \sum_{m=1}^{\infty} \frac{e^{i\delta_m l} J(a, \delta_m) F_-(a, -\delta_m)}{a^* (\delta_m + \delta_r)} \\
 + \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \frac{J(b, \alpha_m) J(a, \delta_p) (ik\eta_2 - i\delta_p) e^{i\delta_p l} f_m}{\pi J(a, \alpha_m) b^* (\alpha_m^2 - \delta_p^2) (\delta_p + \delta_r)} \\
 + \sum_{m=1}^{\infty} \frac{J(c, \beta_m) (ik\eta_3 - i\beta_m) h_m}{\pi J(b, \beta_m) V_2^+(\beta_m) 2\beta_m (\beta_m + \delta_r)} \\
 + \frac{a}{2i} \sum_{m=1}^{\infty} \frac{J(a, \delta_m) e^{i\delta_m l} A_n J_0(\gamma_n)}{a^* (\lambda_n - \delta_m) (\delta_m + \delta_r)}, \quad n, r = 1, 2, \dots, \quad (43)
 \end{aligned}$$

where as previous

$$a^* = J'(b, -\delta_m) V_2^+(\delta_m), \quad b^* = J'(b, -\delta_p) V_2^+(\delta_p).$$

2.5. Analysis of the field

The transmitted field in the region $\rho < a$, $z > 0$ can be obtained by taking inverse Fourier transform of $F_+(\rho, \alpha)$. From Eqs (22) and (23) we obtain

$$u_1(\rho, z) = \frac{1}{2\pi} \int_{\mathcal{L}} \left[\frac{J_0(K\rho)}{J(a, \alpha)} \Phi_1^+(a, \alpha) - F_-(\rho, \alpha) \right] e^{-i\alpha z} d\alpha, \quad (44)$$

where \mathcal{L} is a straight line parallel to the real α axis, lying in the strip $\text{Im}(-k) < \text{Im}(\alpha) < \text{Im}(k)$. If the integral in Eq. (44) are calculated by using residue theorem then the transmission coefficient for any mode is as:

$$\begin{aligned}
 \mathfrak{T} = & \left\{ i \frac{a\pi}{2} \sum_{m=1}^{\infty} \frac{e^{i\delta_m l} J(a, \delta_m) F_-(a, -\delta_m)}{a^*(\delta_m - \tau_r)} \right. \\
 & + i \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \frac{J(b, \alpha_m) J(a, \delta_p) (ik\eta_2 - i\delta_p) e^{i\delta_p l} f_m}{\pi J(a, \alpha_m) b^* (\alpha_m^2 - \delta_p^2) (\delta_p - \tau_r)} \\
 & + i \sum_{m=1}^{\infty} \frac{J(c, \beta_m) (ik\eta_3 - i\beta_m) h_m}{\pi J(b, \beta_m) V_2^+(\beta_m) 2\beta_m (\beta_m - \tau_r)} \\
 & \left. + \frac{a\pi}{2} \sum_{m=1}^{\infty} \frac{J(a, \delta_m) e^{i\delta_m l} A_n J_0(\gamma_n)}{a^* (\lambda_n - \delta_m) (\delta_m - \tau_r)} \right\} \\
 & \times \frac{Y(c, \tau_r) J(b, \tau_r) V_2^+(\tau_r) e^{-i\tau_r l}}{J'(c, \tau_r)}, \quad (45)
 \end{aligned}$$

where again

$$a^* = J'(b, -\delta_m) V_2^+(\delta_m), \quad b^* = J'(b, -\delta_p) V_2^+(\delta_p).$$

3. Scattering matrices technique

In this part, the problem is investigated over two infinite ducts with area expansions $z = 0$ and $z = l$ respectively. The first geometry is defined by $\{\rho = a, z \in (-\infty, 0)\} \cup \{\rho \in (a, b), z = 0\} \cup \{\rho = b, z \in (0, \infty)\}$, and the second geometry is defined by $\{\rho = b, z \in (-\infty, l)\} \cup \{\rho \in (b, c), z = l\} \cup \{\rho = c, z \in (l, \infty)\}$. The problem is reformulated according to geometries and scattering matrices are found by the help of Wiener-Hopf equation (DEMIR, 2017). Matrix sizes vary according to the number of incoming, reflected or transmitted modes. In this study, the number of modes taking into account is denoted by n , r , and p in the first ($z < 0$), second ($0 < z < l$), and third ($z > l$) regions, respectively.

