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Application of the Lagrange-Sylvester formula to the computation of the solutions of state equations of fractional linear systems

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Abstract. The Lagrange-Sylvester formula is applied to the computation of the solutions of state equations of fractional continuous-time and discrete-time linear systems. The solutions are given as finite sums with their numbers of components equal to the degrees of the minimal characteristics polynomials of state matrices of the systems. Procedures for computations of the solutions are given and illustrated by numerical examples of continuous-time and discrete-time fractional linear systems.

Key words: computation; solution; Lagrange-Sylvester formula; fractional; continuous-time; discrete-time; linear; system.

1. Introduction

The first definition of the fractional derivative was introduced by Liouville and Riemann at the end of the 19th century [1–3] and another one was proposed in the 20th century by Caputo [1–5]. This idea has been used by engineers for modeling different processes [4,5]. Mathematical fundamentals of fractional calculus are given in the monographs [1,3]. The positive fractional linear systems have been investigated in [4,6–13]. The positive linear systems with different fractional orders have been addressed in [7, 8, 13]. The solution to the state equation of descriptor fractional continuous-time linear systems with two different fractional orders has been introduced in [8]. The decentralized stabilization of descriptor fractional positive continuous-time linear systems with delays has been investigated in [14] and the stabilization of positive descriptor fractional discrete-time linear systems with two different fractional orders by a decentralized controller in [13].

In this paper, the Lagrange–Sylvester formula will be applied to the computation of state equations of fractional continuous-time and discrete-time linear systems.

The paper is organized as follows. In Section 2 some preliminaries concerning fractional linear continuous-time and discrete-time systems are recalled and solutions to the state linear equations are given. The Lagrange–Sylvester formula is presented in Section 3. The main result of the paper, the application of the Lagrange–Sylvester formula to the computation of the solutions to state equations of the fractional linear systems is given in Section 4. Concluding remarks are given in Section 5.

The following notation will be used: $\Re$ – the set of real numbers; $\Re^{n\times m}$ – the set of $n\times m$ real matrices; I_n – the $n\times n$ identity matrix; A^T denotes the transpose of the matrix A.

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2. Preliminaries

Consider the continuous-time fractional linear system

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}x(t) = Ax(t)+Bu(t), \qquad 0<\alpha\le 1, \tag{1}$$

where $x(t)\in\Re^n$ is the state vector; $u(t)\in\Re^m$ is the input vector; and $A\in\Re^{n\times n}, B\in\Re^{n\times m}$. In this paper the Caputo definition will be used [1,3,4]

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}}{(t-\tau)^{\alpha+1-n}}d\tau, \qquad n-1<\alpha\le n\in N=\{1,2,\ldots\}, \tag{2}$$

where $\alpha\in\Re_+$ is the order of fractional derivative; $f^{(n)}(\tau) = \frac{\mathrm{d}^n f(\tau)}{\mathrm{d}\tau^n}$ and $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\mathrm{d}t$ is the gamma function; $\mathrm{Re}(x)>0$.

The solution to Eq. (1) has the form [4,5]

$$x(t) = \Phi_0(t)x_0(0)+\int_0^t \Phi(t-\tau)Bu(\tau)\,\mathrm{d}\tau, \tag{3}$$

where

$$\Phi_0(t) = \sum_{k=0}^\infty \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \tag{4}$$

$$\Phi(t) = \sum_{k=0}^\infty \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}. \tag{5}$$

Consider the fractional discrete-time linear system

$$\Delta^\alpha x_{i+1} = Ax_i+Bu_i, \quad i\in Z_+=\{0,1,\ldots\}, \quad 0<\alpha<1, \tag{6}$$

where

$$\Delta^{\alpha} x_i = \sum_{k=0}^{i} (-1)^k \binom{\alpha}{k} x_{i-k},$$

$$\binom{\alpha}{k} = \begin{cases} 1 & \text{for } k=0 \\ \dfrac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!} & \text{for } k=1,2,\ldots \end{cases} \tag{7}$$

is the α order difference and $x_i \in \Re^n$, $u_i \in \Re^m$ are the state and input vectors, $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$.

