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Sufficient conditions for uniform global asymptotic stabilization of affine discrete-time systems with periodic coefficients

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Affine discrete-time control periodic systems are considered. The problem of global asymptotic stabilization of the zero equilibrium of the closed-loop system by state feedback is studied. It is assumed that the free dynamic system has the Lyapunov stable zero equilibrium. The method for constructing a damping control is extended from time-invariant systems to time varying periodic affine discrete-time systems. By using this approach, sufficient conditions for uniform global asymptotic stabilization for those systems are obtained. Examples of using the obtained results are presented.

Key words: discrete-time systems, periodic systems, affine systems, uniform global asymptotic stabilization, state feedback

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1. Introduction

Consider a nonlinear control discrete-time system

$$x(t+1) = f(t, x(t), u(t)). \tag{1}$$

Here $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^r$ is the control input, $t \in \mathbb{Z}$, $f(t, 0, 0) \equiv 0$. We investigate the problem of asymptotic stabilization of system (1): one needs to construct a feedback control $u(t) = \widehat{u}(t, x(t))$ with $\widehat{u}(t, 0) \equiv 0$ in system (1) such that the equilibrium $x = 0$ of the closed-loop system

$$x(t+1) = f(t, x(t), \widehat{u}(t, x(t)))$$

is asymptotically stable.

Stabilization problems for nonlinear time-invariant discrete-time systems were studied in [1–9]. Pole placement problems for time-varying discrete-time systems were studied in [10] for periodic systems, and in [11–14] for arbitrary non-periodic time-varying systems.

In [2], sufficient conditions for global asymptotic stabilization of bilinear discrete time-invariant systems were obtained. In [3], sufficient conditions were obtained for global asymptotic stabilization of affine discrete time-invariant systems (1) ($f(t, x, u) = f(x) + g(x)u$), see also [1]. In the paper [4], sufficient conditions for global asymptotic stabilization were obtained for general discrete time-invariant nonlinear systems (1) ($f(t, x, u) \equiv f(x, u)$). In the present research, we extend the results of [1] to affine time-varying periodic discrete systems. We use the Krasovsky–LaSalle principle for periodic discrete-time systems. The corresponding results have been obtained before for bilinear discrete-time homogeneous (see [15]) and non-homogeneous (see [16]) periodic systems. For systems with continuous time, similar results have been obtained in [17] and [18].

2. Main results

Consider a discrete-time affine time-varying system

$$x(t+1) = f(t, x(t)) + g(t, x(t))u(t), \quad (t, x, u) \in \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^r, \tag{2}$$

where $f : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^n$, $f(t, 0) \equiv 0$, $g : \mathbb{Z} \times \mathbb{R}^n \to M_{n,r}$. Here $M_{n,r}$ is the space of real $n \times r$-matrices. Denote $M_n := M_{n,n}$; T is the transposition; $I \in M_n$ the identity matrix. We suppose that f and g are continuous in x. We understand inequalities for symmetric matrices in the sense of quadratic forms. Consider the corresponding free system

$$x(t+1) = f(t, x(t)). \tag{3}$$

Let $\xi(t) = \xi(t, t_0, x_0)$, $t \geqslant t_0$ ($t_0 \in \mathbb{Z}$), be a solution of system (3) with an initial condition $\xi(t_0) = x_0$. Let us denote

$$f^0(t_0, x_0) := x_0, \qquad f^1(t_0, x_0) := f(t_0, x_0),$$
$$f^2(t_0, x_0) := f(t_0 + 1, f(t_0, x_0)), \quad \dots, \quad f^{i+1}(t_0, x_0) := f(t_0 + i, f^i(t_0, x_0)).$$

Then $\xi(t_0 + i) = f^i(t_0, x_0)$, $i = 0, 1, 2, \dots$.

Suppose that system (2) is periodic, i.e., there exists an $\omega \in \mathbb{N}$ such that $f(t + \omega, x) = f(t, x)$, $g(t + \omega, x) = g(t, x)$ for all $t \in \mathbb{Z}$, $x \in \mathbb{R}^n$. Suppose that there exists a $P(t) \in M_n$ satisfying conditions

$$P(t + \omega) = P(t), \qquad P^T(t) = P(t) > 0, \tag{4}$$

and the following condition holds for all $t \in \mathbb{Z}$, $x \in \mathbb{R}^n$:

$$f^T(t, x)P(t + 1)f(t, x)c \leqslant x^T P(t)x. \tag{5}$$

Conditions (4) and (5) ensure the Lyapunov (non-asymptotic) stability of the equilibrium $x = 0$ for system (3).

Let us construct the Lyapunov function

$$V(t, x) = x^T P(t)x, \qquad x \in \mathbb{R}^n, \quad P(t) \in M_n, \tag{6}$$

with $P(t)$ satisfying (4) and (5) for all $t \in \mathbb{Z}$, $x \in \mathbb{R}^n$. Set

$$G(t, x) = I + \frac{1}{2} g^T(t, x)P(t + 1)g(t, x), \tag{7}$$

where $I \in M_r$. Then $G(t, x) \in M_r$, $G(t, x) = G^T(t, x) \geqslant I > 0$, $G(t, x) = G(t + \omega, x)$, and $G^{-1}(t, x)$ is defined for all $t \in \mathbb{Z}$, $x \in \mathbb{R}^n$. Let us construct the control function

$$\widehat{u}(t, x) = -G^{-1}(t, x)g^T(t, x)P(t + 1)f(t, x). \tag{8}$$

Then $\widehat{u}(t + \omega, x) = \widehat{u}(t, x)$, $t \in \mathbb{Z}$, $x \in \mathbb{R}^n$, $\widehat{u}(t, 0) \equiv 0$, and

$$\left(g^T(t, x)P(t + 1)f(t, x)\right)^T = -\widehat{u}^T(t, x)G(t, x). \tag{9}$$

