Approximate controllability of second order infinite dimensional systems

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In the paper approximate controllability of second order infinite dimensional system with damping is considered. Applying linear operators in Hilbert spaces general mathematical model of second order dynamical systems with damping is presented. Next, using functional analysis methods and concepts, specially spectral methods and theory of unbounded linear operators, necessary and sufficient conditions for approximate controllability are formulated and proved. General result may be used in approximate controllability verification of second order dynamical system using known conditions for approximate controllability of first order system. As illustrative example using Green function approach approximate controllability of distributed dynamical system is also discussed.

Key words: infinite dimensional systems, approximate controllability, Green's function approach, flexible Kirchhoff–Love plate

1. Introduction

Controllability is one of the fundamental concepts in mathematical control theory [7]. Roughly speaking, controllability generally means, that it is possible to steer a dynamical system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. In the literature, there are many different definitions of controllability which depend on class of dynamical system [1, 7, 9–11, 13, 14, 16, 18, 20]. Recently, fixed-point theorems are also used for semilinear controllability problems [15, 16]. For infinite dimensional dynamical systems, it is necessary to distinguish between the notions of approximate and exact

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controllability [7, 10, 23, 23–30]. It follows directly from the fact, that in infinite-dimensional spaces there exist linear subspaces that are not closed.

The present paper is devoted to study approximate controllability of linear infinite-dimensional second order dynamical systems with damping. For such dynamical systems direct verification of approximate controllability is rather difficult and complicated [8, 12]. Therefore, using frequency-domain method [11], it is shown that approximate controllability of second order dynamical system can be verified by the approximate controllability condition for suitably defined simplified first order dynamical system. Finally using Green function approach obtained results are applied for investigation of approximate controllability for flexible mechanical structure.

2. System description

Let V and U denote separable Hilbert spaces. Let $A: V \supset D(A) \to V$ be a linear generally unbounded self-adjoint and positive-definite linear operator with dense domain $D(A)$ in V and compact resolvent $R(s, A) = (sI - A)^{-1}$ for all s in the resolvent set $\rho(A)$. Then, operator A has the following properties [1, 7, 8, 12, 16, 18]:

1. Operator A has only pure discrete point spectrum $\sigma_p(A)$ consisting entirely of isolated real positive eigenvalues s_i such that

$$0 < s_1 < s_2 < \ldots < s_i < s_{i+1} < \ldots < \lim_{i \to \infty} s_i = +\infty.$$

2. Each eigenvalue s_i has finite multiplicity $n_i < \infty$ $(i = 1, 2, \ldots)$ equal to the dimensionality of the corresponding eigenmanifold.

3. Eigenvectors $v_{ik} \in D(A)$ $(i = 1, 2, \ldots; k = 1, 2, \ldots, ni)$ form a complete orthonormal system in the separable Hilbert space V.

4. A has spectral representation

$$Av = \sum_{i=1}^{\infty} s_i \sum_{k=1}^{n_i} \langle v, v_{ik} \rangle_V v_{ik} \quad \text{for} \quad v \in D(A).$$

5. Fractional powers A^{α} $(0 < \alpha \leqslant 1)$ of the operator A can be defined as follows

$$A^{\alpha} v = \sum_{i=1}^{\infty} s_i^{\alpha} \sum_{k=1}^{n_i} \langle v, v_{ik} \rangle_V v_{ik} \quad \text{for} \quad v \in D\left(A^{\alpha}\right),$$

where

$$D(A^\alpha) = \left\{ v \in V : \sum_{i=1}^{\infty} s_i^{2\alpha} \sum_{k=1}^{n_i} |\langle v, v_{ik} \rangle_V|^2 < \infty \right\}.$$

Operators A^α ($0 < \alpha \leqslant 1$) are self-adjoint, positive-definite with dense domains in V and generate analytic semigroups on V.

Let us consider linear infinite-dimensional control system described by the following abstract second order differential equation

$$\left(e_2 A + e_1 A^{\frac{1}{2}} + e_0 I \right) \ddot{v}(t) + 2 \left(c_2 A + c_1 A^{\frac{1}{2}} + c_0 I \right) \dot{v}(t)$$
$$+ \left(d_2 A + d_1 A^{\frac{1}{2}} + d_0 I \right) v(t) = Bu(t), \qquad (1)$$

where $e_2 \geqslant 0$, $e_1 \geqslant 0$, $e_0 \geqslant 0$, $e_2 + e_1 + e_0 > 0$, $c_2 \geqslant 0$, $c_1 \geqslant 0$, $c_0 \geqslant 0$, d_1 and d_0 unrestricted in sign, $d_2 > 0$ are real given constants.