Substituting (7) into (6) we obtain

$$x_{i+1} + \sum_{k=1}^{i+1} (-1)^k \binom{\alpha}{k} x_{i-k+1} = Ax_i + Bu_i, \quad i \in Z_+ \tag{8}$$

and

$$x_{i+1} = A_\alpha x_i + \sum_{k=2}^{i+1} (-1)^{k+1} \binom{\alpha}{k} x_{i-k+1} + Bu_i, \quad i \in Z_+, \tag{9}$$

where $A_\alpha = A + \alpha I_n$.

Theorem 1. [1,4] The solution to Eq. (9) has the form

$$x_i = \Phi_i x_0 + \sum_{k=0}^{i-1} \Phi_{i-k-1} Bu_k, \tag{10}$$

where the matrices Φ_i are determined by the equation

$$\Phi_{i+1} = A_\alpha \Phi_i + \sum_{k=2}^{i+1} (-1)^{k+1} \binom{\alpha}{k} \Phi_{i-k+1}, \quad \Phi_0 = I_n. \tag{11}$$

Using (11) for $i = 1,2,\ldots$ it is easy to show that

$$\Phi_i = A_\alpha^i - (i-1)\binom{\alpha}{2} A_\alpha^{i-2} + (i-2)\binom{\alpha}{3} A_\alpha^{i-3} - \ldots + (-1)^{i-1} \binom{\alpha}{i} I_n \quad \text{for } i=1,2,\ldots \tag{12}$$

3. Lagrange–Sylvester formula

Consider the matrix $A \in \Re^{n\times n}$ with the minimal characteristic polynomial

$$\Psi(\lambda) = (\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\ldots(\lambda-\lambda_r)^{m_r}, \tag{13}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the eigenvalues of the matrix A and $\sum_{i=1}^{r} m_i = m \le n$. It is assumed that the function $f(\lambda)$ is well-defined on the spectrum $\sigma_A = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ of the matrix A, i.e.

$$\left. \begin{aligned} & f(\lambda_k),\ f^{(1)}(\lambda_k) = \left.\frac{\mathrm{d} f(\lambda)}{\mathrm{d}\lambda}\right|_{\lambda=\lambda_k}, \ldots, \\ & f^{(m_k-1)}(\lambda_k) = \left.\frac{\mathrm{d}^{m_k-1} f(\lambda)}{\mathrm{d}\lambda^{m_k-1}}\right|_{\lambda=\lambda_k} \end{aligned} \right\}, \quad k=1,\ldots,r \tag{14}$$

are finite [15,16].

In this case the matrix $f(A)$ is well-defined and it is given by the Lagrange–Sylvester formula [15,16]

$$f(A) = \sum_{i=1}^{r} Z_{i1} f(\lambda_i) + Z_{i2} f^{(1)}(\lambda_i) + \ldots + Z_{im_i} f^{(m_i-1)}(\lambda_i), \tag{15}$$

where

$$Z_{ij} = \sum_{k=j-1}^{m_i-1} \frac{\Psi_i(A)(A-\lambda_i I_n)^k}{(k-j+1)!(j-1)!} \frac{\mathrm{d}^{k-j+1}}{\mathrm{d}\lambda^{k-j+1}} \left[\frac{1}{\Psi_i(\lambda)}\right]_{\lambda=\lambda_i}, \tag{16}$$

and

$$\Psi_i(\lambda) = \frac{\Psi(\lambda)}{(\lambda-\lambda_i)^{m_i}}, \quad i=1,\ldots,r. \tag{17}$$

In a particular case when the eigenvalues λ_1, λ_2, …, λ_n of the matrix A are distinct ($\lambda_i \neq \lambda_j$, $i \neq j$) and

$$\phi(\lambda) = \Psi(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)\ldots(\lambda-\lambda_n), \tag{18}$$

then the formula (15) has the form

$$f(A) = \sum_{k=1}^{n} Z_k f(\lambda_k), \tag{19}$$

where

$$Z_k = \prod_{\substack{i=1 \\ i\neq k}}^{n} \frac{A-\lambda_i I_n}{\lambda_k - \lambda_i}. \tag{20}$$