Let us substitute

$$u(t) = \widehat{u}(t, x(t)) \tag{10}$$

into (2). The closed-loop system has the form

$$x(t + 1) = f(t, x(t)) + g(t, x(t))\widehat{u}(t, x(t)). \tag{11}$$

Denote by $F(t, x(t))$ the right-hand side of (11). Then we have $F(t+\omega, x) = F(t, x), t \in \mathbb{Z}, x \in \mathbb{R}^n$, i.e., (11) is a periodic system. Let us consider the difference $\Delta_F V(t, x(t)) = V(t+1, x(t+1)) - V(t, x(t))$ of the Lyapunov function (6) along trajectories of (11). We have

$$\begin{aligned}\Delta_F V(t, x(t)) = {} & f^T(t, x(t))P(t+1)f(t, x(t)) \\ & - x^T(t)P(t)x(t) + 2\mu(t, x(t))\widehat{u}(t, x(t)),\end{aligned}$$

where

$$\mu(t, x) = f^T(t, x)P(t+1)g(t, x) + \frac{1}{2}\widehat{u}^T(t, x)g^T(t, x)P(t+1)g(t, x). \tag{12}$$

Substituting (9) for the first summand in (12) and taking into account (7), we obtain that $\mu(t, x) = -\widehat{u}^T(t, x)$. Thus,

$$\begin{aligned}\Delta_F V(t, x(t)) = {} & f^T(t, x(t))P(t+1)f(t, x(t)) \\ & - x^T(t)P(t)x(t) - 2\widehat{u}^T(t, x(t))\widehat{u}(t, x(t)).\end{aligned} \tag{13}$$

Hence, by (5), we obtain that $\Delta_F V(t, x(t)) \leqslant 0$. Thus, the zero equilibrium of (11) is Lyapunov stable. Let us consider the set

$$E(V) = \{(t, x) \in \mathbb{Z} \times \mathbb{R}^n \colon \Delta_F V(t, x) = 0\}.$$

By (13) and (5), the set $E(V)$ coincides with

$$\widehat{E}(V) = \{(t, x) \in \mathbb{Z} \times \mathbb{R}^n \colon f^T(t, x)P(t+1)f(t, x) - x^T P(t)x = |\widehat{u}(t, x)| = 0\}.$$

Set

$$\begin{aligned}\Omega_0(V) &= \{(t, x) \in \mathbb{Z} \times \mathbb{R}^n \colon f^T(t, x)P(t+1)f(t, x) = x^T P(t)x\}, \\ S_0(V) &= \{(t, x) \in \mathbb{Z} \times \mathbb{R}^n \colon f^T(t, x)P(t+1)g(t, x) = 0\}, \\ E_0(V) &= \Omega_0(V) \cap S_0(V).\end{aligned}$$

By (8), $\widehat{E}(V) = E_0(V)$. Denote by $M(V)$ the largest positive invariant set of (11) relative to $E(V)$, i.e., $M(V)$ is the union of all semi-trajectories $x(t)$, $t \geqslant t_0$ ($t_0 \in \mathbb{Z}$), of (11) such that $(t, x(t)) \in E(V)$ for all $t \geqslant t_0$. Then we have $0 \in M(V)$ because $\xi_0(t) \equiv 0$, $t \geqslant t_0$, is a solution of (11) and $\Delta_F V(t, 0) \equiv 0$, $t \geqslant t_0$. If $M(V) = \{0\}$, then the zero equilibrium of (11) is uniformly globally asymptotically (UGA) stable due to the Krasovsky–La Salle invariance principle for discrete-time periodic systems. The UGA stability of the zero equilibrium means that (see, e.g., [19, Ch. I, Sect. 2.11], [20, Sect. 13.6]) it is uniformly stable and uniformly globally attractive.

Denote by $\xi(t) = \xi(t, t_0, x_0)$, $t \geqslant t_0$ ($t_0 \in \mathbb{Z}$), a solution of system (11) with an initial condition $\xi(t_0) = x_0$. Suppose that $\xi(t) \in M(V)$, $t \geqslant t_0$. Since $E(V) = \widehat{E}(V) = E_0(V)$, we obtain

$$f^T(t, \xi(t))P(t+1)f(t, \xi(t)) = \xi^T(t)P(t)\xi(t), \quad t \geqslant t_0, \tag{14}$$

$$f^T(t, \xi(t))P(t+1)g(t, \xi(t)) = 0, \quad t \geqslant t_0, \tag{15}$$

and hence, by (8), $\widehat{u}(t, \xi(t)) = 0$, $t \geqslant t_0$. By (11), $\xi(t+1) = f(t, \xi(t))$, $t \geqslant t_0$, i.e., $\xi(t)$ is a solution of the free system (3). Hence, $\xi(t_0+i) = f^i(t_0, x_0)$. Substituting this equality into (14), (15), we get the equalities

$$\begin{aligned}\left(f^{i+1}(t_0, x_0)\right)^T P(t_0+i+1)\left(f^{i+1}(t_0, x_0)\right) &= \\ = \left(f^i(t_0, x_0)\right)^T P(t_0+i)\left(f^i(t_0, x_0)\right)&, \quad i \geqslant 0,\end{aligned} \tag{16}$$

$$\left(f^{i+1}(t_0, x_0)\right)^T P(t_0+i+1)g(t_0+i, f^i(t_0, x_0)) = 0, \quad i \geqslant 0. \tag{17}$$

Denote by $M_0(V)$ the largest positive invariant set of the free system (3) relative to $E_0(V)$. Thus, we obtain that $\xi(t) \in M_0(V)$, $t \geqslant t_0$. It follows that $M(V) \subset M_0(V)$. Therefore, if $M_0(V) = \{0\}$, then $M(V) = \{0\}$. The condition $M_0(V) = \{0\}$ means that for any $t_0 \in \mathbb{Z}$ identities (16), (17) hold only if $x_0 = 0$. In the last sentence, the phrase "for any $t_0 \in \mathbb{Z}$" can be replaced by the phrase "for some $t_0 \in \mathbb{Z}$". Let us proof this fact.

