Finite-Difference Operators for 2D problems

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Abstract. This paper presents the concept of using algorithms for reducing the dimensions of finite-difference equations of two-dimensional (2D) problems, for second-order partial differential equations. Solutions are predicted as two-variable functions over the rectangular domain, which are periodic with respect to each variable and which repeat outside the domain. Novel finite-difference operators, of both the first and second orders, are developed for such functions. These operators relate the value of derivatives at each point to the values of the function at all points distributed uniformly over the function domain. A specific feature of the novel operators follows from the arrangement of the function values as well as the values of derivatives, which are rectangular matrices instead of vectors. This significantly reduces the dimensions of the finite-difference operators to the numbers of points in each direction of the 2D area. The finite-difference equations are created exemplary for elliptic equations. An original iterative algorithm is proposed for reducing the process of solving finite-difference equations to the multiplication of matrices.

Key words: second-order partial differential equations, 2D finite-difference operators, finite-difference equations, iterative algorithms.

1. Introduction

The finite difference method (FDM) is an important approach to solving partial differential equations (PDEs). In this approach, partial derivatives are approximated using finite-difference operators (FDOs) that engage the solution values at adjacent points, and finite-difference equations (FDEs) are created based on such operators. A wide variety of such discrete operators exists and has been presented in several books on numerical methods [1–6].

The FDM is still the subject of investigation of several works; a few exemplary works conducted over the past few years are mentioned below. The FDM was used in [7] to solve the three-dimensional (3D) magnetic-field problem. In [8], it was proved that the FDM with hexahedral elements and the edge-element method, when applied to magnetic-field 3D problems, exhibited common features. The new structures of finite-difference schemes were proposed in [9, 10]. In [11], high-order finite-difference schemes were proposed for the Navier–Stokes two-dimensional (2D) equations. Combinations of the FDM with other approaches have also been observed. The FDM and the finite-element method were combined in [12]. However, in the above-mentioned books and papers, the FDOs were defined based on the solution values at points adjacent to the point at which the derivative was determined. New types of FDOs were developed and tested to solve the steady-state solutions of nonlinear dynamic systems directly when the solutions were periodic [13–15] or double-periodic [16]. These operators re-

lated the values of the derivatives at each point to the values of the function, at all points distributed uniformly over the function domain. The same types of FDOs were adapted to solve the one-dimensional (1D) boundary-value problems of ordinary differential equations [17, 18] if the solution in the function domain repeated outside, i.e. it is periodic.

A previous study [19] extended that type of operators to 2D problems for second-order PDEs. In such cases, solutions were predicted as two-variable functions over the rectangular domain which repeated outside that area. Thus, solutions could be predicted as two-variable functions that are periodic with respect to each variable. The first and second-order partial FDOs were presented in [19]. They combined the values of partial derivatives and two-variable functions for points distributed uniformly over a rectangular area. The values of derivatives at each point were related to the values of the function at all points over the area. However, those FDOs had structures in which the unknown values of the solution were arranged in a vector having a dimension equal to the product of the numbers of points in each direction. The relative FDEs for 2D problems were expected to have the same dimensions.

A modified FDO is presented in this paper. The difference follows from the arrangement of the unknown values, both of a solution and its derivatives, onto rectangular matrices, and not vectors as before. The elements of those matrices correlate strictly with the locations of individual points in the rectangular area. It significantly reduces the dimensions of the FDOs to the numbers of points in each direction in the 2D area. The FDEs are crating based on new operators for exemplary elliptic equations. The matrices in such equations have significantly smaller dimensions. An original iterative algorithm is proposed for reducing the solving process to the multiplication of matrices only.

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2. Novel discrete differential operators

The discrete 2D differential operators have been developed in [19] for the two-variable double-periodic function $z(x,y) = z(x+2\pi,y) = z(x,y+2\pi)$ if it can be approximated by a Fourier series with a limited number of terms:

$$z(x,y) = \sum_{r=-R}^{R} \sum_{s=-S}^{S} z_{r,s} \cdot e^{jrx} \cdot e^{jsy}. \tag{1}$$

When this function is determined in the area $-\pi < x < \pi$, $-\pi < y < \pi$, unique relations between function values and their Fourier coefficients can be found by choosing a set of $(2R+1) \cdot (2S+1)$ points, located regularly in the function domain:

$$x_n = n \cdot \alpha, \quad -R < n < R, \quad \alpha = 2\pi/(2R+1),$$

$$y_m = m \cdot \beta, \quad -S < m < S, \quad \beta = 2\pi/(2S+1).$$

The function values $z_{n,m} = z(x_n, y_m)$ are determined by the quadratic forms:

$$z_{n,m} = \begin{bmatrix} b^{-S \cdot m} \\ \vdots \\ b^{-1 \cdot m} \\ 1 \\ b^{1 \cdot m} \\ \vdots \\ b^{S \cdot m} \end{bmatrix}^T \begin{bmatrix} z_{-R,S} & \cdots & z_{-1,S} & z_{0,S} & z_{1,S} & \cdots & z_{R,S} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ z_{-R,1} & \cdots & z_{-1,1} & z_{0,1} & z_{1,1} & \cdots & z_{R,1} \\ z_{-R,0} & \cdots & z_{-1,0} & z_{0,0} & z_{1,0} & \cdots & z_{R,0} \\ z_{-R,-1} & \cdots & z_{-1,-1} & z_{0,-1} & z_{1,-1} & \cdots & z_{R,-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ z_{-R,-S} & \cdots & z_{-1,-S} & z_{0,-S} & z_{1,-S} & \cdots & z_{R,-S} \end{bmatrix} \cdot \begin{bmatrix} a^{-R \cdot n} \\ \vdots \\ a^{-1 \cdot n} \\ 1 \\ a^{1 \cdot n} \\ \vdots \\ a^{R \cdot n} \end{bmatrix}, \tag{2}$$