It is assumes that the operator $B \colon U \to V$ is linear and its adjoint operator $B^* \colon V \to U$ is $A^{\frac{1}{2}}$-bounded [1,2,11], i.e. $D(B^*) \supset D\left(A^{\frac{1}{2}} \right)$ and there is a positive real number M such that

$$\|B^* v\| \leqslant M \left(\|v\|_V + \left\| A^{\frac{1}{2}} v \right\|_V \right) \quad \text{for} \quad v \in D\left(A^{\frac{1}{2}} \right).$$

Moreover, it is assumed that the admissible controls u belong to

$$L^2_{loc} \left([0, \infty), U \right).$$

It is well known [2–6] that for each $t_1 > 0$ and $u \in L^2_{loc}([0, \infty), U)$, abstract ordinary differential equation (1) with initial conditions

$$v(0) \in D(A), \quad \dot{v}(0) \in V$$

has a unique solution

$$v(t; v(0), \dot{v}(0), u) \in C^2([0, t_1], V)$$

such that $V(t) \in D(A)$ and $\dot{v}(t) \in D(A)$ for $t \in (0, t_1]$.

Moreover, for $v(0) \in V$ there exists so-called "mild solution" for Eq. (1) in the product space $W = V \times V$ with inner product defined as follows $\langle v, w \rangle_W = \langle [v_1, v_2], [w_1, w_2] \rangle_W = \langle v_1, w_1 \rangle_V + \langle v_1, w_1 \rangle_V$.

In order to transform second order equation (1) into first order equations in the Hilbert space W, let us make the substitution [1–6, 20]

$$v(t) = w_1(t), \quad \dot{v}(t) = w_2(t).$$

Then, equation (1) becomes

$$w(t) = Fw(t) + Gu(t) \tag{2}$$

where

$$F = \begin{vmatrix} 0 & I \\ F_1 & F_2 \end{vmatrix}, \quad w = \begin{vmatrix} w_1 \\ w_2 \end{vmatrix}, \quad G = \begin{vmatrix} 0 \\ \left(e_2 A + e_2 A^{\frac{1}{2}} + e_0 I \right)^{-1} B \end{vmatrix},$$

$$F_1 = -\left(e_2 A + e_1 A^{\frac{1}{2}} + e_0 I \right)^{-1} \left(d_2 A + d_1 A^{\frac{1}{2}} + d_0 I \right),$$

$$F_2 = -2 \left(e_2 A + e_1 A^{\frac{1}{2}} + e_0 I \right)^{-1} \left(c_2 A + c_1^{\frac{1}{2}} + c_0 I \right)$$

Remark 1 *Since the operators A and $\left(A^{\frac{1}{2}} \right)$ are self-adjoint and under assumptions on coefficients e_i $(i = 1, 2, 3)$, sequence*

$$\left\{ \frac{1}{e_2 s_i + e_1 \sqrt{s_i} + e_0} \in \mathbb{R}, \quad i = 1, 2, \dots \right\}$$

converges towards zero, it is easy to see that operator $\left(e_2 A + e_1 A^{\frac{1}{2}} + e_0 I \right)^{-1}$ is self-adjoint, positive-definite and bounded on V.

Taking advantage of relation $\langle v_1, F^* v_2 \rangle_W = \langle F v_1, v_2 \rangle_W$, we can obtain the adjoint operator F^* as follows:

$$F^* = \begin{vmatrix} 0 & -\left(d_2 A + d_1 A^{\frac{1}{2}} + d_0 I \right) \left(e_2 A + e_1 A^{\frac{1}{2}} + e_0 I \right)^{-1} \\ I & -2 \left(c_2 A + c_1^{\frac{1}{2}} + c_0 I \right) \left(e_2 A + e_1 A^{\frac{1}{2}} + e_0 I \right)^{-1} \end{vmatrix}.$$

Similarly, the adjoint for operator G can be obtained as

$$G^* = \left[0, B^* \left(e_2 A + e_2 A^{\frac{1}{2}} + e_0 I \right)^{-1} \right].$$

Remark 2 *It should be pointed out that properties of operators F and F^* depend strongly on values of coefficients c_i, d_i, e_i, $i = (0, 1, 2)$ [2–6, 12]. In particular:*

  1) *if $c_2 = c_1 = c_0 = 0$ and additionally $e_2 \neq 0$ or $(e_2 = 0$ and $d_2 = 0$, $e_1 \neq 0)$ or $(e_2 = e_1 = 0$, $d_2 = d_1 = 0)$, then, operator F is bounded and generates an analytic group of linear bounded operators on the Hilbert space $W = V \times V$.*

2) if ($e_2 = 0$ and $d_2 \neq 0$) or ($d_2 = 0$ and $e_2 = e_1 = 0$ and $d_1 \neq 0$), then, operator F is unbounded and generates a group of linear bounded operators on the Hilbert space $W = V \times v$ which cannot be analytic [18].

3) if ($e_2 = 0$ and $c_2 \neq 0$) or ($e_2 = e_1 = 0$ and ($c_2 \neq 0$ or $c_1 \neq 0$)), then, operator F is unbounded and generates an analytic semigroup of linear bounded operators on the Hilbert space $W = V \times V$.

4) if $e_2 \neq 0$ or ($e_2 = e_1 = 0$ and $c_2 = c_1 = 0$ and $d_2 = d_1 = 0$) or ($e_2 = 0$ and $c_2 = 0$ and $d_2 = 0$ and $e_1 \neq 0$), then, operator F is bounded and generates an analytic semigroup of linear bounded operators on the Hilbert space $W = V \times V$.