4. Numerical results

In this section, transmission coefficient is calculated numerically and some graphs are represented comparatively with scattering matrices technique and direct solution of Wiener-Hopf equations. The graphs are obtained by considering the change of radii and impedances. Infinite series in the formulas are truncated at some number N and after some observations, truncation number is chosen as $N = 10$ (see Fig. 2). While the transmission coefficient is found for direct solution, the fundamental mode is taking into account, that is in Eq. (45) $n = r = 1$. In Figs 3–5, as the value of first region radius a increases, value of the transmission coefficient starts as a higher point and continues to decrease. Similar behaviour is observed for the second region radius b while the reverse for the third

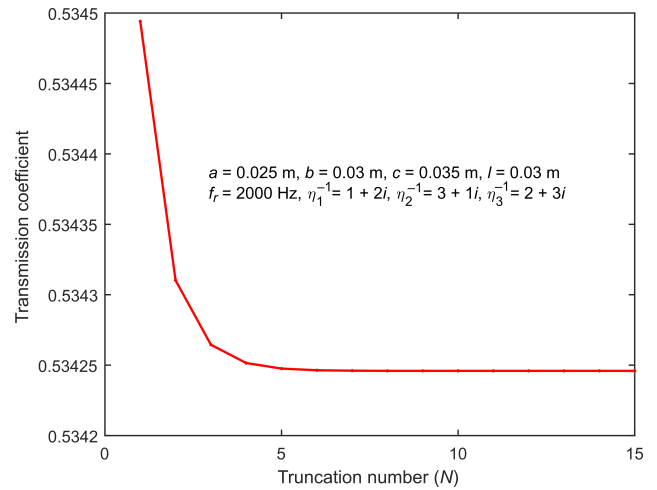


Fig. 2. Transmission coefficient *versus* truncation number N .

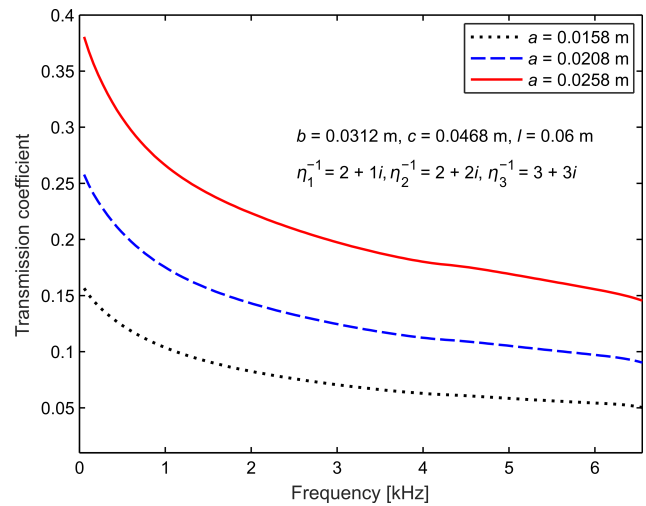


Fig. 3. Transmission coefficient *versus* frequency, for different values of radius a .

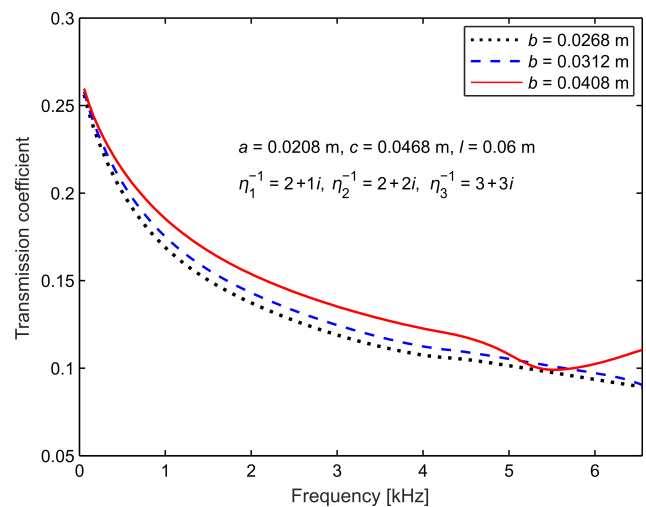


Fig. 4. Transmission coefficient *versus* frequency, for different values of radius b .