where $a = e^{j \cdot \alpha}$ and $b = e^{j \cdot \beta}$. Consequently, the relations for the whole point set can be written in the matrix form as:

$$\mathbf{z} = \mathbf{B} \cdot \mathbf{Z} \cdot \mathbf{A}. \tag{3}$$

The explicit forms of these matrices are given in the Appendix. The inverse relation can be obtained rather easily:

$$\mathbf{Z} = \mathbf{B}^{-1} \cdot \mathbf{z} \cdot \mathbf{A}^{-1}, \tag{4}$$

because the matrices $\mathbf{A}$ and $\mathbf{B}$ are of the Hermitian type:

$$\mathbf{A}^{-1} = \frac{1}{2R+1}(\mathbf{A}^*)^T, \quad \mathbf{B}^{-1} = \frac{1}{2S+1}(\mathbf{B}^*)^T.$$

2.1. First-order discrete partial differential operators.

The first partial derivatives, if they exist, have the Fourier series forms of:

$$\frac{\partial z(x,y)}{\partial x} = \sum_{s=-S}^{S} \left(\sum_{r=-R}^{R} \left(j \cdot r \cdot Z_{r,s} \cdot e^{j \cdot r \cdot x} \right) \right) \cdot e^{j \cdot s \cdot y}, \tag{5a}$$

$$\frac{\partial z(x,y)}{\partial y} = \sum_{r=-R}^{R} \left(\sum_{s=-S}^{S} \left(j \cdot s \cdot Z_{r,s} \cdot e^{j \cdot s \cdot y} \right) \right) \cdot e^{j \cdot r \cdot x}. \tag{5b}$$

The relations between the Fourier coefficients of the function and its first derivatives take the forms:

$$\mathbf{Z}'_x = j \cdot \mathbf{Z} \cdot \mathbf{R}, \quad \mathbf{Z}'_y = j \cdot \mathbf{S} \cdot \mathbf{Z}. \tag{6}$$

The matrices $\mathbf{Z}'_x$ and $\mathbf{Z}'_y$ are arranged similarly to the matrix $\mathbf{Z}$. Their detailed forms are provided in Appendix. The matrices $\mathbf{R}$ and $\mathbf{S}$ are diagonal and take the forms of:

$$\mathbf{R} = \text{diag}\begin{bmatrix} -R & \cdots & -1 & 0 & 1 & \cdots & R \end{bmatrix},$$

$$\mathbf{S} = \text{diag}\begin{bmatrix} -S & \cdots & -1 & 0 & 1 & \cdots & S \end{bmatrix}.$$

Combining (3), (4), and (6), we obtain:

$$\mathbf{z}'_x = \mathbf{z} \cdot \mathbf{D}_x, \quad \mathbf{D}_x = j \cdot (\mathbf{A}^{-1} \cdot \mathbf{R} \cdot \mathbf{A}), \tag{7a}$$

$$\mathbf{z}'_y = \mathbf{D}_y \cdot \mathbf{z}, \quad \mathbf{D}_y = j \cdot (\mathbf{B} \cdot \mathbf{S} \cdot \mathbf{B}^{-1}). \tag{7b}$$

The matrices $\mathbf{D}_x$ and $\mathbf{D}_y$ are square but have different dimensions: $2R+1$ and $2S+1$, respectively. Both matrices are singular (because one from the eigenvalues equals zero) and skew-Hermitian (because the other eigenvalues are purely imaginary). The explicit forms of these matrices are presented in the Appendix.

2.2. Second-order discrete partial differential operators.

The discrete partial differential operators of the second order can be obtained by repeating the differentiating operations. The operator with respect to "x" follows from the operation:

$$\mathbf{z}''_{xx} = \mathbf{z}'_x \cdot (j \cdot \mathbf{A}^{-1} \cdot \mathbf{R} \cdot \mathbf{A}) = -\mathbf{z} \cdot (\mathbf{A}^{-1} \cdot \mathbf{R} \cdot \mathbf{A}) \cdot (\mathbf{A}^{-1} \cdot \mathbf{R} \cdot \mathbf{A}),$$

$$\mathbf{z}''_{xx} = -\mathbf{z} \cdot (\mathbf{A}^{-1} \cdot \mathbf{R}^2 \cdot \mathbf{A}),$$

where $\mathbf{R}^2 = \text{diag}\begin{bmatrix} R^2 & \cdots & 1 & 0 & 1 & \cdots & R^2 \end{bmatrix}$, and analogously, the operator with respect to "y" follows from:

$$\mathbf{z}''_{yy} = (j \cdot \mathbf{B} \cdot \mathbf{S} \cdot \mathbf{B}^{-1}) \cdot \mathbf{z}'_y = -(\mathbf{B} \cdot \mathbf{S} \cdot \mathbf{B}^{-1}) \cdot (\mathbf{B} \cdot \mathbf{S} \cdot \mathbf{B}^{-1}) \cdot \mathbf{z},$$

$$\mathbf{z}''_{yy} = -(\mathbf{B} \cdot \mathbf{S}^2 \cdot \mathbf{B}^{-1}) \cdot \mathbf{z},$$

where $\mathbf{S}^2 = \text{diag}\begin{bmatrix} S^2 & \cdots & 1 & 0 & 1 & \cdots & S^2 \end{bmatrix}$.