5) if $c_2 = e_2 = 0$ and $e_1 \neq 0$ and $d_2 \neq 0$, then, operator F is unbounded and generates an C_0-semigroup of linear unbounded operators on the Hilbert space $W = V \times V$ which is not analytic.

These statements are important for investigation of controllability. In the sequel, in addition to the second-order equation (1), and first order dynamical system (2) we shall also consider the simplified first order differential equation

$$\dot{v}(t) = -A^\alpha v(t) + Bu(t) \tag{3}$$

where $\alpha \in (0, \infty)$.

In the next sections we shall also discuss simplified version of dynamical systems (1), (2) and (3) with finite-dimensional control space $U = \mathbb{R}^m$. In this special case, for convenience, we shall introduce the following notations

$$B = \begin{bmatrix} b_1, \ldots, b_j, \ldots, b_m \end{bmatrix}, \qquad u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_j(t) \\ \vdots \\ u_m(t) \end{bmatrix},$$

where $b_j \in V$ and $u_j \in L^2_{loc}(0, \infty)$ for $j = 1, 2, \ldots, m$.

Let us observe that in this special case linear operator B is finite-dimensional and therefore, it is compact operator [1, 7, 17, 19]. Using eigenvectors v_{ik} ($i = 1, 2, \ldots,$ $k = 1, 2, \ldots, n_i$), we introduce for finite-dimensional operator B the

following notation [7, 18]

$$B_i = \begin{vmatrix} \langle b_1, v_{i1} \rangle_V & \langle b_2, v_{i1} \rangle_V & \ldots & \langle b_j, v_{i1} \rangle_V & \vdots & \langle b_m, v_{i1} \rangle_V \\ \langle b_1, v_{i2} \rangle_V & \langle b_2, v_{i1} \rangle_V & \ldots & \langle b_j, v_{i2} \rangle_V & \vdots & \langle b, v_{i2} \rangle_V \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \langle b_1, v_{ik} \rangle_V & \langle b_2, v_{ik} \rangle_V & \ldots & \langle b_j, v_{ik} \rangle_V & \vdots & \langle b_m, v_{ik} \rangle_V \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \langle b_1, v_{in_i} \rangle_V & \langle b_2, v_{in_i} \rangle_V & \ldots & \langle b_j, v_{in_i} \rangle_V & \vdots & \langle b_m, v_{in_i} \rangle_V \end{vmatrix} \tag{4}$$

B_i are $n_i \times m$-dimensional constant matrices which play an important role in controllability investigations [7, 8, 10, 18]. For the case when eigenvalues s_i are simple, i.e., $n_i = 1$, B_i are m-dimensional row vectors

$$B_i = [\langle b_1, v_1 \rangle_V \ldots \langle b_j, v_i \rangle_V \ldots \langle b_m, v_i \rangle_V]. \tag{5}$$

3. Approximate controllability

For infinite-dimensional dynamical systems, two general kinds of controllability, i.e. approximate (weak) and exact (strong) controllability may be introduced [1, 7, 10, 16, 18].

In the present paper we shall concentrate on approximate controllability for the second order dynamical system (1).

Definition 1 *[1, 7, 16] Dynamical system (1) is said to be approximately controllable in the time interval $[0, t_1]$ if for any initial condition $w(0) \in V \times V$, any given final condition $w_f \in V \times V$ and each positive real number ε, there exists an admissible control $u \in L^2_{loc}((0, t_1], U)$ such that*

$$\|w(t_1; w(0), u) - w_f\|_{V \times V} \leqslant \varepsilon. \tag{6}$$

Definition 2 *[1, 7, 16] Dynamical system (1) is said to be approximately controllable in finite time (or, briefly, approximately controllable) if for any initial condition $w(0) \in V \times V$, any given final condition $w_f \in V \times V$ and each positive real number ε, there exist a finite time $t_1 < \infty$ (depending generally on $w(0)$ and w_f) and an admissible control $u \in L^2_{loc}((0, t_1], U)$ such that the inequality (6) holds.*

Remark 3 *When the semigroup associated with the dynamical system (1) is analytic, then approximate controllability in finite time coincides with approximate controllability in each time interval $[0, t_1]$ [1, 7, 16, 18].*

Remark 4 *When the semigroup associated with the dynamical system (1) is compact or the control operator B is compact, then dynamical system (1) is never exactly controllable in any infinite-dimensional state space [1, 7, 17, 19].*

Now, let us recall two well-known lemmas [11, 15] concerning approximate controllability of first order dynamical systems (2) and (3), which will be useful in the sequel.

Lemma 1 *[11] Dynamical system (2) is approximately controllable if and only if for any $z \in \mathbb{C}$, there exists no nonzero $w \in D\left(F^*\right)$ such that*

$$\left| \begin{matrix} F^* - zI \\ G^* \end{matrix} \right| w = 0. \tag{7}$$

Similarly, dynamical system (3) is approximately controllable if and only if for any complex number s, there exists no nonzero $w \in D\left(A^\alpha\right)$ such that

$$\left| \begin{matrix} -A^\alpha - sI \\ B^* \end{matrix} \right| v = 0. \tag{8}$$

Lemma 2 *[15] Dynamical system (3) is approximately controllable if and only if it is approximately controllable for some $\alpha \in (0, \infty)$.*

From Lemma 2 it follows that, in the simplest case, for approximate controllability investigation it is enough to take $\alpha = 1$.