It is easy to show [16] that the matrices (16) satisfy the equalities

$$\sum_{i=1}^{r} Z_{i1} = I_n, \tag{21}$$

$$Z_{ij}Z_{kl} = 0 \quad \text{for } i \neq k, \tag{22}$$

$$Z_{i1}Z_{kl} = Z_{ij} \quad \text{for } i=1,\ldots,r,\ j=1,\ldots,m, \tag{23}$$

$$Z_{i1}^k = Z_{i1} \quad \text{for } k=1,2\ldots,\ i=1,\ldots,r, \tag{24}$$

$$Z_{ij} = \frac{1}{(j-1)!}(A-I_n\lambda_i)^{j-1} Z_{i1} \quad \text{for } 1,\ldots,r, j=1,\ldots,m, \tag{25}$$

In particular case the matrices (20) satisfy the equalities

$$\sum_{k=1}^{n} Z_k = I_n, \tag{26}$$

$$Z_i Z_j = 0 \quad \text{for} \quad i \neq j, \quad i,j=1,\ldots,n, \tag{27}$$

$$Z_i^k = Z_i \quad \text{for} \quad k=1,2,\ldots,\ i=1,\ldots,n \tag{28}$$

4. Computation of the solutions to the state equations of fractional linear systems

4.1. Continuous-time linear systems. In this section, the Lagrange–Sylvester formula will be applied to compute the solution (3) to Eq. (1). Let the minimal characteristic polynomial of matrix A has the form (13). Applying the Lagrange–Sylvester formula (15) to (3) we obtain

$$x(t)=\sum_{i=1}^{r}Z_{i1}[M_{i1}^{(0)}(\lambda_i,t,x_0)+M_{i2}^{(0)}(\lambda_i,t,u)] + Z_{i2}\left[M_{i1}^{(1)}(\lambda_i,t,x_0)+M_{i2}^{(1)}(\lambda_i,t,u)\right]+\ldots + Z_{im_i}\left[M_{i1}^{(m_i-1)}(\lambda_i,t,x_0)+M_{i2}^{(m_i-1)}(\lambda_i,t,u)\right], \tag{29a}$$

where Z_{ij} is defined by (16) and

$$M_{i1}^{(j)}(\lambda_i,t,x_0)=\sum_{k=0}^{\infty}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\left.\frac{\mathrm{d}^j\lambda^k}{\mathrm{d}\lambda^j}\right|_{\lambda=\lambda_i}x_0,$$
$$M_{i2}^{(j)}(\lambda_i,t,u)=\sum_{k=0}^{\infty}\int_0^t\frac{(t-\tau)^{(k+1)\alpha-1}}{\Gamma(k\alpha+1)\alpha} \tag{29b}$$
$$1.5ex]\cdot\left.\frac{\mathrm{d}^j\lambda^k}{\mathrm{d}\lambda^j}\right|_{\lambda=\lambda_i}Bu(\tau)\,\mathrm{d}\tau$$

for $j=0,1,\ldots,m_{i-1}$, $\quad i=1,\ldots,r$.

The solution (3) to Eq. (1) can be computed by the use of the following procedure:

Procedure 1.

Step 1. Compute the minimal characteristic polynomial (13) of the matrix A and its eigenvalues $\lambda_1,\ldots,\lambda_r$ and $m_1,\ldots,m_r$.

Step 2. Compute the matrices Z_{ij} defined by (16).

Step 3. Using (29a) compute the desired solution (3) to Eq. (1).

In a particular case when the minimal characteristic polynomial of the matrix A has the form (18) then the solution (3) to Eq. (1) is given by

$$x(t)=\sum_{i=1}^{n}Z_k[M_0(\lambda_k,t,x_0)+M(\lambda_k,t,u)], \tag{30a}$$

where

$$M_0(\lambda_k,t,x_0)=\sum_{j=0}^{\infty}\frac{\lambda_k^j t^{j\alpha}}{\Gamma(j\alpha+1)}x_0,$$
$$M(\lambda_k,t,u)=\sum_{j=0}^{\infty}\int_0^t\frac{\lambda_k^j(t-\tau)^{(j+1)\alpha-1}}{\Gamma[(j+1)\alpha]}Bu(\tau)\,\mathrm{d}\tau, \tag{30b}$$

and Z_k is defined by (20).

From (16) the following conclusion follows.

Conclusion 1. The matrices Z_{ij} depend only on the matrix A.