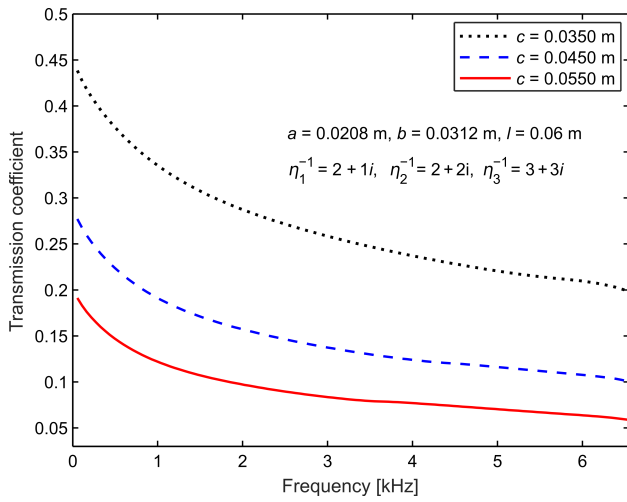


Fig. 5. Transmission coefficient *versus* frequency, for different values of radius c .

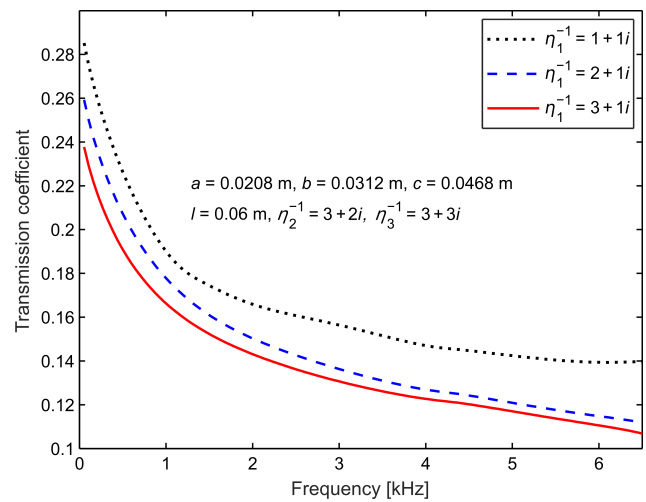


Fig. 7. Transmission coefficient *versus* frequency, for different values of $\text{Re} \eta_1^{-1}$.

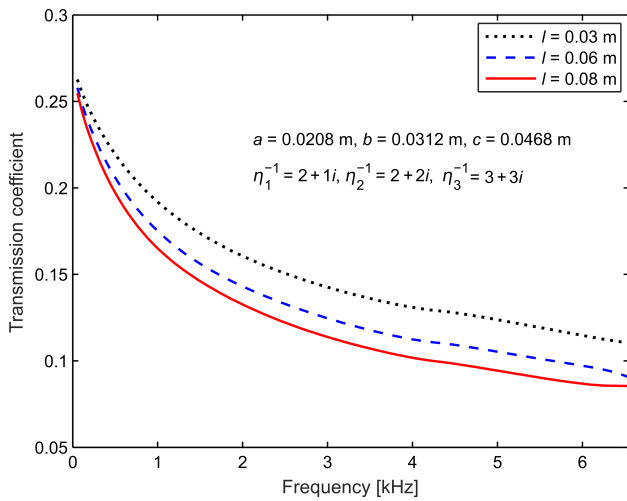


Fig. 6. Transmission coefficient *versus* frequency, for different values of length l .