The mixed operator follows from:

$$\mathbf{z}''_{xy} = \mathbf{z}'_y \cdot (j \cdot \mathbf{A}^{-1} \cdot \mathbf{R} \cdot \mathbf{A}) = -(\mathbf{B} \cdot \mathbf{S} \cdot \mathbf{B}^{-1}) \cdot \mathbf{z} \cdot (\mathbf{A}^{-1} \cdot \mathbf{R} \cdot \mathbf{A}),$$

$$\mathbf{z}''_{yx} = (j \cdot \mathbf{B} \cdot \mathbf{S} \cdot \mathbf{B}^{-1}) \cdot \mathbf{z}'_x = -(\mathbf{B} \cdot \mathbf{S} \cdot \mathbf{B}^{-1}) \cdot \mathbf{z} \cdot (\mathbf{A}^{-1} \cdot \mathbf{R} \cdot \mathbf{A}).$$

Finally, the following relations are obtained:

$$\mathbf{z}''_{xx} = \mathbf{z} \cdot \mathbf{D}_{xx}, \quad \mathbf{z}''_{yy} = \mathbf{D}_{yy} \cdot \mathbf{z}, \quad \mathbf{z}''_{xy} = \mathbf{D}_y \cdot \mathbf{z} \cdot \mathbf{D}_x. \quad (8)$$

The operators $\mathbf{D}_{xx}$ and $\mathbf{D}_{yy}$ are defined as:

$$\mathbf{D}_{xx} = -(\mathbf{A}^{-1} \cdot \mathbf{R}^2 \cdot \mathbf{A}) \quad \text{and} \quad \mathbf{D}_{yy} = -(\mathbf{B} \cdot \mathbf{S}^2 \cdot \mathbf{B}^{-1}). \quad (9)$$

These two matrices are also square and have the same dimensions as the first-order operators, i.e. $2R+1$ and $2S+1$, respectively. They are singular and symmetrical because their eigenvectors are real. The explicit forms of the matrices $\mathbf{D}_{xx}$ and $\mathbf{D}_{yy}$ are given in the Appendix.

All types of discrete operators for two-variable periodic function have been presented in [1]. They are based on the same assumptions as that of the Fourier series limitations and are determined for the same point set presented in this paper. However, the values of the function, its derivatives, and Fourier coefficients have been arranged into vectors with dimensions $(2R+1) \cdot (2S+1)$. Consequently, all discrete operators have the form of square matrices with the same dimensions, which can be rather high.

An alternative form of discrete operators developed in this section is based on the arrangement of those values in the matrices. The values of the function and its derivatives, as well as the respective Fourier coefficients, constitute rectangular matrices with dimensions related to the numbers of points for the "*x*" and "*y*" directions. All matrices in relations (7) and (8) have dimensions $2R+1$ or $2S+1$, or combinations of those numbers. Thus, the new discrete operators have relatively smaller dimensions.

3. Application examples

3.1. Elementary linear equation. Let us consider a linear elliptical equation:

$$a_x \frac{\partial^2 z(x,y)}{\partial x^2} + a_y \frac{\partial^2 z(x,y)}{\partial y^2} = g(x,y), \quad (10)$$

with constant coefficients a_x and a_y. Application of the second-order differential operators (8) leads to finite-difference equations of the form:

$$a_x \cdot \mathbf{z} \cdot \mathbf{D}_{xx} + a_y \cdot \mathbf{D}_{yy} \cdot \mathbf{z} = \mathbf{g}. \quad (11)$$

The matrices $\mathbf{D}_{xx}$ and $\mathbf{D}_{yy}$ are square, but have different dimensions $2R+1$ and $2S+1$, as mentioned in Section 2, and are singular because each of them has one eigenvalue that equals zero. The matrices $\mathbf{z}$ and $\mathbf{g}$ are rectangular (see Appendix) with dimensions $(2R+1) \times (2S+1)$. Equation (11) cannot be solved directly because of the different location of the matrix $\mathbf{z}$ with respect to the matrices $\mathbf{D}_{xx}$ and $\mathbf{D}_{yy}$. To solve the equation, it is modified into the form:

$$a_x \cdot \mathbf{z} \cdot \mathbf{D}_{xx} + a_y \cdot \mathbf{D}_{yy} \cdot \mathbf{z} + a_x \cdot \mathbf{z} \cdot \mathbf{D}_{x0} = \mathbf{g} + a_x \cdot \mathbf{z} \cdot \mathbf{D}_{x0},$$

$$a_x \cdot \mathbf{z} \cdot \mathbf{D}_{xx} + a_y \cdot \mathbf{D}_{yy} \cdot \mathbf{z} + a_y \cdot \mathbf{D}_{y0} \cdot \mathbf{z} = \mathbf{g} + a_y \cdot \mathbf{D}_{y0} \cdot \mathbf{z}.$$

