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Research paper

Nonlinear double-beam system dynamics

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Abstract: Double-beam model is considered in many investigations both theoretical and typically engineering ones. One can find different studies concerning analysis of such structures behaviour, especially in the cases where the system is subjected to dynamic excitations. This kind of model is successfully considered as a reliable representation of railway track. Inclusion of nonlinear physical and geometrical properties of rail track components has been justified by various computational studies and theoretical analyses. In order to properly describe behaviour of real structures their nonlinear properties cannot be omitted. Therefore a necessity to search appropriate analytical nonlinear models is recognized and highlighted in published literature. This paper presents essential extension of previously carried out double-beam system analysis. Two nonlinear factors are taken into account and parametrical analysis of the semi-analytical solution is undertaken with special emphasis on different range of parameters describing nonlinear stiffness of foundation and layer between beams. This study is extended by preliminary discussion regarding the dynamic effects produced by a series of loads moving along the upper beam. A new solution for the case of several forces acting on the upper beam with different frequencies of their variations in time is presented and briefly discussed.

Keywords: double-beam, nonlinear dynamics, semi-analytical solution

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1. Introduction

The double-beam model is used in modelling of various structures. One of them is a rail track which can be represented as a two-layer system [1-3]. In this system, the first layer represents rails and the second one describes sleepers, both mathematically modelled by coupled and modified Euler-Bernoulli beam equations. This infinitely long two-layer model has been already used in the analysis of rail track dynamics in the case of nonlinear foundation. Results for vertical vibrations generated by moving train, obtained by using the Fourier transform combined with Adomian's decomposition [4-6] and analytical wavelet based approximation [7-9], are validated by comparison with experimental measurements done on real railway track [2, 10].

The method of solution was also validated for various systems, including a one-layer model of rail track, described by the Euler-Bernoulli beam representing rails resting on viscoelastic foundation with nonlinear stiffness [10]. Due to hybrid character of this heuristic method and overlapping analytical approximations one should carefully consider conditions for correctness and convergence of solution depending on particular application [9, 11, 12]. These issues were also discussed in past papers. On the other hand, classical approaches are insufficient to obtain nonlinear solution exact enough. The system can be solved by using semi-numerical tools, e.g. FEM, but these are burden with uncontrolled errors and, in addition, are recognized as computationally and time consuming methods, mainly due to a need of calculations repetition for each change of the system physical parameters.

The mentioned difficulties lead to a necessity of seeking more reliable models and solution methods, especially in the area of analytical approaches, with possibility of control over solution correctness and applicability of built systems. Therefore a detailed study of the double-beam system possessing two nonlinear factors must be undertaken in order to recognize its all important features before application to realistic engineering structures analysis [12-14]. The infinitely long double-beam model discussed in this paper was previously analytically solved in linear case [15]. The linear system was next extended by assumption describing viscoelastic foundation as nonlinear one [13]. After that, nonlinear factor was included in the layer between beams, with the system resting on linear viscoelastic foundation, which corresponds to properties of fastening systems in rail track [14]. The applied nonlinearity was represented by cubic function [16, 17]. The Adomian's decomposition combined with wavelet based approximation allowed obtaining reliable solution with controlled accuracy which could be directly used in the analysis of rail track behaviour. In recently published paper, the double-beam system with two nonlinearities included both in foundation and in viscoelastic layer connecting beams was successfully solved [12]. Modified convergence conditions for approximate wavelet based solution were also proposed to control an accuracy of results. However, a detailed parametrical analysis was left as an open problem.

In this paper, the formulated and solved previously nonlinear infinitely long double-beam system is studied in details with regard to parameters describing nonlinear stiffness of foundation and layer between beams. This study can be recognized as a first step towards the model applicability analysis and it is an essential extension of already published results. Preliminary discussion concerning the dynamic response of the system to a series of loads moving along upper beam leads to recognition of possibilities to represent real rail track behaviour, where track bed and fastening systems possess nonlinear stiffness. The main novelty of present paper is a solution for the case with several forces moving along the upper beam and distributed on some interval, among which some can vary in time with different frequency. Reliable solution in this case makes the model ready to apply directly in railway engineering, where the whole train with loads coming from the vehicle axles must be taken into account. This fact, along with a source of nonlinearities arising from laboratory investigations (stiffness of rail track foundation and fastening systems) makes the developed model and its solution a good tool for the track dynamics analysis.

One should underline that vertical vibrations are the main goal chosen during initial studies concerning railway track dynamic behaviour. Other characteristics are out of scope of the investigation presented in this paper and need different way of modelling. However, the vertical vibrations of rail track are the main factor deciding about its stability and safety. The analysed models give also possibility to calculate, besides the beams dynamic deflection, vibration velocity and acceleration which leads to direct comparison with experimental measurements.