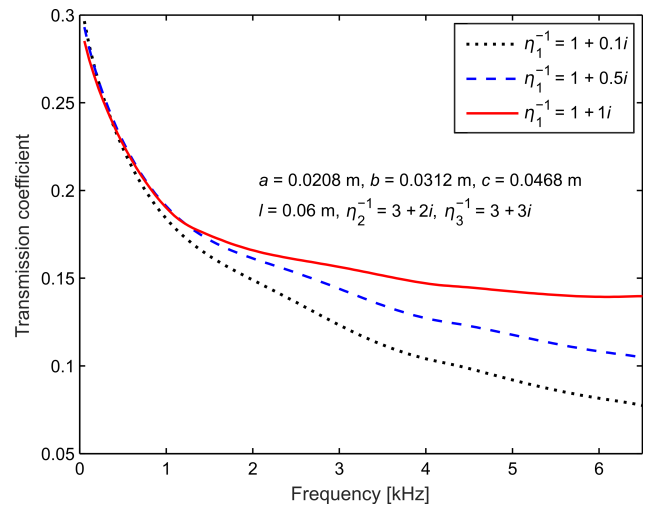


Fig. 8. Transmission coefficient *versus* frequency, for different values of $\text{Im} \eta_1^{-1}$.

region radius is observed. Figure 6 shows the transmission coefficient with different values of length l which is the distance between the steps. It is seen that the transmission coefficient decreases with increasing value of l , as expected. In Figs 7–12, transmission coefficient graphics are presented with different values of impedances for resistance and reactance. As the resistance of η_1^{-1} decreases, the transmission coefficient increases. Conversely, the transmission coefficient decreases with decreasing value of the resistance of η_3^{-1} . In Fig. 9, it is seen that after starting at the same point the transmission coefficient values decreases with increasing value of the resistance of η_2^{-1} , and after a certain frequency, it takes the form of increasing transmission coefficient with increasing value of the resistance of η_2^{-1} . In Fig. 8, it is observed that the transmission coefficient decreases up to a certain frequency range, as the reactance of η_1^{-1} increases. Beyond this frequency range a reversed behaviour is observed. In Fig. 10,

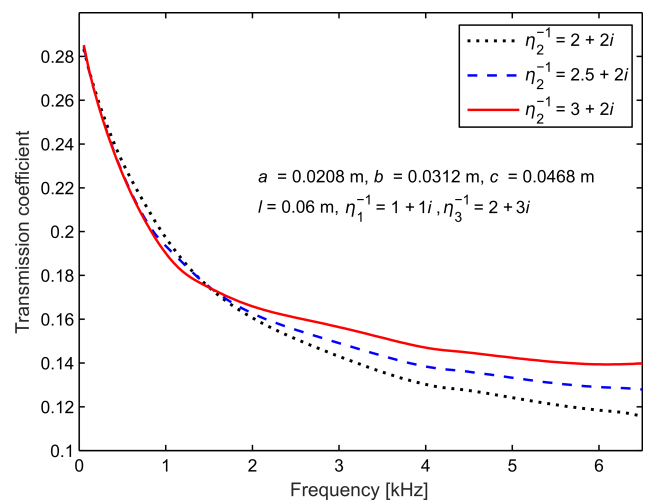


Fig. 9. Transmission coefficient *versus* frequency, for different values of $\text{Re} \eta_2^{-1}$.

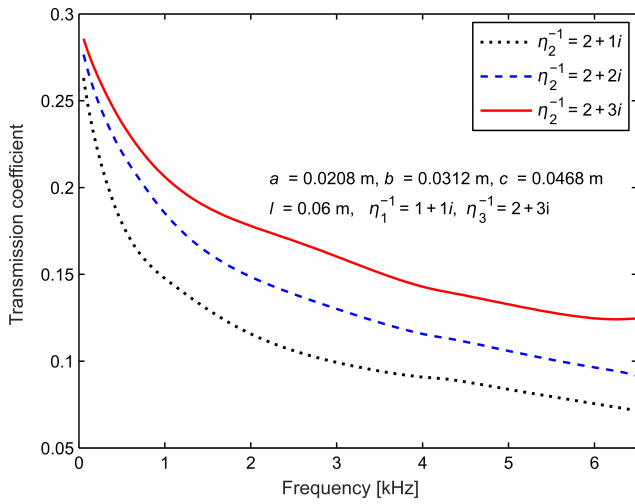


Fig. 10. Transmission coefficient *versus* frequency, for different values of $\text{Im} \eta_2^{-1}$.