Lemma 1 *The following statements are equivalent.*
1. *For any $t_0 \in \mathbb{Z}$ identities* (16)*,* (17) *hold only if $x_0 = 0$.*
2. *For some $t_0 \in \mathbb{Z}$ identities* (16)*,* (17) *hold only if $x_0 = 0$.*

Proof. The implication ($1 \Rightarrow 2$) is obvious. Let us prove the implication ($2 \Rightarrow 1$). Suppose that for some $t_0 \in \mathbb{Z}$ identities (16), (17) hold only if $x_0 = 0$. Consider an arbitrary $t_1 \in \mathbb{Z}$. There exists a $k \in \mathbb{N}$ such that $t_0 + k\omega \geqslant t_1$. Denote $t_2 := t_0 + k\omega$, $j_0 := t_2 - t_1 \geqslant 0$. Suppose that identities

$$\begin{aligned}\left(f^{i+1}(t_1, x_1)\right)^T P(t_1+i+1)\left(f^{i+1}(t_1, x_1)\right) &= \\ = \left(f^i(t_1, x_1)\right)^T P(t_1+i)\left(f^i(t_1, x_1)\right)&,\end{aligned} \tag{18}$$

$$\left(f^{i+1}(t_1, x_1)\right)^T P(t_1+i+1)g(t_1+i, f^i(t_1, x_1)) = 0 \tag{19}$$

hold for $i \geqslant 0$. Then (18), (19) hold for $i \geqslant j_0$ as well. Denote $x_2 = f^{j_0}(t_1, x_1)$, $j := i - j_0 \geqslant 0$. We have

$$f^i(t_1, x_1) = f^{j+j_0}(t_1, x_1) = f^j(t_1 + j_0, f^{j_0}(t_1, x_1)) = f^j(t_2, x_2).$$

Similarly,

$$
\begin{aligned}
g(t_1+i, f^i(t_1,x_1) = g(t_1+j_0+j, f^j(t_2,x_2)) = g(t_2+j, f^j(t_2,x_2)),\\
P(t_1+i) = P(t_1+j_0+j) = P(t_2+j),\\
P(t_1+i+1) = P(t_2+j+1).
\end{aligned}
$$

Hence, it follows from (18), (19) that identities

$$
\begin{aligned}
\Big(f^{j+1}(t_2,x_2)\Big)^T P(t_2+j+1)\Big(f^{j+1}(t_2,x_2)\Big) = \\
= \Big(f^{j}(t_2,x_2)\Big)^T P(t_2+j)\Big(f^{j}(t_2,x_2)\Big),
\end{aligned}
\tag{20}
$$

$$
\Big(f^{j+1}(t_2,x_2)\Big)^T P(t_2+j+1) g(t_2+j, f^j(t_2,x_2)) = 0 \tag{21}
$$

hold for $j \geqslant 0$. Since system (2) and the matrix $P(t)$ are ω-periodic, it follows from (20), (21) that identities

$$
\begin{aligned}
\Big(f^{j+1}(t_0,x_2)\Big)^T P(t_0+j+1)\Big(f^{j+1}(t_0,x_2)\Big) = \\
= \Big(f^{j}(t_0,x_2)\Big)^T P(t_0+j)\Big(f^{j}(t_0,x_2)\Big),
\end{aligned}
$$

$$
\Big(f^{j+1}(t_0,x_2)\Big)^T P(t_0+j+1) g(t_0+j, f^j(t_0,x_2)) = 0
$$

hold for $j \geqslant 0$. By assumption, it follows that $x_2 = 0$, i.e.,

$$
f^{j_0}(t_1,x_1) = 0. \tag{22}
$$

Consider equalities (18) at $i = 0, \dots, j_0 - 1$. From these equalities, we obtain

$$
\begin{aligned}
x_1^T P(t_1) x_1 = \Big(f^1(t_1,x_1)\Big)^T P(t_1+1)\Big(f^1(t_1,x_1)\Big) = \dots \\
\dots = \Big(f^{j_0}(t_1,x_1)\Big)^T P(t_1+j_0)\Big(f^{j_0}(t_1,x_1)\Big).
\end{aligned}
\tag{23}
$$

It follows from (22) and (23) that $x_1^T P(t_1) x_1 = 0$. Since $P(t) > 0$ for all $t \in \mathbb{Z}$, we have $x_1 = 0$. The lemma is proved. □

Thus, the following theorem is proved.

Theorem 1 *Let system* (2) *be ω-periodic. Suppose that there exists a matrix $P(t)$ satisfying conditions* (4), (5) *for all $t \in \mathbb{Z}$, $x \in \mathbb{R}^n$. Suppose that for some $t_0 \in \mathbb{Z}$ identities* (16), (17) *hold only if $x_0 = 0$. Then the state feedback control* (10), (8) *UGA stabilizes the origin of* (2).

Remark 1 *Suppose that $\omega = 1$. Thus, $f(t,x) \equiv f(x)$, $g(t,x) \equiv g(x)$, i.e., system (2) is time-invariant, and $P(t) \equiv P$. Then Theorem 1 coincides with [1, Theorem 3.1]. Thus, Theorem 1 is a generalization of [1, Theorem 3.1] on global asymptotic stabilization from time-invariant systems to time-varying periodic systems.*

Remark 2 *The question of when conditions (4) and (5) are satisfied is an important and difficult one. In fact, this is the question of the existence of the Lyapunov function. Conditions (4) and (5) are sufficient conditions for Lyapunov (non-asymptotic) stability of the free system (3). Suppose that system (3) is linear, i.e., $f(t,x) = A(t)x$ that is system (3) has the form*

$$x(t+1) = A(t)x(t), \tag{24}$$

where $A(t+\omega) = A(t)$. Then condition of Lyapunov (non-asymptotic) stability is both necessary and sufficient for the fulfillment of conditions (4) and (5). This fact was proved, e.g., in [15], under the assumption that the linear periodic system (24) is reducible. It can be shown that the reducibility requirement can be removed. Moreover, the method for constructing the matrix $P(t)$ for the linear system (24) is constructive. This construction method makes it possible to find a wide class C of periodic matrices $P(t)$ satisfying condition (4) and the condition

$$A^T(t)P(t+1)A(t) - P(t) \leqslant 0. \tag{25}$$

The size of this class depends on the number k of multipliers of system (24) lying strictly inside the unit circle. The larger k, the wider the class C.