In the first equation, the term $a_x \cdot \mathbf{z} \cdot \mathbf{D}_{x0}$ has been added to both sides of (11). Analogously, the term $a_y \cdot \mathbf{D}_{y0} \cdot \mathbf{z}$ has been added to create the second equation. The matrices $\mathbf{D}_{x0}$ and $\mathbf{D}_{y0}$ change the zero's eigenvalues to non-zero values, when added to the matrices $\mathbf{D}_{xx}$ and $\mathbf{D}_{yy}$.

$$\mathbf{D}_{0x} = -(\mathbf{A}^{-1} \cdot \mathbf{R}_0 \cdot \mathbf{A}), \quad \mathbf{D}_{0y} = -(\mathbf{B} \cdot \mathbf{S}_0 \cdot \mathbf{B}^{-1}), \quad (13)$$

where

$$\mathbf{R}_0 = \mathrm{diag}\begin{bmatrix} 0 & \cdots & 0 & d_0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\mathbf{S}_0 = \mathrm{diag}\begin{bmatrix} 0 & \cdots & 0 & d_0 & 0 & \cdots & 0 \end{bmatrix},$$

and have the forms:

$$\mathbf{D}_{0x} = \frac{-d_0}{2R+1} \cdot \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix};$$

$$\mathbf{D}_{0y} = \frac{-d_0}{2S+1} \cdot \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

An arbitrary value, d_0, is the new eigenvalue, substituting for the zero eigenvalue of the matrices $\mathbf{D}_{xx}$ and $\mathbf{D}_{yy}$. It leads to two equations with non-singular matrices $(\mathbf{D}_{xx} + \mathbf{D}_{0x})$ and $(\mathbf{D}_{yy} + \mathbf{D}_{0y})$, which can be solved iteratively in a sequence:

$$\mathbf{z}_x^{i+1} \cdot (\mathbf{D}_{xx} + \mathbf{D}_{0x}) = (\mathbf{g} + \mathbf{z}_x^i \cdot \mathbf{D}_{0x} - (a_y/a_x) \cdot \mathbf{z}_y^i \cdot \mathbf{D}_{yy}),$$

$$(\mathbf{D}_{yy} + \mathbf{D}_{0y}) \cdot \mathbf{z}_y^{i+1} = (\mathbf{g} + \mathbf{D}_{0y} \cdot \mathbf{z}_y^i - (a_x/a_y) \cdot \mathbf{z}_x^{i+1} \cdot \mathbf{D}_{xx}).$$

The new values $\mathbf{z}_x^{i+1}$ and $\mathbf{z}_y^{i+1}$ allow for repeated iteration. The variables of those equations are denoted as $\mathbf{z}_x$ and $\mathbf{z}_y$, respectively. Iterations should be completed when $\mathbf{z}_x = \mathbf{z}_y = \mathbf{z}$ at a predefined accuracy. However, it is not necessary to solve them because the inverse matrices $(\mathbf{D}_{xx} + \mathbf{D}_{0x})^{-1}$ and $(\mathbf{D}_{yy} + \mathbf{D}_{0y})^{-1}$ can be determined analytically. Using Eqs. (9) and (13), it follows that:

$$\mathbf{D}_{xx} + \mathbf{D}_{0x} = -(\mathbf{A}^{-1} \cdot (\mathbf{R}^2 + \mathbf{R}_0) \cdot \mathbf{A}),$$

$$\mathbf{D}_{yy} + \mathbf{D}_{0y} = -(\mathbf{B} \cdot (\mathbf{S}^2 + \mathbf{S}_0) \cdot \mathbf{B}^{-1}),$$

where

$$\mathbf{R}^2 + \mathbf{R}_0 = \mathrm{diag}\begin{bmatrix} R^2 & \cdots & 1 & d_0 & 1 & \cdots & R^2 \end{bmatrix},$$

$$\mathbf{S}^2 + \mathbf{S}_0 = \mathrm{diag}\begin{bmatrix} S^2 & \cdots & 1 & d_0 & 1 & \cdots & S^2 \end{bmatrix}.$$

Thus, the inverse matrices take the forms:

$$(\mathbf{D}_{xx} + \mathbf{D}_{0x})^{-1} = \mathbf{C}_{xx} + \mathbf{C}_{0x},$$

$$(\mathbf{D}_{yy} + \mathbf{D}_{0y})^{-1} = \mathbf{C}_{yy} + \mathbf{C}_{0y},$$

with:

$$\mathbf{C}_{xx} = -\mathbf{A}^{-1} \cdot \mathrm{diag} \left[1/R^2 \quad \cdots \quad 1/1 \quad 0 \quad 1/1 \quad \cdots \quad 1/R^2 \right] \cdot \mathbf{A},$$

$$\mathbf{C}_{yy} = -\mathbf{B} \cdot \mathrm{diag} \left[1/S^2 \quad \cdots \quad 1/1 \quad 0 \quad 1/1 \quad \cdots \quad 1/S^2 \right] \cdot \mathbf{B}^{-1},$$

$$\mathbf{C}_{0x} = \frac{-1}{(2R+1) \cdot d_0} \cdot \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix};$$

$$\mathbf{C}_{0y} = \frac{-1}{(2S+1) \cdot d_0} \cdot \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

The matrices $\mathbf{C}_{xx}$ and $\mathbf{C}_{yy}$ have exactly the same forms as the matrices $\mathbf{D}_{xx}$ and $\mathbf{D}_{yy}$. Their detailed forms are described in the Appendix.