Now, using the frequency-domain method [11], we shall formulate and prove necessary and sufficient condition for approximate controllability for dynamical system (1), which is the main result of the paper.

Theorem 1 *Second order dynamical system (1) is approximately controllable if and only if corresponding first order dynamical system (3) is approximately controllable for some $\alpha \in (0, \infty)$.*

Proof. By Lemma 2, in order to prove Theorem 1 it is sufficient to show the equivalence of the conditions (7) and (8) for some $\alpha \in (0, \infty)$ Therefore, in the proof, we shall take $\alpha = \frac{1}{2}$.

(7) $\rightarrow$ (8). By contradiction. To establish a contradiction, suppose that for some s, there exists a nonzero $v \in D\left(A^{\frac{1}{2}}\right)$ satisfying equality (8), i.e.

$$A^{\frac{1}{2}} = -sv \quad \text{and} \quad B^*v = 0 \tag{9}$$

Thus, $-s$ is an eigenvalue of positive-definite operator $A^{\frac{1}{2}}$. Hence, s is real and negative. The vector $v \neq 0$ is the associated eigenvector. Taking into account the

form of the linear operators F^* and G^*, equation (7) produce the following set of equalities:

$$- zw_1 - \left(d_2 A + d_1 A^{\frac{1}{2}} + d_0 I \right) \left(e_2 A + e_1 A^{\frac{1}{2}} + e_0 I \right)^{-1} w_2 = 0, \qquad (10)$$

$$w_1 = 2 \left(c_2 A + c_1 A^{\frac{1}{2}} + c_0 I \right) \left(e_2 A + e_1 A^{\frac{1}{2}} + e_0 I \right)^{-1} w_2 + zw_2, \qquad (11)$$

$$\left(e_2 A + e_1 A^{\frac{1}{2}} + e_0 I \right)^{-1} w_2 = 0. \qquad (12)$$

Hence, for $w_2 = v \neq 0$, from (9) and (10) it follows

$$w_1 = 2 \left(c_2 A + c_1 A^{\frac{1}{2}} + c_0 I \right) \left(e_2 A + e_1 A^{\frac{1}{2}} + e_0 I \right)^{-1} v + zv =$$
$$= \left(c_2 s^2 + c_1 s + c_0 \right) \left(e_2 s^2 + e_1 s + e_0 \right)^{-1} v + zv \qquad (13)$$

Substituting (11) into (10) and taking into account (9), yields

$$2 \left(c_2 s^2 + c_1 s + c_0 \right) \left(e_2 s^2 + e_1 s + e_0 \right)^{-1} v + z^2 v +$$
$$+ \left(d_2 s^2 + d_1 s + d_0 \right) \left(e_2 s^2 + e_1 s + e_0 \right)^{-1} v = 0. \qquad (14)$$

Thus, we obtain the following second order algebraic equation with respect to z:

$$\left(e_2 s^2 + e_1 s + e_0 \right) z^2 + 2 \left(c_2 s^2 + c_1 s + c_0 \right) z + \left(d_2 s^2 + d_1 s + d_0 \right) = 0. \qquad (15)$$

Its solutions z_1 and z_2 are given by

$$z_1 = \frac{-c_2 s^2 - c_1 s - c_0 + \Delta}{e_2 s^2 + e_1 s + e_0}, \quad z_2 = \frac{-c_2 s^2 - c_1 s - c_0 - \Delta}{e_2 s^2 + e_1 s + e_0}$$

where

$$\Delta = \left(\left(c_2 s^2 + c_1 s + c_0 \right)^2 - \left(e_2 s^2 + e_1 s + e_0 \right) \left(d_2 s^2 + d_1 s + d_0 \right) \right)^{\frac{1}{2}}$$

Thus, the nonzero vectors

$$w_j = \begin{bmatrix} 2 \left(c_2 A + c_1 A^{\frac{1}{2}} + c_0 I \right) \left(e_2 A + e_1 A^{\frac{1}{2}} + e_0 I \right)^{-1} v + z_j v \\ v \end{bmatrix} \in W, \quad j = 1, 2,$$

satisfy (7). This provides the contradiction.

$(8) \rightarrow (7)$. By contradiction. Suppose that for some z there exists a nonzero vector $w = [w_1, w_2]^T \in D(F^*)$ satisfying equality (7), i.e.,

$$- zw_1 - \left(d_2 A + d_1 A^{\frac{1}{2}} + d_0 I \right) \left(e_2 A + e_1 A^{\frac{1}{2}} + e_0 I \right)^{-1} w_2 = 0, \qquad (16)$$

$$w_1 = 2 \left(c_2 A + c_1 A^{\frac{1}{2}} + c_0 I \right) \left(e_2 A + e_1 A^{\frac{1}{2}} + e_0 I \right)^{-1} w_2 + zw_2, \qquad (17)$$

$$\left(e_2 A + e_1 A^{\frac{1}{2}} + e_0 I \right)^{-1} w_2 = 0. \qquad (18)$$

From (18) it directly follows that, for $w_2 = 0$, we have $w_1 = 0$. Thus, $w_2 \neq 0$ and hence, we can take $w_2 = v$. Taking this into account and substituting (17) into (16), yields

$$2 \left(c_2 A + c_1 A^{\frac{1}{2}} + c_0 I \right) \left(e_2 A + e_1 A^{\frac{1}{2}} + e_0 I \right)^{-1} v + zv +$$
$$+ \left(c_2 s^2 + c_1 s + c_0 \right) \left(e_2 s^2 + e_1 s + e_0 \right)^{-1} v + zv = 0. \qquad (19)$$