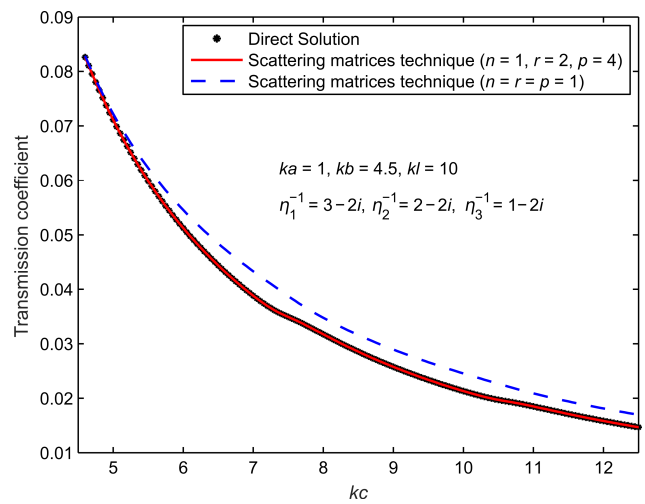


Fig. 13. Transmission coefficient *versus* radius kc .

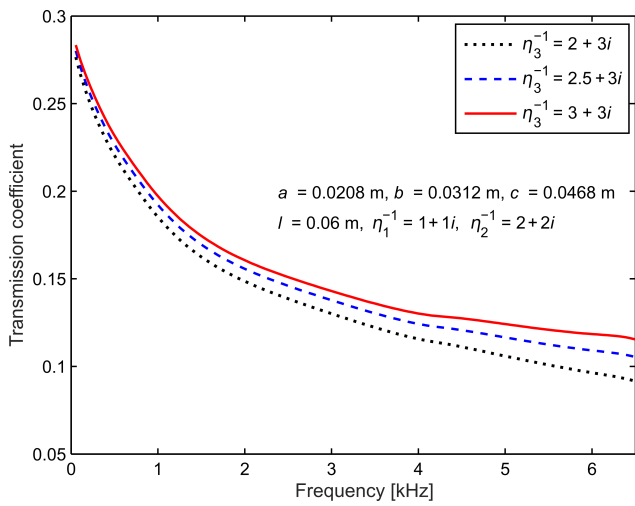


Fig. 11. Transmission coefficient *versus* frequency, for different values of $\text{Re} \eta_3^{-1}$.

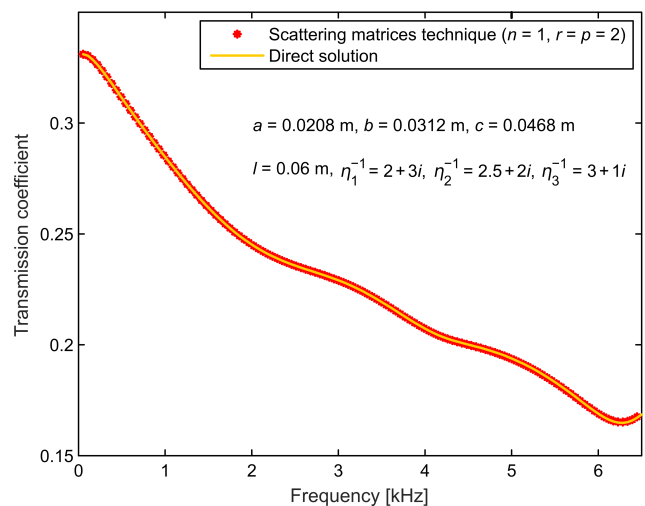


Fig. 14. Transmission coefficient *versus* frequency.