In the general case, if the system is nonlinear, then the condition of (non-asymptotic) stability is not sufficient for the existence of a nonstrict Lyapunov function, and even more so in the form of a quadratic function. How to construct a Lyapunov function satisfying conditions (4) and (5) in the general case is an open question. One method might be as follows. Using the system (3), we construct system (24) of linear approximation. Next, we construct a class C of matrices $P(t)$ satisfying conditions (4) and (25). From this class, one can try to choose the matrix $P(t)$ satisfying conditions (4) and (5). We apply this method, in particular, further in Examples 1 and 2.

Next, we obtain sufficient conditions for the equality $M_0(V) = \{0\}$ to be fulfilled. Suppose that $f(t,x)$ is of class C^1 in x. Let us define $K(t,x) = \dfrac{\partial f(t,x)}{\partial x}$. Let us construct the following matrices:

$$N_1(\tau,x) = g(\tau,x),$$

$$N_{i+1}(\tau,x) = \left[K(\tau+i, f^i(\tau,x)) \cdot N_i(\tau,x), g(\tau+i, f^i(\tau,x))\right], \quad i \geqslant 1.$$

We have $N_i(\tau+\omega,x) = N_i(\tau,x) \in M_{n,ir}$ for any $i \in \mathbb{N}$, $\tau \in \mathbb{Z}$, $x \in \mathbb{R}^n$.

Theorem 2 *Let system (2) be ω-periodic. Let f be of class C^1 in x. Suppose that there exists a matrix $P(t)$ satisfying conditions (4), (5) for all $t \in \mathbb{Z}$, $x \in \mathbb{R}^n$. Suppose that the following condition holds:*

$$\exists\, t_0 \in \mathbb{Z} \;\; \forall x \in \mathbb{R}^n \setminus \{0\} \;\; \exists\, \nu \geqslant 1 \;\; \operatorname{rank} N_\nu(t_0, x) = n. \tag{26}$$

Then the state feedback control (10), (8) UGA stabilizes the origin of (2).

Proof. Let us prove that $M_0(V) = \{0\}$ under the assumptions of the theorem. Then, by the proof of Theorem 1, the theorem will be proved. We prove this statement by contradiction. Suppose $M_0(V) \neq \{0\}$. Then there exist $t_1 \in \mathbb{Z}$ and $x_1 \in \mathbb{R}^n$, $x_1 \neq 0$, such that the solution $\xi(t) = \xi(t, t_1, x_1)$ of the free system (3) with the initial condition $\xi(t_1) = x_1$ satisfies the condition $\xi(t, t_1, x_1) \in M_0(V)$ for all $t \geqslant t_1$. Hence, identities (18), (19) hold for $i \geqslant 0$. Let us show that $\xi(t, t_1, x_1) \neq 0$ for any $t \geqslant t_1$. Suppose that $\xi(t_2, t_1, x_1) = 0$ for some $t_2 \geqslant t_1$. Hence, $f^{t_2 - t_1}(t_1, x_1) = 0$. By (18) at $i = 0, \ldots, t_2 - t_1 - 1$, we obtain, similarly to (23), the following equalities:

$$\begin{aligned} x_1^T P(t_1) x_1 &= \left(f^1(t_1, x_1)\right)^T P(t_1 + 1)\left(f^1(t_1, x_1)\right) = \ldots \\ \ldots &= \left(f^{t_2 - t_1}(t_1, x_1)\right)^T P(t_2)\left(f^{t_2 - t_1}(t_1, x_1)\right) = 0. \end{aligned}$$

Since $P(t_1) > 0$, we have $x_1 = 0$. This is contradiction. Hence, $\xi(t, t_1, x_1) \neq 0$, $t \geqslant t_1$. Let us construct the number $t_0 \in \mathbb{Z}$ from condition (26). By periodicity of $N_i(\tau, x)$, one can assume without loss of generality that $t_0 > t_1$. Set $x_0 = \xi(t_0, t_1, x_1)$. Hence,

$$\xi(t, t_0, x_0) \neq 0, \qquad t \geqslant t_0. \tag{27}$$

Let us construct for $x_0 \neq 0$ the number $\nu \geqslant 1$ from condition (26) such that $\operatorname{rank} N_\nu(t_0, x_0) = n$. Since $\xi(t, t_0, x_0) \in M_0(V)$ for all $t \geqslant t_0$, equalities (16), (17) hold.

Consider the function

$$\varphi(t, x) = f^T(t, x) P(t + 1) f(t, x) - x^T P(t) x.$$

We have $\Omega_0(V) = \{(t, x) \in \mathbb{Z} \times \mathbb{R}^n : \varphi(t, x) = 0\}$. The function $\varphi(t, x)$ attains its maximum at any point $(\widetilde{t}, \widetilde{x}) \in \Omega_0(V)$ because $\varphi(t, x) \leqslant 0$ for all $(t, x) \in \mathbb{Z} \times \mathbb{R}^n$, by (5). Consequently, $(\partial\varphi/\partial x)(\widetilde{t}, \widetilde{x}) = 0$ for any $(\widetilde{t}, \widetilde{x}) \in \Omega_0(V)$. We have

$$\begin{aligned} \frac{\partial \varphi(t, x)}{\partial x} &= 2 f^T(t, x) P(t + 1) \frac{\partial f(t, x)}{\partial x} - 2 x^T P(t) \\ &= 2\left(f^T(t, x) P(t + 1) K(t, x) - x^T P(t)\right). \end{aligned}$$