After transformation, equation (19) takes the form

$$\left(z^2 e - 2zc_2 + d_2 \right) Av + \left(z^2 e_1 + 2zc_1 + d_1 \right) A^{\frac{1}{2}} v +$$
$$+ \left(z^2 e_0 + 2zc_0 + d_0 \right) v = 0. \qquad (20)$$

In order to solve equation (20) with respect to $A^{\frac{1}{2}}$, let us consider the following two cases:

1. If $z^2 e - 2zc_2 + d_2 = 0$ and $z^2 e_1 + 2zc_1 + d_1 \neq 0$, then (20)

$$A^{\frac{1}{2}} v = -\frac{z^2 e_0 + 2zc_0 + d_0}{z^2 e_1 + 2zc_1 + d_1}. \qquad (21)$$

2. If $z^2 e - 2zc_2 + d_2 \neq 0$, then solving (15) as a second order algebraic equation with respect to $A^{\frac{1}{2}}$, we obtain

$$A^{\frac{1}{2}} v = s_1 v \quad \text{or} \quad A^{\frac{1}{2}} v = s_2 v, \qquad (22)$$

where

$$s_1 = \frac{-2 \left(z^2 e_1 + 2zc_1 + d_1 \right) + \Delta}{2 \left(z^2 e - 2zc_2 + d_2 \right)} \quad \text{or} \quad s_2 = \frac{-2 \left(z^2 e_1 + 2zc_1 + d_1 \right) - \Delta}{2 \left(z^2 e - 2zc_2 + d_2 \right)}$$

with

$$\Delta = \left(\left(z^2 e_1 + 2zc_1 + d_1 \right)^2 - 4 \left(z^2 e - 2zc_2 + d_2 \right) \left(z^2 e_0 + 2zc_0 + d_0 \right) \right)^{\frac{1}{2}}.$$

Therefore, equalities (15), (16) imply that there exists $v \neq 0$ satisfying (8) which provides contradiction. Hence Theorem 1 follows.   □

Corollary 1 *Suppose that one of the conditions 1a, 2 or 3 given in Remark 2 is satisfied. Then dynamical system (1) is approximately controllable in any time interval $[0, t_1]$ if and only if dynamical system (3) is approximately controllable in finite time.*

Proof. Since for the case when one of the conditions 1a, 2 or 3 is satisfied operator F generates analytic semigroup (group) then approximate controllability of dynamical system (2) and hence also of dynamical system (1) is equivalent to its approximate controllability in any time interval $[0, t_1]$. Therefore, Corollary 1 immediately follows from Theorem 1.   □

Corollary 2 *Suppose that the space of control values is finite-dimensional, i.e. $U = R^m$. Then the dynamical system (1) is approximately controllable in any time interval $[0, t_1]$ if and only if*

$$\text{rank } B_i = n_i \quad for \quad i = 1, 2, 3 \ldots \tag{23}$$

where matrices $B_i (i = 1, 2, 3, \ldots)$ are given by (4).

Proof. Corollary 2 is a direct consequence of the Theorem 1, Corollary 2 and well-known results [7, 16–18] concerning approximate controllability of infinite-dimensional dynamical systems with finite-dimensional controls.   □

Corollary 3 *Suppose that $U = R^m$ and moreover that $n_i = 1$ for $i = 1, 2, 3, \ldots$. Then the dynamical system (1) is approximately controllable in any time interval $[0, t_1]$ if and only if*

$$\sum_{j=1}^{m} \langle b_j, v_i \rangle^2 \neq 0 \quad for \quad i = 1, 2, 3 \ldots \tag{24}$$

Proof. From Corollary 2 immediately follows that for the case when $n_i = 1$ for $i = 1, 2, 3, \ldots$, the dynamical system (1) is approximately controllable in any time interval if and only if each m-dimensional row vector $B_i (i = 1, 2, \ldots)$ given by (5) has at least one nonzero entry. Thus, Corollary 3 follows.   □

In the next section we shall use the general controllability results in order to check approximate controllability of dynamical system modeling mechanical flexible structure.

3.1.   Approximate controllability of (3) using the Green's function approach

In this section, we will use the Green's function approach developed in [31–33] to study the approximate controllability of first-order system (3) depending on α.