Example 1. Using Procedure 1 compute the solution (3) to Eq. (1) with the matrices

$$\text{Case 1.}\quad A_1=\begin{bmatrix}-2 & 1\\ 2 & -3\end{bmatrix},\quad B_1=\begin{bmatrix}1\\2\end{bmatrix},\quad 0<\alpha<1, \tag{31a}$$

$$\text{Case 2.}\quad A_2=\begin{bmatrix}-2 & 1\\ 0 & -2\end{bmatrix},\quad B_2=\begin{bmatrix}1\\1\end{bmatrix},\quad 0<\alpha<1, \tag{31b}$$

the constant input $u(t)=1$ and nonzero initial conditions $x_0=[1,\ 2]^T$.

Applying Procedure 1 in Case 1 we obtain:

Step 1. The characteristic (minimal) polynomial of the matrix A given by (31a) has the form

$$\det[I_2\lambda-A_1]=\begin{vmatrix}\lambda+2 & -1\\ -2 & \lambda+3\end{vmatrix} =\lambda^2+5\lambda+4 \tag{32}$$

and its eigenvalues are: $\lambda_1=-1$, $\lambda_2=-4$.

Step 2. Using (20) we compute the matrices

$$Z_1=\frac{1}{\lambda_1-\lambda_2}[A_1-I_2\lambda_2]=\frac{1}{3}\begin{bmatrix}2 & 1\\ 2 & 1\end{bmatrix},$$
$$Z_2=\frac{1}{\lambda_2-\lambda_1}[A_1-I_2\lambda_1]=\frac{1}{3}\begin{bmatrix}1 & -1\\ -2 & 2\end{bmatrix}. \tag{33}$$

Step 3. Using (30b), the matrix B, u and the initial condition x_0 we compute

$$M_0(\lambda_1,t,x_0)=\sum_{j=0}^{\infty}\frac{(-1)^j t^{j\alpha}}{\Gamma(j\alpha+1)}x_0,$$

$$M_0(\lambda_2,t,x_0)=\sum_{j=0}^{\infty}\frac{(-4)^j t^{j\alpha}}{\Gamma(j\alpha+1)}x_0,$$

$$M(\lambda_1,t,u)=\sum_{j=0}^{\infty}\int_0^t\frac{(-1)^j(t-\tau)^{(j+1)\alpha-1}}{\Gamma[(j+1)\alpha]}Bu(\tau)\,\mathrm{d}\tau =\sum_{j=0}^{\infty}\begin{bmatrix}1\\2\end{bmatrix}\frac{(-1)^j t^{(j+1)\alpha}}{[(j+1)\alpha-1]\,\Gamma[(j+1)\alpha]}, \tag{34}$$

$$M(\lambda_2,t,u)=\sum_{j=0}^{\infty}\int_0^t\frac{(-4)^j(t-\tau)^{(j+1)\alpha-1}}{\Gamma[(j+1)\alpha]}Bu(\tau)\,\mathrm{d}\tau =\sum_{j=0}^{\infty}\begin{bmatrix}1\\2\end{bmatrix}\frac{(-4)^j t^{(j+1)\alpha}}{[(j+1)\alpha-1]\Gamma[(j+1)\alpha]}.$$

The desired solution is given by

$$x(t)=Z_1[M_0(\lambda_1,t,x_0)+M(\lambda_1,t,u)] +Z_2[M_0(\lambda_2,t,x_0)+M(\lambda_2,t,u)]. \tag{35}$$

Similarly, applying Procedure 1 in Case 2 we obtain:

Step 1. The characteristic (minimal) polynomial of the matrix A given by (31b) has the form

$$\det[I_2\lambda - A_2] = \begin{vmatrix} \lambda+2 & -1 \\ 0 & \lambda+2 \end{vmatrix} = (\lambda+2)^2 \tag{36}$$

and its eigenvalues are: $\lambda_1 = \lambda_2 = -2$.