Therefore for any $m \in \mathbb{N}$ and for any function $(s, y) \mapsto z(s, y) \in M_{n,m}$ the equality

$$\begin{gathered}\left(f^T(\widetilde{t}, \widetilde{x}) P(\widetilde{t}+1) K(\widetilde{t}, \widetilde{x}) - \widetilde{x}^T P(\widetilde{t})\right) z(s, y) = 0,\\ (\widetilde{t}, \widetilde{x}) \in \Omega_0(V), \qquad s \in \mathbb{Z}, \qquad y \in \mathbb{R}^n,\end{gathered} \tag{28}$$

holds. Equality (17) for $i = 0$ implies that

$$f^T(t_0, x_0) P(t_0 + 1) g(t_0, x_0) = 0.$$

This means that the row vector $f^T(t_0, x_0)P(t_0 + 1)$ is orthogonal to the columns of the matrix $N_1(t_0, x_0)$.

Let us prove, by induction, the following assertion ($\mathcal{A}$):

for all $k \in \mathbb{N}$ the row vector $\left(f^k(t_0, x_0)\right)^T P(t_0 + k)$ is orthogonal to the columns of the matrix $N_k(t_0, x_0)$.

The basis for $k = 1$ is proved. Assume ($\mathcal{A}$) holds for $k = i$, i.e.,

$$\left(f^i(t_0, x_0)\right)^T P(t_0 + i) N_i(t_0, x_0) = 0. \tag{29}$$

By (16), the relation $(t, \xi(t, t_0, x_0)) \in \Omega_0(V)$ holds for all $t \geqslant t_0$. Substituting $i \cdot r$ for m, t_0 for s, x_0 for y, $t_0 + i$ for $\widetilde{t}$, $\xi(t_0 + i, t_0, x_0) =: f^i(t_0, x_0)$ for $\widetilde{x}$, $N_i(t_0, x_0)$ for $z(s, y)$ in (28), we get

$$\begin{aligned}\Big[\left(f(t_0 + i, f^i(t_0, x_0)\right)^T &P(t_0 + i + 1) K(t_0 + i, f^i(t_0, x_0))\\ &- \left(f^i(t_0, x_0)\right)^T P(t_0 + i)\Big] N_i(t_0, x_0) = 0.\end{aligned}$$

Taking into account the induction assumption (29), we obtain

$$\left(f^{i+1}(t_0, x_0)\right)^T P(t_0 + i + 1) K(t_0 + i, f^i(t_0, x_0)) \cdot N_i(t_0, x_0) = 0. \tag{30}$$

It follows from (30) and (17) that

$$\begin{aligned}&\left(f^{i+1}(t_0, x_0)\right)^T P(t_0 + i + 1)\cdot\\ &\qquad \cdot [K(t_0 + i, f^i(t_0, x_0)) N_i(t_0, x_0), g(t_0 + i, f^i(t_0, x_0))] = 0.\end{aligned}$$

Thus, assertion ($\mathcal{A}$) holds for $k = i + 1$. By induction, assertion ($\mathcal{A}$) holds for all $k \in \mathbb{N}$. In particular, for $k = \nu$, we have

$$\left(f^\nu(t_0, x_0)\right)^T P(t_0 + \nu) N_\nu(t_0, x_0) = 0.$$

Since rank $N_\nu(t_0, x_0) = n$, we obtain

$$\left(f^\nu(t_0, x_0)\right)^T P(t_0 + \nu) = 0.$$

Since $P(t) > 0$, we get $f^\nu(t_0, x_0) = 0$, i.e., $\xi(t_0 + \nu, t_0, x_0) = 0$. This contradicts (27). The theorem is proved. □

Remark 3 *Theorem 1 and Theorem 2 generalize results obtained before for periodic bilinear homogeneous [15, Theorem 8 and Theorem 10] and non-homogeneous [16, Theorem 1 and Theorem 2] systems.*

Remark 4 *Suppose that system is linear, i.e.,*

$$f(t,x) = A(t)x, \qquad g(t,x) = B(t). \tag{31}$$

Then condition (26) *means that system* (2), (31) *is completely reachable (see Theorem 5 in [16]).*

Theorem 1 and Theorem 2 give sufficient conditions for uniform global asymptotic stabilization of the origin of (2). The similar theorems can be proved for the problem of uniform local asymptotic (ULA) stabilization of the origin of (2). We give below the formulations of these theorems omitting the rigorous proofs. These proofs can be obtained by following the method of proving Theorems 1 and 2. The differences will be that inequality (5) is assumed to be fulfilled not for all $x \in \mathbb{R}^n$ but for $x \in \mathcal{D}$ where $\mathcal{D} \subset \mathbb{R}^n$ is some neighborhood of the origin (i.e., an open set containing the origin); also, identities (16), (17) are considered not for any $x_0 \in \mathbb{R}^n$ but for $x_0 \in \mathcal{W}$ where $\mathcal{W} \subset \mathbb{R}^n$ is some neighborhood of the origin.

Theorem 3 *Let system* (2) *be ω-periodic. Suppose that there exists a matrix $P(t)$ satisfying conditions* (4), (5) *for all $t \in \mathbb{Z}$ and for all $x \in \mathcal{D}$ where $\mathcal{D} \subset \mathbb{R}^n$ is some neighborhood of the origin. Suppose that, for some neighborhood $\mathcal{W} \subset \mathbb{R}^n$ of the origin, for some $t_0 \in \mathbb{Z}$, the fulfillment of the identities* (16) *and* (17) *for some $x_0 \in \mathcal{W}$ implies $x_0 = 0$. Then the state feedback control* (10), (8) *ULA stabilizes the origin of* (2).