3.2. The Green's function approach

Before applying the Green's function approach, let us briefly describe it. Assume that the control system under study is described by the higher order equation

$$\frac{d^n w}{dt^n} + N\left(\frac{d^{n-1} w}{dt^{n-1}}, \ldots, w\right) = f(u, t), \quad t > 0, \tag{25}$$

subject to Cauchy conditions

$$\left.\frac{d^k w}{dt^k}\right|_{t=0} = w_k, \quad k = 0, 1, \ldots, n-1. \tag{26}$$

Here, N is a non-linearity guaranteeing the existence of unique solution to (25), (26) for given right-hand side f.

It has numerically established (see [34, 35] and references therein) that the general solution to (25), (26) can be represented in terms of nonlinear Green's function as follows:

$$w(t) = \sum_{k=0}^{\infty} g_k \int_0^t \tau^k G(\tau) f(u, t - \tau) \, d\tau. \tag{27}$$

Here, G is the nonlinear Green's function of (25), (26) satisfying

$$\frac{d^n G}{dt^n} + N\left(\frac{d^{n-1} G}{dt^{n-1}}, \ldots, G\right) = s\delta(t),$$

$$\left.\frac{d^k G}{dt^k}\right|_{t=0} = 0, \quad k = 0, 1, \ldots, n-1,$$

coefficients g_k are determined in terms of $w^{(m)}(0)$, $m \geqslant n$, obtained by differentiating both sides of (25) with respect to t and evaluating the resulting equation at $t = 0$.

Then, once (27) is obtained, the approximate controllability of (25) is studied by estimating the residue

$$\mathcal{R}_T = \left| \sum_{k=0}^{\infty} g_k \int_0^t \tau^k G(\tau) f(u, T - \tau) \, d\tau - w^f \right|,$$

where w^f is the desired final state. Accordingly, if there exist admissible controls providing

$$\mathcal{R}_T \leqslant \varepsilon,$$

the system is approximately controllable at T.

3.3. Approximate controllability of (3) via the Green's function approach

We now apply the Green's function approach to derive approximate controllabiity conditions for (3). The following assertion holds.

Theorem 2 *For approximate controllability of (3) to the final state v_f at finite time $t = t_1$ it is sufficient that for $\alpha \in (0, \infty)$, $\varepsilon > 0$, t_1, $v(0)$, and v_f,*

$$\|u\| \leqslant \frac{\varepsilon - \|V^\alpha (t_1) v(0) - v^f\|}{\|V^\alpha B\|} \, .$$

Proof. The Green's function solution of (3) reads as

$$v(t) = V^\alpha(t)v(0) + \int_0^t V^\alpha (t - \tau) \, Bu(\tau) \, \mathrm{d}\tau,$$

where

$$V^\alpha(t) = \exp\left[-A^\alpha t\right].$$

Let v^f be the final state to be achieved at finite time $t = t_1$. Then, the approximate controllability of (3) is established by controls satisfying

$$\mathcal{R}_T(u) = \left\| V^\alpha (t_1) \, v(0) + \int_0^{t_1} V^\alpha (t_1 - \tau) \, Bu(\tau) \, \mathrm{d}\tau - v_f \right\| \leqslant \varepsilon.$$

Making use of the triangle and Minkowski inequalities, we estimate

$$\mathcal{R}_T(u) \leqslant \left\| V^\alpha (t_1) \, v(0) - v^f \right\| + \left\| \int_0^{t_1} V^\alpha (t_1 - \tau) \, Bu(\tau) \, \mathrm{d}\tau \right\| \leqslant$$

$$\leqslant \left\| V^\alpha (t_1) \, v(0) - v^f \right\| + \|V^\alpha B\| \|u\|.$$

Therefore, admissible controls satisfying

$$\|u\| \leqslant \frac{\varepsilon - \left\| V^\alpha (t_1) \, v(0) - v^f \right\|}{\|V^\alpha B\|}$$

ensure the approximate controllability of (3) in t_1.   □

4. Illustrative example

The example corresponds to two-dimensional in space system. Let us consider distributed parameter dynamical system with higher order spatial operator described by the following partial differential equation:

$$
e_2 \frac{\partial^5 v\,(t, x_1, x_2)}{\partial t^2 \partial x_1^4} + 2e_2 \frac{\partial^5 v\,(t, x_1, x_2)}{\partial t^2 \partial x_1^2 \partial x_2^2} + e_2 \frac{\partial^5 v\,(t, x_1, x_2)}{\partial t^2 \partial x_2^4} -
$$

$$
- e_1 \frac{\partial^3 v\,(t, x_1, x_2)}{\partial t^2 \partial x_1^2} - e_1 \frac{\partial^3 v\,(t, x_1, x_2)}{\partial t^2 \partial x_2^2} + e_0 \frac{\partial v\,(t, x_1, x_2)}{\partial t^2} +
$$

$$
+ c_2 \frac{\partial^5 v\,(t, x_1, x_2)}{\partial t \partial x_1^4} + 2 \frac{\partial^5 v\,(t, x_1, x_2)}{\partial t^2 \partial x_1^2 \partial x_2^2} + c_2 \frac{\partial^5 v\,(t, x_1, x_2)}{\partial t \partial x_2^4} -
$$