Step 2. Using (16) we compute the matrices

$$\begin{aligned} Z_{11} &= \sum_{k=0}^{1} \Psi_1(A)(A-\lambda_1 I_2)^k \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\left[\frac{1}{\Psi_1(\lambda)}\right]_{\lambda=\lambda_1} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ Z_{12} &= \sum_{k=1}^{1} \Psi_1(A)(A-\lambda_1 I_2)^k \frac{\mathrm{d}^{k-1}}{\mathrm{d}\lambda^{k-1}}\left[\frac{1}{\Psi_1(\lambda)}\right]_{\lambda=\lambda_1} \\ &= A - \lambda I_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{37}$$

since $\Psi_1(\lambda) = \dfrac{(\lambda-1)^2}{(\lambda-1)^2} = 1$.

Step 3. Using (30b), the matrix B, u and the initial condition x_0 we compute

$$\begin{aligned} M_0(\lambda_1,t,x_0) &= \sum_{j=0}^{\infty} \frac{(-2)^j t^{j\alpha}(j+1)}{\Gamma(j\alpha+1)} x_0, \\ M(\lambda_1,t,u) &= \sum_{j=0}^{\infty} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \int_0^t \frac{(-2)^{j-1}(1-2)\tau^{(j+1)\alpha-1}}{\Gamma(j\alpha+1)}\,\mathrm{d}\tau. \end{aligned} \tag{38}$$

The desired solution is given by

$$x(t) = Z_{11}M_0(\lambda_1,t,x_0) + Z_{12}M(\lambda_1,t,u). \tag{39}$$

4.2. Discrete-time linear systems. In this section, the Lagrange–Sylvester formula will be applied to compute the solution (10) to Eq. (9). Let the minimal characteristic polynomial of the matrix A have the form (13). Applying the Lagrange–Sylvester formula (15) to (10) we obtain

$$\begin{aligned} x_i = &\sum_{k=1}^{r} Z_{k1}\left[\overline{M}_{k1}^{(0)}(\lambda_k,i,x_0) + \overline{M}_{k2}^{(0)}(\lambda_k,i,u_i)\right] \\ &+ Z_{k2}\left[\overline{M}_{k2}^{(1)}(\lambda_k,t,x_0) + \overline{M}^{(1)}(\lambda_k,i,u)\right] + \ldots \\ &+ Z_{km_k}\left[\overline{M}_{k1}^{(m_k-1)}(\lambda_k,i,x_0) + \overline{M}_{k2}^{(m_k-1)}(\lambda_k,i,u_i)\right] \\ &\text{for } i = 1,2,\ldots, \end{aligned} \tag{40a}$$

where Z_{kj} is defined by (16) and

$$\begin{aligned} \overline{M}_{k1}^{(j)}(\lambda_k,i,x_0) &= \left[\lambda_k^i(i-1)\binom{\alpha}{2}\lambda_k^{i-2} + (i-2)\binom{\alpha}{3}\lambda_k^{i-3} - \ldots + (-1)^{i-1}\binom{\alpha}{i}\right] x_0, \\ \overline{M}_{k2}(\lambda_j,i,u_i) &= \sum_{j=0}^{i-1}\left[\lambda_k^{i-k-1} - (i-k-2)\binom{\alpha}{2}\lambda_k^{i-k-2} + (i-k-3)\binom{\alpha}{3}\lambda_k^{i-k-3} - \ldots + (-1)^{i-k-2}\binom{\alpha}{i-k-1}\right] Bu_j \\ &\text{for } j = 0,1,\ldots m_k-1,\ k = 1,\ldots,r. \end{aligned} \tag{40b}$$

The solution (10) to Eq. (7) (or (9)) can be computed by the use of the following procedure:

Procedure 2.

Step 1. Compute the minimal characteristic polynomial (13) of the matrix A and its eigenvalues and $m_1,\ldots,m_r$.

Step 2. Compute the matrices Z_{ij} using (16).

Step 3. Using (40a) to compute the desired solution (10) to Eq. (7).