Theorem 4 *Let system* (2) *be ω-periodic. Let f be of class C^1 in x. Suppose that there exists a matrix $P(t)$ satisfying conditions* (4), (5) *for all $t \in \mathbb{Z}$ and for all $x \in \mathcal{D}$ where $\mathcal{D} \subset \mathbb{R}^n$ is some neighborhood of the origin. Suppose that, for some neighborhood $\mathcal{W} \subset \mathbb{R}^n$ of the origin, the following condition holds:*

$$\exists\, t_0 \in \mathbb{Z} \;\; \forall x \in \mathcal{W} \setminus \{0\} \;\; \exists\, \nu \geqslant 1 \;\; \operatorname{rank} N_\nu(t_0, x) = n. \tag{32}$$

Then the state feedback control (10), (8) *ULA stabilizes the origin of* (2).

3. Examples

Example 1 Consider system (2) with

$$
\begin{aligned}
& n=2, \quad r=1, \quad \omega=2, \\
& f_0(x)=\begin{bmatrix}\sin x_1 \\ x_2\end{bmatrix}, \qquad f_1(x)=\begin{bmatrix}0 \\ x_2\end{bmatrix}, \\
& f(t,x)=\begin{cases} f_0(x), & t=2s, \\ f_1(x), & t=2s-1,\end{cases} \qquad s\in\mathbb{Z}, \\
& g_0(x)=\begin{bmatrix}x_1x_2 \\ x_1^2+x_2^2\end{bmatrix}, \qquad g_1(x)=\begin{bmatrix}1+x_1^2+x_2^2 \\ x_1x_2\end{bmatrix}, \\
& g(t,x)=\begin{cases} g_0(x), & t=2s, \\ g_1(x), & t=2s-1,\end{cases} \qquad s\in\mathbb{Z}.
\end{aligned}
\tag{33}
$$

Note that the system of linear approximation at the equilibrium $x = 0$ has the form

$$
\begin{aligned}
& x(t+1)=A(t)x(t)+B(t)u(t), \quad x\in\mathbb{R}^2, \quad u\in\mathbb{R}^1, \\
& A(0)=\begin{bmatrix}1&0\\0&1\end{bmatrix}, \quad A(1)=\begin{bmatrix}0&0\\0&1\end{bmatrix}, \quad A(t+\omega)=A(t), \\
& B(0)=\begin{bmatrix}0\\0\end{bmatrix}, \qquad B(1)=\begin{bmatrix}1\\0\end{bmatrix}, \qquad B(t+\omega)=B(t).
\end{aligned}
\tag{34}
$$

It is clear that system (34) is not asymptotically stabilizable by any linear feedback control $u(t) = U(t)x(t)$.

We have

$$
\begin{aligned}
& f^T(0,x)f(0,x)=\sin^2 x_1+x_2^2\leqslant x_1^2+x_2^2=x^Tx, \\
& f^T(1,x)f(1,x)=x_2^2\leqslant x_1^2+x_2^2=x^Tx.
\end{aligned}
$$

Hence, (5) holds for all $x \in \mathbb{R}^n$ and (by periodicity) for all $t \in \mathbb{Z}$ if $P(t) \equiv I$. In particular, the free nonlinear system is Lyapunov stable. Note that the free nonlinear system is not asymptotically stable because for every $x_0 = \operatorname{col}(0, \beta)$, where $\beta \neq 0$, the solution $\xi(t) = \xi(t, 0, x_0)$ of the free system with the initial condition $\xi(0) = x_0$ satisfies $\xi(t) \equiv x_0$, $t \geqslant 0$ (i.e., does not tend to 0 as $t \to \infty$).

Let us apply Theorem 2. Suppose $t_0 := 0$. We have

$$
N_1(0,x)=\begin{bmatrix}x_1x_2 \\ x_1^2+x_2^2\end{bmatrix},
$$

$$
\begin{aligned}
N_2(0,x) &= \left[K(1, f(0,x))N_1(0,x), g(1, f(0,x))\right] \\
&= \begin{bmatrix} 0 & 1+\sin^2 x_1 + x_2^2 \\ x_1^2 + x_2^2 & x_2 \sin x_1 \end{bmatrix}.
\end{aligned}
$$

Hence, rank $N_2(0,x) = 2 \quad \forall x \neq 0$. Thus, conditions of Theorem 2 are fulfilled. Constructing $G(t,x)$ by (7), and $\widehat{u}(t,x)$ by (8), we obtain

$$
G(t,x) = \begin{cases} 1 + \dfrac{1}{2}\left(x_1^2 x_2^2 + (x_1^2 + x_2^2)^2\right), & t = 2s, \\ 1 + \dfrac{1}{2}\left(x_1^2 x_2^2 + (1 + x_1^2 + x_2^2)^2\right), & t = 2s-1, \end{cases}
$$

$$
\widehat{u}(t,x) = \begin{cases} -\dfrac{2\left[x_1 x_2 \sin x_1 + x_2(x_1^2 + x_2^2)\right]}{2 + x_1^2 x_2^2 + (x_1^2 + x_2^2)^2}, & t = 2s, \\ -\dfrac{2x_1 x_2^2}{2 + x_1^2 x_2^2 + (1 + x_1^2 + x_2^2)^2}, & t = 2s-1, \end{cases} \tag{35}
$$

$s \in \mathbb{Z}$. By Theorem 2, feedback control (10), (35) UGA stabilizes the origin of system (2), (33).