Finally, the iterations reduce to multiplications of matrices:

$$\mathbf{z}_x^{i+1} = (\mathbf{g} + \mathbf{z}_x^i \cdot \mathbf{D}_{0x} - (a_y/a_x) \cdot \mathbf{z}_y^i \cdot \mathbf{D}_{yy}) \cdot (\mathbf{C}_{xx} + \mathbf{C}_{0x}), \quad (14a)$$

$$\mathbf{z}_y^{i+1} = (\mathbf{C}_{yy} + \mathbf{C}_{0y}) \cdot (\mathbf{g} + \mathbf{D}_{0y} \cdot \mathbf{z}_y^i - (a_x/a_y) \cdot \mathbf{z}_x^{i+1} \cdot \mathbf{D}_{xx}). \quad (14b)$$

The starting matrix $\mathbf{z}_x^0$ can be determined by solving the 1D equation:

$$a_x \frac{\partial^2 z_m(x)}{\partial x^2} = g(x, y_m),$$

with respect to variable x for individual values of y_m. Analogously, the matrix $\mathbf{z}_y^0$ can be determined solving the 1D equation:

$$a_y \frac{\partial^2 z_n(y)}{\partial y^2} = g(x_n, y),$$

with respect to variable y for individual values of x_m. The values $\mathbf{z}_x^0 = \mathbf{z}_y^0 = \mathbf{0}$ can also be considered.

3.2. Non-linear equation. Many engineering problems lead to non-linear elliptical equations of the form:

$$\frac{\partial}{\partial x} \left(a_x(z, x, y) \frac{\partial z(x, y)}{\partial x} \right)$$
$$+ \frac{\partial}{\partial y} \left(a_y(z, x, y) \frac{\partial z(x, y)}{\partial y} \right) = g(x, y). \quad (15)$$

In such cases, the problem of solving should be stated as follows: introducing functions $u(x, y)$ and $v(x, y)$

$$\frac{\partial z(x, y)}{\partial x} = z_x'(x, y) = \frac{u(x, y)}{a_x(z, x, y)}, \quad (16a)$$

$$\frac{\partial z(x, y)}{\partial y} = z_y'(x, y) = \frac{v(x, y)}{a_y(z, x, y)}, \quad (16b)$$

Equation (15) can be written as:

$$\frac{\partial u(x, y)}{\partial x} + \frac{\partial v(x, y)}{\partial y} = g(x, y). \quad (16c)$$

Equations (16) constitute a set of three first-order partial differential equations with three unknown functions $u(x, y)$, $v(x, y)$, and $z(x, y)$. The respective finite-difference equations take the forms:

$$\mathbf{z} \cdot \mathbf{D}_x = \mathbf{z}_x', \quad (17a)$$

$$\mathbf{D}_y \cdot \mathbf{z} = \mathbf{z}_y', \quad (17b)$$

$$\mathbf{u} \cdot \mathbf{D}_x + \mathbf{D}_y \cdot \mathbf{v} = \mathbf{g}, \quad (17c)$$

when using the first-order operators (7). The matrices $\mathbf{z}_x'$ and $\mathbf{z}_y'$ contain quotients on the right-hand sides in (16a) and (16b), calculated at the grid points and arranged as in matrix $\mathbf{z}$.

To solve equation set (17), Eq. (17c) should be written in two forms, analogously, similar to Eq. (11).

$$\mathbf{u} \cdot (\mathbf{D}_x + \mathbf{D}_{x0}) = \mathbf{g} + \mathbf{u} \cdot \mathbf{D}_{x0} - \mathbf{D}_y \cdot \mathbf{v},$$

$$(\mathbf{D}_y + \mathbf{D}_{y0}) \cdot \mathbf{v} = \mathbf{g} + \mathbf{D}_{y0} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{D}_x.$$

The matrices $(\mathbf{D}_{xx} + \mathbf{D}_{0x})$ and $(\mathbf{D}_{yy} + \mathbf{D}_{0y})$ are now nonsingular; thus, these equations can be solved iteratively:

$$\mathbf{u}^{i+1} \cdot (\mathbf{D}_x + \mathbf{D}_{0x}) = (\mathbf{g} + \mathbf{u}^i \cdot \mathbf{D}_{0x} - \mathbf{D}_y \cdot \mathbf{v}^i), \quad (18a)$$

$$(\mathbf{D}_y + \mathbf{D}_{0y}) \cdot \mathbf{v}^{i+1} = (\mathbf{g} + \mathbf{D}_{0x}\mathbf{v}^i - \mathbf{u}^i \cdot \mathbf{D}_x). \quad (18b)$$

Repeating the considerations in Section 3A, the final solutions can be presented in direct forms, corresponding to (14):

$$\mathbf{u}^{i+1} = (\mathbf{g} + \mathbf{u}^i \cdot \mathbf{D}_{0x} - \mathbf{D}_y \cdot \mathbf{v}^i) \cdot (\mathbf{C}_x + \mathbf{C}_{0x}), \quad (19a)$$

$$\mathbf{v}^{i+1} = (\mathbf{C}_y + \mathbf{C}_{0y}) \cdot (\mathbf{g} + \mathbf{D}_{0y} \cdot \mathbf{v}^i - \mathbf{u}^{i+1} \cdot \mathbf{D}_x), \quad (19b)$$

where

$$\mathbf{C}_x = -\mathbf{j} \cdot \mathbf{A}^{-1} \cdot \mathbf{R}^{-1} \cdot \mathbf{A}, \qquad \mathbf{C}_y = -\mathbf{j} \cdot \mathbf{B} \cdot \mathbf{S}^{-1} \cdot \mathbf{B}^{-1}.$$

The matrices $\mathbf{C}_x$ and $\mathbf{C}_y$ have exactly the same forms as the matrices $\mathbf{D}_x$ and $\mathbf{D}_y$. Their detailed forms are described in the Appendix.