In a particular case when the minimal characteristic polynomial of the matrix A has the form (18) then the solution (10) to Eq. (7) is given by

$$x_i = \sum_{k=1}^{n} Z_k[\overline{M}_0(\lambda_k,i,x_0) + \overline{M}(\lambda_k,i,u)], \quad i = 1,2,\ldots, \tag{41a}$$

where

$$\begin{aligned} \overline{M}_0(\lambda_k,i,x_0) &= \left[\begin{matrix} \lambda_k^i - (i-1)\binom{\alpha}{2}\lambda_k^{i-2} + (i-2)\binom{\alpha}{3}\lambda_k^{i-3} \\ + \ldots + (-1)^{i-1}\binom{\alpha}{i} \end{matrix}\right] x_0, \\ \overline{M}(\lambda_k,t,u) &= \sum_{l=0}^{i-1}\left[\lambda_k^{i-l-1} - (i-l-2)\binom{\alpha}{2}\lambda_k^{i-l-3} + (i-l-3)\binom{\alpha}{3}\lambda_k^{i-l-4} - \ldots + (-1)\binom{\alpha}{i-l-1}\right] Bu_l \\ &\text{for } i = 1,2,\ldots,\ k = 1,\ldots,n, \end{aligned} \tag{41b}$$

Example 2. Using Procedure 2 compute the solution (10) to Eq. (9) with the matrices

$$\begin{aligned} A &= \begin{bmatrix} 0.1 & 0.2 \\ 0.05 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \alpha = 0.3, \\ A_\alpha &= A + I_n\alpha = \begin{bmatrix} 0.4 & 0.2 \\ 0.05 & 0.4 \end{bmatrix}, \end{aligned} \tag{42}$$

the constant input $u_i = 1$ and nonzero initial conditions $x_0 = [2,\ 2]^T$.

Applying Procedure 2 we obtain:

Step 1. The characteristic (minimal) polynomial of the matrix A given by (42) has the form

$$\det[I_2\lambda - A_\alpha] = \begin{vmatrix} \lambda - 0.4 & -0.2 \\ -0.05 & \lambda - 0.4 \end{vmatrix} = \lambda^2 - 0.8\lambda + 0.15 \tag{43}$$

and its eigenvalues are: $\lambda_1 = 0.3$, $\lambda_2 = 0.5$.

Step 2. Using (42) we compute the matrices

$$\begin{aligned} Z_1 &= \frac{A_\alpha - I_2\lambda_2}{\lambda_1 - \lambda_2} = \begin{bmatrix} 0.5 & -1 \\ -0.25 & 0.5 \end{bmatrix}, \\ Z_2 &= \frac{A_\alpha - I_2\lambda_1}{\lambda_2 - \lambda_1} = \begin{bmatrix} 0.5 & 1 \\ 0.25 & 0.5 \end{bmatrix}. \end{aligned} \tag{44}$$

Step 3. Using (30b), the matrix B, $u_i = 1$ and the initial condition x_0 we compute

$$\begin{aligned} \overline{M}_0(\lambda_k, i, x_0) &= \left[\lambda_j^i - (l-1)\binom{\alpha}{2}\lambda_k^{l-2} \right. \\ &\quad \left. + (l-2)\binom{\alpha}{3}\lambda_k^{l-3} - \ldots + (-1)^{i-1}\binom{\alpha}{i}\right] x_0, \\ \overline{M}(\lambda_j, i, u) &= \sum_{l=0}^{i-1}\left[\lambda_j^{i-l-1} - (i-l-2)\binom{\alpha}{2}\lambda_j^{i-l-3} \right. \\ &\quad + (i-l-3)\binom{\alpha}{3}\lambda_j^{i-l-4} - \ldots \\ &\quad \left. + (-1)\binom{\alpha}{i-l-1}\right] Bu_l, \quad j = 1,2 \end{aligned} \tag{45}$$

and the desired solution

$$\begin{aligned} x_i &= Z_1[M_0(\lambda_1, i, x_0) + M(\lambda_1, i, u)] \\ &\quad + Z_2[M_0(\lambda_2, i, x_0) + M(\lambda_2, i, u)]. \end{aligned} \tag{46}$$

5. Concluding remarks

The Lagrange–Sylvester formula has been applied to the computation of the solutions of state equations of fractional continuous-time and discrete-time linear systems. The solutions have been given as the finite sums of the components with their numbers equal to the degrees of the minimal characteristic polynomials of the state matrices of the systems. Procedures for computations of the solutions have been given and illustrated by numerical examples of continuous-time and discrete-time fractional linear systems. The considerations can be extended to fractional linear systems with delays and to different fractional orders continuous-time and discrete-time linear systems.

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