Example 2 Consider system (2) with

$$
\begin{aligned}
&n = 2, \quad r = 1, \quad \omega = 2, \\
&f_0(x) = \begin{bmatrix} x_1 \sin x_2 \\ x_2 e^{-x_1^2} \end{bmatrix}, \quad f_1(x) = \begin{bmatrix} x_1 \cos x_2 \\ x_2 \end{bmatrix}, \\
&f(t,x) = \begin{cases} f_0(x), & t = 2s, \\ f_1(x), & t = 2s-1, \end{cases} \quad s \in \mathbb{Z}, \\
&g_0(x) = \begin{bmatrix} x_1^2 - x_2^2 \\ 2x_1 x_2 \end{bmatrix}, \quad g_1(x) = \begin{bmatrix} -2x_1 x_2 \\ x_1^2 + x_2^2 \end{bmatrix}, \\
&g(t,x) = \begin{cases} g_0(x), & t = 2s, \\ g_1(x), & t = 2s-1, \end{cases} \quad s \in \mathbb{Z}.
\end{aligned} \tag{36}
$$

We have

$$
\begin{aligned}
f^T(0,x) f(0,x) &= x_1^2 \sin^2 x_2 + x_2^2 e^{-2x_1^2} \leqslant x_1^2 + x_2^2 = x^T x, \\
f^T(1,x) f(1,x) &= x_1^2 \cos^2 x_2 + x_2^2 \leqslant x_1^2 + x_2^2 = x^T x.
\end{aligned}
$$

Hence, (5) holds for all $x \in \mathbb{R}^n$ and (by periodicity) for all $t \in \mathbb{Z}$ if $P(t) \equiv I$. In particular, the free nonlinear system is Lyapunov stable. Note that the free nonlinear system is not asymptotically stable because for every $x_0 = \mathrm{col}\,(0, \beta)$, where $\beta \neq 0$, the solution $\xi(t) = \xi(t, 0, x_0)$ of the free system with the initial condition $\xi(0) = x_0$ satisfies $\xi(t) \equiv x_0$, $t \geqslant 0$ (i.e., does not tend to 0 as $t \to \infty$).

Suppose $t_0 := 1$. Denote $x_0 = \mathrm{col}\,(\alpha, \beta)$. Suppose that (16) and (17) hold. Let us show that this implies $x_0 = 0$ necessarily. Assume the contrary. From (16) at $i = 0, 1$ and (17) at $i = 0$, it follows that

$$\alpha^2 + \beta^2 = \alpha^2 \cos^2 \beta + \beta^2 = \alpha^2 \sin^2 \beta + \beta^2 e^{-2\alpha^2}, \tag{37}$$

$$\alpha \cos \beta(-2\alpha\beta) + \beta(\alpha^2 + \beta^2) = 0. \tag{38}$$

From (37), it follows that

$$\alpha^2(1 - \cos^2 \beta) = 0, \tag{39}$$

$$\alpha^2(1 - \sin^2 \beta) + \beta^2(1 - e^{-2\alpha^2}) = 0. \tag{40}$$

From (39), it follows that $\alpha = 0$ or $\cos^2 \beta = 1$. If $\alpha = 0$ then it follows from (38) that $\beta = 0$, hence, $x_0 = 0$. This is contradiction. Hence, $\alpha \neq 0$. Therefore $\cos^2 \beta = 1$. Then $\alpha^2(1 - \sin^2 \beta) + \beta^2(1 - e^{-2\alpha^2}) \geqslant \alpha^2 > 0$. This contradicts (40). Thus, (16) and (17) hold only if $x_0 = 0$. Thus, conditions of Theorem 1 are fulfilled. Constructing $G(t, x)$ by (7), and $\widehat{u}(t, x)$ by (8), we obtain

$$G(t,x) = \begin{cases} 1 + \dfrac{1}{2}\left(x_1^2 + x_2^2\right)^2, & t = 2s, \\ 1 + \dfrac{1}{2}\left((x_1^2 + x_2^2)^2 + 4x_1^2x_2^2\right), & t = 2s-1, \end{cases}$$

$$\widehat{u}(t,x) = \begin{cases} -\dfrac{2\left[(x_1^2 - x_2^2)x_1 \sin x_2 + 2x_1x_2^2e^{-x_1^2}\right]}{2 + (x_1^2 + x_2^2)^2}, & t = 2s, \\ -\dfrac{2\left[(-2x_1x_2)x_1 \cos x_2 + (x_1^2 + x_2^2)x_2\right]}{2 + (x_1^2 + x_2^2)^2 + 4x_1^2x_2^2}, & t = 2s-1, \end{cases} \tag{41}$$

$s \in \mathbb{Z}$. By Theorem 1, feedback control (10), (41) UGA stabilizes the origin of system (2), (36).

Example 3 Let some organisms, interacting with the environment in the wild, have a dependence of the reproduction rate on the number, which is determined by a non-monotonic curve, so that the reproduction rate is low when the number of organisms is small, as well as when the number approaches a certain limit, above which reproduction stops completely. The maximum is reached at a certain intermediate number. Such dependence can arise as an adaptation to the scarcity of food resources. More accurate modeling of the population of such organisms naturally requires taking into account the volume of these food resources. However, this information may not always be available. Information on the exact values of the breeding factor may also not be available. Under these conditions, it is reasonable to model the dependence of the reproduction rate on the abundance using a function similar to the Tent Map without introducing an equation for food resources into the model.

Let us denote by $y = y(t) \in [0, 1]$ the dimensionless population size, the free dynamics of which is tracked at integer points $t \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ in time and is described by the equation

$$y(t+1) = p(y(t)), \quad t \in \mathbb{N}_0, \tag{42}$$

where $p\colon [0, 1] \to [0, 1]$ is a function like the Tent Map (see [21, Section 15.4]) slightly modified:

$$p(x) = \begin{cases} 2x, & x \in [0, 1/2), \\ 3/2 - x, & x \in [1/2, 1]. \end{cases}$$

System (42) has the equilibrium $H = 3/4$. Let an arbitrary initial condition be given: $y_0 = 3/4 - a$, where $a \in (0, 1/4)$. System (42) with the initial condition $y(0) = y_0$ has the solution

$$\widehat{y}(t) = \begin{cases} 3/4 - a, & t = 0, 2, \ldots, \\ 3/4 + a, & t = 1, 3, \ldots. \end{cases} \tag{43}$$

Here we consider solutions only to the right, for $t \in \mathbb{N}_0$; the periodicity of a function $\alpha(t)$ is understood as $\alpha(t+\omega) = \alpha(t)$ for $t \in \mathbb{N}_0$. Note that all Theorems given above are valid if we consider the time t not on the whole axis $\mathbb{Z}$ but only on the right semiaxis: $t \in \mathbb{N}_0$. The solution $\widehat{y}(t)$ is ω-periodic with $\omega = 2$. The solution $\widehat{y}(t)$ is (non-asymptotically) stable.