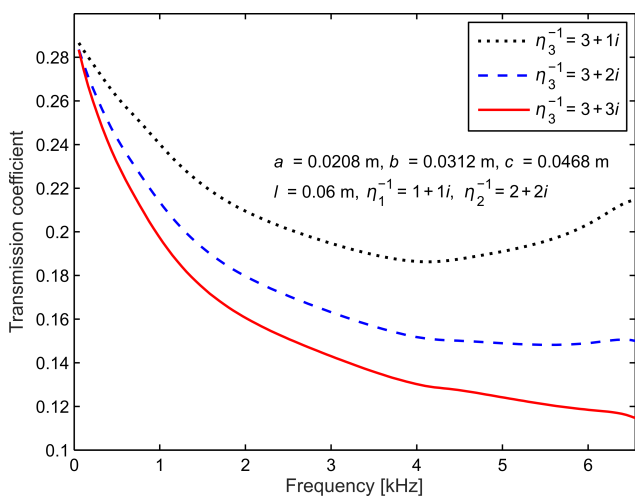


Fig. 12. Transmission coefficient *versus* frequency, for different values of $\text{Im} \eta_3^{-1}$.

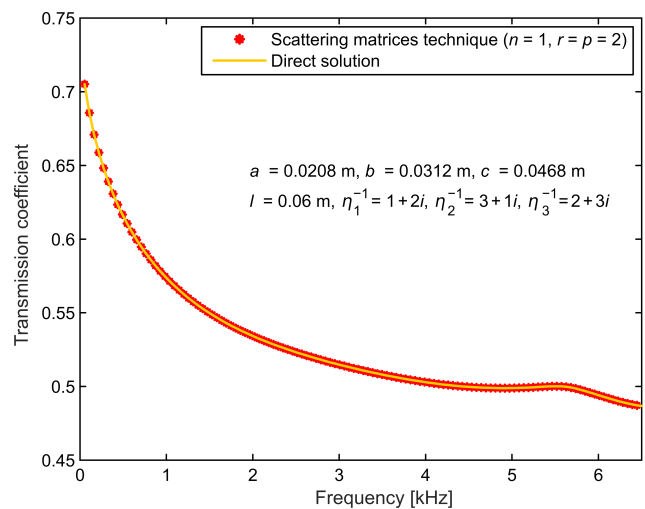


Fig. 15. Transmission coefficient *versus* frequency.

increased values of the reactance of η_2^{-1} , the transmission coefficient also increases. But, in Fig. 12, it is seen that transmission coefficient decreases with increasing value of the reactance of η_3^{-1} . In Fig. 13, the transmission coefficient is calculated for only fundamental mode and for fundamental mode together with higher order modes. Results are compared with direct solution. To see the effect of adjusting the matrix dimensions on the transmission coefficient, result obtained by taking only fundamental mode is shown in the graph, too. When the matrix dimensions are adjusted according to propagated mode number and graph is obtained, it is seen that there is a perfect agreement with the direct solution. That's why we need to adjust the dimensions of the matrices. Figures 14 and 15 are given as a comparison of direct solution and scattering matrices technique for different radii and impedance values.

5. Conclusion

Model of the problem consists of two stepped duct which walls are treated by acoustically absorbent lining. Mainly, the problem is handled with Wiener-Hopf technique. Scattering matrices technique is used for validation of results. For convenience, the problem is divided into regions in the direct solution and two modified Wiener-Hopf equations were obtained whose solutions involve infinitely many expansion coefficients satisfying an infinite system of linear algebraic equations. In the technique of scattering matrices, the main geometry is considered as two sub-geometries and transmission and reflection matrices are obtained with the help of these geometries. Then the transmission coefficient formula is obtained from these matrices for the problem containing two points of discontinuity. Numerical results are obtained for different radii, different impedances and distance between steps. In addition, in the method of scattering matrices, fundamental mode and higher order mode comparison are made to emphasize the importance of adjusting the matrix dimensions. As a future work, we will generalise the transmission coefficient formula which is found by the help of scattering matrices, for n -stepped lined duct. In this way, we will be able to examine the scattering from a waveguide which radius is linearly varying.

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