Before starting a new iteration, Eqs. (17a) and (17b) should be used to recalculate the values of the functions $a_x(z, x, y)$ and $a_y(z, x, y)$ in (16a) and (16b), if they depend on the function $z(x, y)$. For this, Eqs. (17a) and (17b) should be modified into the forms:

$$\mathbf{z}_x^{i+1} \cdot (\mathbf{D}_x + \mathbf{D}_{0x}) = \mathbf{z}_x'^{i+1} + \mathbf{z}_x^i \cdot \mathbf{D}_{0x}, \quad (20a)$$

$$(\mathbf{D}_y + \mathbf{D}_{0y}) \cdot \mathbf{z}_y^{i+1} = \mathbf{z}_y'^{i+1} + \mathbf{D}_{0y} \cdot \mathbf{z}_y^i. \quad (20b)$$

The elements $z'^{i+1}_{x,n,m}$ and $z'^{i+1}_{y,n,m}$ of matrices $\mathbf{z}'^{i+1}_x$ and $\mathbf{z}'^{i+1}_y$ should be calculated from the relationships:

$$z'^{i+1}_{x,n,m} = \frac{u^{i+1}_{n,m}}{a^i_{x,n,m}} = \frac{u^{i+1}(x_n,y_m)}{a_x(z^i_{n,m},x_n,y_m)}, \tag{21a}$$

$$z'^{i+1}_{y,n,m} = \frac{v^{i+1}_{n,m}}{a^i_{y,n,m}} = \frac{v^{i+1}(x_n,y_m)}{a_y(z^i_{n,m},x_n,y_m)}. \tag{21b}$$

The explicit solutions of Eq. (20) have the forms:

$$\mathbf{z}^{i+1}_x = (\mathbf{z}'^i_x + \mathbf{z}^i_x \cdot \mathbf{D}_{0x}) \cdot (\mathbf{C}_x + \mathbf{C}_{0x}), \tag{22a}$$

$$\mathbf{z}^{i+1}_y = (\mathbf{C}_y + \mathbf{C}_{0y}) \cdot (\mathbf{z}'^i_y + \mathbf{D}_{0y} \cdot \mathbf{z}^i_y). \tag{22b}$$

The matrices $\mathbf{z}^{i+1}_x$ and $\mathbf{z}^{i+1}_y$ can differ in successive iterations. The iterations should be stopped when $\mathbf{z}_x = \mathbf{z}_y = \mathbf{z}$, at a predefined accuracy.

The starting matrices $\mathbf{z}^0$, $\mathbf{u}^0$, and $\mathbf{v}^0$ are necessary for this case. The starting matrix $\mathbf{z}^0$ can be determined by solving a 1D equation with respect to the variable x:

$$a_x(z_m,x,y_m)\frac{\partial z_m(x)}{\partial x} = g(x,y_m), \tag{23a}$$

or with respect to the variable y:

$$a_y(z_n,x_n,y)\frac{\partial z_n(y)}{\partial y} = g(x_n,y). \tag{23b}$$

The matrices $\mathbf{u}^0$ and $\mathbf{v}^0$ can be determined based on solutions (23a) or (23b), respectively. The starting matrices can be predicted heuristically.

4. Conclusions

The paper presented a new type of finite-difference operator designed to solve 2D problems described by second-order partial differential equations. The novelty of the operators followed from the arrangement of the unknown values of the solution as rectangular matrices covering the locations of discretization points exactly in the rectangular area of the solution. The finite-difference operators for all partial derivatives, both the first and second orders, were presented. They constituted singular square matrices with reduced dimensions, related to the numbers of discretization points in each direction. Two examples showed how to create finite-difference equations for typical second-order elliptic equations. Although those equations had dimensions that were reduced almost twice, they had specific structures that could not be solved directly. This paper proposed algorithms dedicated to solving them, which reduced the problem to the multiplication of matrices.

However, the algorithm should be numerically tested, and results should be compared with other methods. The aim of this paper was to present only the idea of the algorithm based on new finite-difference operators.

Appendix

Detail forms of matrices

The matrix of function's values

$$\mathbf{z} = \begin{bmatrix} z_{-R,S} & \cdots & z_{-1,S} & z_{0,S} & z_{1,S} & \cdots & z_{R,S} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ z_{-R,1} & \cdots & z_{-1,1} & z_{0,1} & z_{1,1} & \cdots & z_{R,1} \\ z_{-R,0} & \cdots & z_{-1,0} & z_{0,0} & z_{1,0} & \cdots & z_{R,0} \\ z_{-R,-1} & \cdots & z_{-1,-1} & z_{0,-1} & z_{1,-1} & \cdots & z_{R,-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ z_{-R,-S} & \cdots & z_{-1,-S} & z_{0,-S} & z_{1,-S} & \cdots & z_{R,-S} \end{bmatrix};$$

$\mathbf{z}'_x$, $\mathbf{z}'_y$, $\mathbf{z}''_{xx}$, $\mathbf{z}''_{yy}$ and $\mathbf{z}''_{xy}$ have the same structures.