Suppose we are dealing with some kind of anthropogenic impact on the population, which is expressed in the influence on the change in its numbers. In the general case, this influence can be constant or variable. One of the typical cases of variable influence is periodic, associated with various economic and biological cycles. In the simplest case, it can be associated with a change in the seasons of the year or time of day, and, accordingly, its simplest dependence on time has the form of a periodic switch from zero to maximum intensity, which, however, naturally also depends on the number of organisms. For example, the intensity of the catch of free-living individuals obviously increases with the growth of their numbers. We will assume that this anthropogenic impact on the population is described by the term $q(t, y(t))u(t)$, where the function $q(t, y)$ has the form

$$q(t, y) = \begin{cases} 0, & t = 2s, \\ y, & t = 2s+1, \end{cases} \qquad s \in \mathbb{N}_0 .$$

So, we consider the affine discrete-time system

$$y(t+1) = p(y(t)) + q(t, y(t))u(t), \qquad t \in \mathbb{N}_0\,. \tag{44}$$

We solve the problem of ULA stabilization of the solution (43): one needs to construct a state feedback control

$$u(t) = \widetilde{u}(t, y(t)) \tag{45}$$

such that the function $\widehat{y}(t)$ is ULA stable solution of the closed-loop system (44), (45). This means that the anthropogenic impact is aimed at maintaining the asymptotic dynamics of the population in the given mode.

Reduce system (44) to the system in deviations. Let $x = y - \widehat{y}(t)$. We have

$$\begin{aligned} x(t+1) &= y(t+1) - \widehat{y}(t+1) = p(y(t)) - p(\widehat{y}(t)) + q(t, y(t))u(t) \\ &= p(x(t) + \widehat{y}(t)) - p(\widehat{y}(t)) + q(t, x(t) + \widehat{y}(t))u(t). \end{aligned} \tag{46}$$

Denote $f(t,x) := p(x + \widehat{y}(t)) - p(\widehat{y}(t))$, $g(t,x) := q(t, x + \widehat{y}(t))$. Then system (46) takes the form

$$x(t+1) = f(t, x(t)) + g(t, x(t))u(t). \tag{47}$$

It is clear that $f(t,0) \equiv 0$, $t \geqslant 0$. The problem of ULA stabilization of the solution $\widehat{y}(t)$ of system (44) is reduced to the problem of ULA stabilization of the origin of (47). We consider system (47) for $t \geqslant 0$ and for x from some small neighborhood $\mathcal{D}$ of the origin, namely,

$$x \in \mathcal{D} := (a - 1/4, 1/4 - a). \tag{48}$$

It is clear that $g(t+\omega, x) = g(t,x)$ for all $t \geqslant 0$ and $x \in \mathcal{D}$. Under condition (48) we have $x + \widehat{y}(t) \in (1/2, 1)$ for all $t \geqslant 0$. Therefore,

$$f(t,x) = p(x + \widehat{y}(t)) - p(\widehat{y}(t)) = 3/2 - (x + \widehat{y}(t)) - (3/2 - \widehat{y}(t)) = -x.$$

Let us check that the conditions of Theorem 4 are satisfied. Obviously, system (47) is ω-periodic. The function f is of class C^1 in x. There exist a matrix $P(t)$ satisfying conditions (4) and (5) for all $t \geqslant 0$ and $x \in \mathcal{D}$, namely, $P(t) \equiv P = 1$. Next, set $\mathcal{W} := \mathcal{D}$. Let $t_0 = 1$ and $\nu = 1$. Then $N_1(t_0, x) = g(t_0, x) = q(1, x + \widehat{y}(1)) = x + \widehat{y}(1) = x + 3/4 + a \neq 0$ for any $x \in \mathcal{W}$. Hence, condition (32) is satisfied. So, all conditions of Theorem 4 are fulfilled. Constructing $G(t,x)$ by (7), and $\widehat{u}(t,x)$ by (8), we obtain

$$\begin{aligned} G(t,x) &= 1 + q^2(t, x + \widehat{y}(t))/2, \\ \widehat{u}(t,x) &= \frac{2xq(t, x + \widehat{y}(t))}{2 + q^2(t, x + \widehat{y}(t))}\,. \end{aligned} \tag{49}$$

By Theorem 4, feedback control (10), (49) ULA stabilizes the origin of system (47).

Let us return to system (44) by using the reverse replacement $y = x + \widehat{y}(t)$. Set $\widetilde{u}(t, y) = \widehat{u}(t, y - \widehat{y}(t))$. Then

$$\widetilde{u}(t, y) = \frac{2(y - \widehat{y}(t))q(t, y)}{2 + q^2(t, y)}. \tag{50}$$

By Theorem 4, feedback control (45), (50) ULA stabilizes the solution $\widehat{y}(t)$ of the closed-loop system (44). The function (50) is ω-periodic.

4. Conclusion

In this paper, we have studied the problem of uniform global asymptotic stabilization of the zero equilibrium for an affine non-stationary discrete-time system with periodic coefficients. Assumption of existing the quadratic periodic Lyapunov function is required that ensure Lyapunov (non-asymptotic) stability of zero equilibrium of the free system. The damping control technique developed earlier for autonomous discrete-time systems extends to periodic discrete-time systems. For periodic discrete-time systems, sufficient conditions of uniform global asymptotic stabilization are obtained generalizing similar conditions for autonomous discrete-time systems. In addition, new sufficient conditions are obtained that include controllability-like rank condition that is expressed in terms of distributions and depends only on coefficients of the system.

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