$$
- c_1 \frac{\partial^3 v\,(t, x_1, x_2)}{\partial t \partial x_1^2} - c_1 \frac{\partial^3 v\,(t, x_1, x_2)}{\partial t \partial x_1^2} + c_0 \frac{\partial v\,(t, x_1, x_2)}{\partial t^2} +
$$

$$
+ d_2 \frac{\partial^4 v\,(t, x_1, x_2)}{\partial x_1^4} + 2d_2 \frac{\partial^4 v\,(t, x_1, x_2)}{\partial x_1^2 \partial x_2^2} + d_2 \frac{\partial^4 v\,(t, x_1, x_2)}{\partial x_2^4} -
$$

$$
- d_2 \frac{\partial^3 v\,(t, x_1, x_2)}{\partial t \partial x_1^2} - d_2 \frac{\partial^3 v\,(t, x_1, x_2)}{\partial t \partial x_2^2} + d_0 v\,(t, x_1, x_2) =
$$

$$
= \sum_{j=1}^{m} b_j\,(t, x_1, x_2)\, u_j(t), \tag{28}
$$

with $t > 0$, $x_1 \in \left(0, \frac{\pi}{a}\right)$, $x_2 \in \left(0, \frac{\pi}{b}\right)$ and initial conditions:

$$
v\,(0, x_1, x_2) = v_0\,(x_1, x_2), \qquad \left. \frac{\partial v\,(t, x_1, x_2)}{\partial t} \right|_{t=0} = v_1\,(x_1, x_2) \tag{29}
$$

and boundary conditions

$$
v\,(t, 0, x_2) = \left. \frac{\partial^2 v\,(t, x_1, x_2)}{\partial_1^2} \right|_{x_1=0} = 0,
$$

$$
v\,(t, x_1, 0) = \left. \frac{\partial^2 v\,(t, x_1, x_2)}{\partial x_2^2} \right|_{x_2=0} = 0, \tag{30}
$$

$$
v\left(t, \frac{\pi}{a}, x_2\right) = \left. \frac{\partial^2 v\,(t, x_1, x_2)}{\partial_1^2} \right|_{x_1=\frac{\pi}{2}} = 0,
$$

$$
v\left(t, x_1, \frac{\pi}{b}\right) = \left. \frac{\partial^2 v\,(t, x_1, x_2)}{\partial x_2^2} \right|_{x_2=\frac{\pi}{b}} = 0. \tag{31}
$$

For $t > 0$, $x_1 \in \left(0, \frac{\pi}{a}\right)$, $x_2 \in \left(0, \frac{\pi}{b}\right)$.

Above, $e_i \geqslant 0$, $c_i \geqslant 0$, $i = 0, 1, 2$, $\sum_{i=0}^{2} c_i > 0$, d_0 and d_1 are unrestricted in sign and $d_2 \geqslant 0$.

Let $\Omega \subset \mathbb{R}^2$ be a rectangular domain defined as follows:

$$\Omega = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in \left(0, \frac{\pi}{a}\right), x_2 \in \left(0, \frac{\pi}{b}\right)\right\}.$$

The above initial-boundary value problem describes the transverse motion of a flexible slender rectangular isotropic and homogeneous Kirchhoff–Love plate in rectangular Cartesian coordinates, for which in plane deformations are neglected. The function $v(t, x_1, x_2)$ denotes the transverse displacement of the plate from the reference and stress-free state at time $t > 0$ and position (x_1, x_2). The lengths of rectangular plate sides are assumed to be equal to $\frac{\pi}{a}$ and $\frac{\pi}{b}$, respectively. The boundary conditions correspond to hinged edges of the plate.

The above partial differential equation with boundary conditions can be represented as a linear abstract differential equation in the Hilbert space $L^2(\Omega, \mathbb{R})$. In order to do that, it is necessary to introduce linear, unbounded differential operator $A \colon L^2(\Omega, \mathbb{R}) D(A) \to L^2(\Omega, \mathbb{R})$, defined by:

$$Av(x_1, x_2) = \frac{\partial^4 v(x_1, x_2)}{\partial x_1^4} + 2\frac{\partial^4 v(x_1, x_2)}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 v(x_1, x_2)}{\partial x_2^4} \tag{32}$$

with domain $D(A)$ corresponding to boundary conditions:

$$D(A) = \left\{ v \colon v \in H^4(\Omega), v(0, x_2) = \frac{\partial^2 v(x_1, x_2)}{\partial x_1^2}\bigg|_{x_1=0} = 0, \right.$$

$$v\left(\frac{\pi}{a}, x_2\right) = \frac{\partial^2 v(x_1, x_2)}{\partial x_1^2}\bigg|_{x_1=\frac{\pi}{a}} = 0, \quad x_2 \in \left(0, \frac{\pi}{b}\right),$$

$$v(x_1, 0) = \frac{\partial^2 v(x_1, x_2)}{\partial x_2^2}\bigg|_{x_2=0} = 0$$

$$v\left(x_1, \frac{\pi}{b}\right) = \frac{\partial^2 v(x_1, x_2)}{\partial x_1^2}\bigg|_{x_1=\frac{\pi}{b}} = 0, \quad x_1 \in \left(0, \frac{\pi}{a}\right) \left. \right\},$$

where $H^4(\Omega)$ is the fourth order Sobolev space on Ω.