The matrix of function's Fourier coefficients

$$\mathbf{Z} = \begin{bmatrix} Z_{-R,S} & \cdots & Z_{-1,S} & Z_{0,S} & Z_{1,S} & \cdots & Z_{R,S} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ Z_{-R,1} & \cdots & Z_{-1,1} & Z_{0,1} & Z_{1,1} & \cdots & Z_{R,1} \\ Z_{-R,0} & \cdots & Z_{-1,0} & Z_{0,0} & Z_{1,0} & \cdots & Z_{R,0} \\ Z_{-R,-1} & \cdots & Z_{-1,-1} & Z_{0,-1} & Z_{1,-1} & \cdots & Z_{R,-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ Z_{-R,-S} & \cdots & Z_{-1,-S} & Z_{0,-S} & Z_{1,-S} & \cdots & Z_{R,-S} \end{bmatrix};$$

$\mathbf{Z}'_x$, $\mathbf{Z}'_y$, $\mathbf{Z}''_{xx}$, $\mathbf{Z}''_{yy}$ and $\mathbf{Z}''_{xy}$ have the same structures.

Matrices of transformations

$$\mathbf{A} = \begin{bmatrix} a^{R^2} & \cdots & a^R & 1 & a^{-R} & \cdots & a^{-R^2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a^R & \cdots & a^1 & 1 & a^{-1} & \cdots & a^{-R} \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ a^{-R} & \cdots & a^{-1} & 1 & a^1 & \cdots & a^R \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a^{-R^2} & \cdots & a^{-R} & 1 & a^R & \cdots & a^{R^2} \end{bmatrix},$$

$$a = e^{j\cdot\alpha}, \quad \alpha = 2\pi/(2R+1);$$

$$\mathbf{B} = \begin{bmatrix} b^{S^2} & \cdots & b^S & 1 & b^{-S} & \cdots & b^{-S^2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b^S & \cdots & b^1 & 1 & b^{-1} & \cdots & b^{-S} \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ b^{-S} & \cdots & b^{-1} & 1 & b^1 & \cdots & b^S \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b^{-S^2} & \cdots & b^{-S} & 1 & b^S & \cdots & b^{S^2} \end{bmatrix},$$

$$b = e^{j\cdot\beta}, \quad \beta = 2\pi/(2S+1).$$

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The first order discrete differential operators

$$\mathbf{D_x} = - \begin{bmatrix} 0 & -d_1^{(1)} & -d_2^{(1)} & \cdots & -d_R^{(1)} & d_R^{(1)} & \cdots & d_2^{(1)} & d_1^{(1)} \\ d_1^{(1)} & 0 & -d_1^{(1)} & -d_2^{(1)} & \cdots & -d_R^{(1)} & d_R^{(1)} & \cdots & d_2^{(1)} \\ d_2^{(1)} & d_1^{(1)} & 0 & -d_1^{(1)} & -d_2^{(1)} & \cdots & -d_R^{(1)} & \cdots & d_3^{(1)} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ d_R^{(1)} & \cdots & d_2^{(1)} & d_1^{(1)} & 0 & -d_1^{(1)} & -d_2^{(1)} & \cdots & -d_R^{(1)} \\ -d_R^{(1)} & d_R^{(1)} & \cdots & d_2^{(1)} & d_1^{(1)} & 0 & -d_1^{(1)} & \cdots & -d_{R-1}^{(1)} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ -d_2^{(1)} & \cdots & -d_R^{(1)} & d_R^{(1)} & \cdots & \cdots & d_1^{(1)} & 0 & -d_1^{(1)} \\ -d_1^{(1)} & -d_2^{(1)} & \cdots & -d_R^{(1)} & d_R^{(1)} & \cdots & d_2^{(1)} & d_1^{(1)} & 0 \end{bmatrix};$$

$$d_r^{(1)} = \frac{2}{2R+1} \sum_{k=1}^{R} k \cdot \sin\left(k \cdot r \cdot \frac{2\pi}{2R+1}\right);$$

$$\mathbf{D_y} = \begin{bmatrix} 0 & -d_1^{(1)} & -d_2^{(1)} & \cdots & -d_S^{(1)} & d_S^{(1)} & \cdots & d_2^{(1)} & d_1^{(1)} \\ d_1^{(1)} & 0 & -d_1^{(1)} & -d_2^{(1)} & \cdots & -d_S^{(1)} & d_S^{(1)} & \cdots & d_2^{(1)} \\ d_2^{(1)} & d_1^{(1)} & 0 & -d_1^{(1)} & -d_2^{(1)} & \cdots & -d_S^{(1)} & \cdots & d_3^{(1)} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ d_S^{(1)} & \cdots & d_2^{(1)} & d_1^{(1)} & 0 & -d_1^{(1)} & -d_2^{(1)} & \cdots & -d_S^{(1)} \\ -d_S^{(1)} & d_S^{(1)} & \cdots & d_2^{(1)} & d_1^{(1)} & 0 & -d_1^{(1)} & \cdots & -d_{S-1}^{(1)} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ -d_2^{(1)} & \cdots & -d_S^{(1)} & d_S^{(1)} & \cdots & \cdots & d_1^{(1)} & 0 & -d_1^{(1)} \\ -d_1^{(1)} & -d_2^{(1)} & \cdots & -d_S^{(1)} & d_S^{(1)} & \cdots & d_2^{(1)} & d_1^{(1)} & 0 \end{bmatrix};$$

$$d_s^{(1)} = \frac{2}{2S+1} \sum_{k=1}^{S} k \cdot \sin\left(k \cdot s \cdot \frac{2\pi}{2S+1}\right).$$