Operator A has the following properties:

1. A is a self-adjoint, positive definite operator with domain $D(A)$ dense in the space $L^2(\Omega, \mathbb{R})$.

2. A has purely discrete point spectrum consisting entirely of the eigenvalues:

$$s_{kn} = a^4 k^4 + 2a^2 k^2 b^2 n^2 + b^4 n^4, \quad m, n = 1, 2, 3, \ldots$$

If the numbers a_2, b_2 are linearly independent over integers, then each eigenvalue of A has multiplicity 1.

3. The set of eigenfunctions $\{v_{kn}, k, n = 1, 2, \ldots\}$ of A forms a complete orthonormal system in the space $L^2(\Omega, \mathbb{R})$ and is given by:

$$v_{kn} = \frac{2\sqrt{ab}}{\pi} \sin akx_1 \sin bnx_2,$$

$$x_1 \in \left(0, \frac{\pi}{a}\right), \quad x_2 \in \left(0, \frac{\pi}{b}\right), \quad k, n = 1, 2, 3, \ldots$$

4. Fractional power $A^\beta, 0 < \beta < 1$ of operator A can be defined. Particularly, square root of A can be expressed as follows:

$$A^{\frac{1}{2}} v(x_1, x_2) = -\frac{\partial^2 v(x_1, x_2)}{\partial x_1^2} - \frac{\partial^2 v(x_1, x_2)}{\partial x_2^2}, \tag{33}$$

for $v \in D\left(A^{\frac{1}{2}}\right)$ where,

$$D\left(A^{\frac{1}{2}}\right) = \left\{v : v \in H^2(\Omega), v(0, x_2) = v\left(\frac{\pi}{a}, x_2\right) = 0, x_2 \in \left(0, \frac{\pi}{b}\right),\right.$$

$$\left. v(x_1, 0) = v\left(x_1, \frac{\pi}{b}\right) = 0, \quad x_1 \in \left(0, \frac{0}{\frac{\pi}{a}}\right) \right\}.$$

It should be pointed out, that all the derivatives are taken in the sense of distributions on Ω.

Applying operators (32) and (33), partial differential equation (28) with initial conditions (29) and boundary conditions (31), (30) can be expressed as an abstract differential equation in the Hilbert space $L^2(\Omega, \mathbb{R})$. To this aim, let us denote $x(t) = v(t, \cdot, \cdot)$ initial conditions $x(0) = v_0(\cdot, \cdot), \dot{x}(0) = v_1(\cdot, \cdot)$.

Therefore, taking into account Corollary 3, it is possible to formulate the following necessary and sufficient condition for controllability in an arbitrary time interval:

Kirchhoff–Love plate bending vibrations described by (28) are controllable in an arbitrary time interval if and only if

$$\sum_{k=1}^{m} \langle b_k, v_{mn} \rangle^2 = \sum_{k=1}^{m} \left(\int_{\Omega} b_k v_{mn} \, \mathrm{d}\Omega \right)^2 =$$

$$= \sum_{k=1}^{m} \left(\int_0^{\frac{\pi}{a}} \int_0^{\frac{\pi}{b}} b_k \, (x_1, x_2) \sin (akx_1) \sin (bnx_2) \, \mathrm{d}x_1 \mathrm{d}x_2 \right)^2 \neq 0,$$

for $k, n = 1, 2, 3 \ldots$.

Remark 5 *In the case, when numbers a_2 and b_2 are not linearly independent over integers, i.e., if there exists some integers l and k such that $a^2 = \frac{k}{l}b^2$, then operator A has infinite multiplicity, i.e., $\sup_i m_i = \infty$. In fact, if $k = l$, then operator A have eigenvalues of arbitrary high multiplicity. In this case, by Corollary 1, system (28) is not approximately controllable by finitely many dimensional controls.*

5.   Conclusions

The present paper contains results concerning approximate controllability of second order abstract infinite dimensional dynamical systems. Using the frequency-domain method [11] and the methods of functional analysis, especially theory of linear unbounded operators, necessary and sufficient conditions for approximate controllability in any time interval are formulated and proved. Moreover, some special cases are also investigated and discussed. Then, the general controllability conditions are applied to investigate approximate controllability any time interval of dynamical system modeling flexible mechanical structure.

Theorem 1 strongly simplifies approximate controllability considerations for second order systems. Moreover, the results presented in the paper are generalization of the controllability conditions given in the literature [1, 8, 11, 15–18] to second order abstract dynamical systems with damping terms. Finally, it should be pointed out, that the obtained results could be extended to cover the case of more complicated second-order abstract dynamical systems [3–6] and dynamical systems with positive controls [21, 22] or for dynamical systems with delays [8, 12].

Taking into account results presented above it can be conclude, that controllability problems for partial differential equations strongly depends on eigenfunctions and eigenvalues and their multiplicities for linear operator A. On the other

hand, eigenvalue problem of the operator A depends on the shape of the domain Ω and its boundary conditions. Moreover, the system can be approximately controllable only if the number of inputs is at least equal to the highest multiplicity of eigenvalues of the operator A.

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