The matrices $\mathbf{C_x}$ and $\mathbf{C_y}$ have the same structures with the elements

$$c_r^{(1)} = -\frac{2}{2R+1} \sum_{k=1}^{R} \frac{1}{k} \cdot \sin\left(k \cdot r \cdot \frac{2\pi}{2R+1}\right),$$

$$c_s^{(1)} = -\frac{2}{2S+1} \sum_{k=1}^{S} \frac{1}{k} \cdot \sin\left(k \cdot s \cdot \frac{2\pi}{2S+1}\right).$$

The second order discrete differential operators

$$\mathbf{D_{xx}} = \begin{bmatrix} d_0^{(2)} & d_1^{(2)} & d_2^{(2)} & \cdots & d_R^{(2)} & d_R^{(2)} & \cdots & d_2^{(2)} & d_1^{(2)} \\ d_1^{(2)} & d_0^{(2)} & d_1^{(2)} & d_2^{(2)} & \cdots & d_R^{(2)} & d_R^{(2)} & \cdots & d_2^{(2)} \\ d_2^{(2)} & d_1^{(2)} & d_0^{(2)} & d_1^{(2)} & d_2^{(2)} & \cdots & d_R^{(2)} & \cdots & d_3^{(2)} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ d_R^{(2)} & \cdots & d_2^{(2)} & d_1^{(2)} & d_0^{(2)} & d_1^{(2)} & d_2^{(2)} & \cdots & d_R^{(2)} \\ d_R^{(2)} & d_R^{(2)} & \cdots & d_2^{(2)} & d_1^{(2)} & d_0^{(2)} & d_1^{(22)} & \cdots & d_{R-1}^{(2)} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ d_2^{(2)} & \cdots & d_R^{(2)} & d_R^{(2)} & \cdots & d_2^{(2)} & d_1^{(2)} & d_0^{(2)} & d_1^{(2)} \\ d_1^{(2)} & d_2^{(2)} & \cdots & d_R^{(2)} & d_R^{(2)} & \cdots & d_2^{(2)} & d_1^{(2)} & d_0^{(2)} \end{bmatrix};$$

$$d_r^{(2)} = -\frac{2}{2R+1} \sum_{k=1}^{S} \left(k^2 \cdot \cos\left(k \cdot r \cdot \frac{2\pi}{2S+1}\right)\right);$$

$$\mathbf{D_{yy}} = \begin{bmatrix} d_0^{(2)} & d_1^{(2)} & d_2^{(2)} & \cdots & d_S^{(2)} & d_S^{(2)} & \cdots & d_2^{(2)} & d_1^{(2)} \\ d_1^{(2)} & d_0^{(2)} & d_1^{(2)} & d_2^{(2)} & \cdots & d_S^{(2)} & d_S^{(2)} & \cdots & d_2^{(2)} \\ d_2^{(2)} & d_1^{(2)} & d_0^{(2)} & d_1^{(2)} & d_2^{(2)} & \cdots & d_S^{(2)} & \cdots & d_3^{(2)} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ d_S^{(2)} & \cdots & d_2^{(2)} & d_1^{(2)} & d_0^{(2)} & d_1^{(2)} & d_2^{(2)} & \cdots & d_S^{(2)} \\ d_S^{(2)} & d_S^{(2)} & \cdots & d_2^{(2)} & d_1^{(2)} & d_0^{(2)} & d_1^{(22)} & \cdots & d_{S-1}^{(2)} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ d_2^{(2)} & \cdots & d_S^{(2)} & d_S^{(2)} & \cdots & d_2^{(2)} & d_1^{(2)} & d_0^{(2)} & d_1^{(2)} \\ d_1^{(2)} & d_2^{(2)} & \cdots & d_S^{(2)} & d_S^{(2)} & \cdots & d_2^{(2)} & d_1^{(2)} & d_0^{(2)} \end{bmatrix};$$

$$d_s^{(2)} = -\frac{2}{2S+1} \sum_{k=1}^{S} \left(k^2 \cdot \cos\left(k \cdot s \cdot \frac{2\pi}{2S+1}\right)\right).$$

The matrices $\mathbf{C_{xx}}$ and $\mathbf{C_{yy}}$ have the same structures with the elements

$$c_r^{(2)} = \frac{2}{2R+1} \sum_{k=1}^{R} \frac{1}{k^2} \cdot \sin\left(k \cdot r \cdot \frac{2\pi}{2R+1}\right),$$

$$c_s^{(2)} = \frac{2}{2S+1} \sum_{k=1}^{S} \frac{1}{k} \cdot \sin\left(k \cdot s \cdot \frac{2\pi}{2S+1}\right).$$

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