2. Model formulation

The theoretical model analysed in this paper, called a double-beam system, is composed of two infinitely long beams connected by nonlinear viscoelastic layer [12-14]. This system lays on foundation which is also nonlinear and viscoelastic (Fig. 1). A set of distributed and harmonically changing in time forces is moving along upper beam generating vibrations that might cause instability of the system.

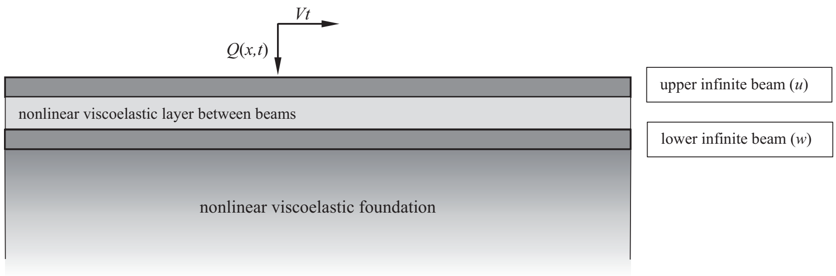


Fig. 1. Nonlinear double-beam model.

Linear case of this structure was already studied and its solution can be obtained quite easily by using classical analytical methods. Inclusion of two nonlinear factors makes it however relatively complex and several approximations must be applied in order to solve partial differential equations. For this purpose, an analytical wavelet approximation using coiflet filters (Eqs. 2.1-2.2) combined with Adomian's decomposition (Eqs. 2.3-2.5) can be used [8, 9, 13]. The following formula defines approximated inverse Fourier transform of the transformed function $\widetilde{w}$ in the physical domain [9]:

(2.1) $$w(x) = \lim_{n\to\infty} w_n(x)$$

(2.2) $$w_n(x) = \frac{1}{2^{n+1}\pi} \prod_{k=1}^{K(N)} \left(\sum_{j=0}^{3N-1} p_j e^{ijx/2^{n+k}}\right) \sum_{k=-\infty}^{+\infty} \widetilde{w}((k+M)2^{-n}) e^{ixk2^{-n}}$$

where $M = \sum_{k=0}^{3N-1} kp_k$ and N is a degree of accuracy for applied wavelet filter (p_k). Adomian's decomposition allows to avoid difficulties related to nonlinearities. It assumes that solution of the problem can be found as an infinite series [18]:

(2.3) $$w(x,t) = \sum_{j=0}^{\infty} w_j(x,t)$$

where the first term is a solution of linear problem and others can be represented by so called Adomian polynomials

(2.4) $$w^3(x,t) = \sum_{k=0}^{\infty} P_k(x,t)$$

(2.5) $$P_k(x) = \frac{1}{k!}\left[\frac{d^k}{d\lambda^k}\left(\sum_{j=0}^{\infty}\lambda^j w(x,t)\right)^3\right]_{\lambda=0}, \ (k = 0,1,2,\dots).$$

The consecutive terms of the series (2.3) can be controlled by the "error index" allowing a proper choice of the approximation with assumed level of accuracy [12-14, 19]. However, the stability of solution with regard to various systems of changing parameters and coefficients appearing in wavelet formulas remains the only reliable criterion for a proper solution (Eqs. 2.1-2.2).
The double-beam model considered in this paper can be mathematically formulated as a system of coupled partial differential equations describing two Euler-Bernoulli beams as follows:

(2.6) $$EI_u\frac{\partial^4 u}{\partial x^4} + m_u\frac{\partial^2 u}{\partial t^2} + c_u\left(\frac{\partial u}{\partial t} - \frac{\partial w}{\partial t}\right) + k_u(u-w) + k_{Nu}u^3 - k_{Nw}w^3 = Q(x,t)$$

(2.7) $$EI_w\frac{\partial^4 w}{\partial x^4} + m_w\frac{\partial^2 w}{\partial t^2} + c_w\frac{\partial w}{\partial t} + k_w w - c_u\left(\frac{\partial u}{\partial t} - \frac{\partial w}{\partial t}\right) - k_u(u-w) + k_{Nw}w^3 - k_{Nu}u^3 = 0.$$

The following standard notations are used:
$u(x,t)$ [m] and $w(x,t)$ [m] – vertical vibrations of upper and lower beam, respectively;
EI_u [Nm^2], m_u [kg/m] – bending stiffness and unit mass of upper beam;
EI_w [Nm^2], m_w [kg/m] – bending stiffness and unit mass of lower beam;
k_{Nu} [N/m^4], k_u [N/m^2], c_u [Ns/m^2] – nonlinear part of stiffness, linear stiffness and viscous damping of the layer connecting beams;
k_{Nw} [N/m^4], k_w [N/m^2], c_w [Ns/m^2] – nonlinear part of stiffness, linear stiffness and viscous damping of the foundation;
$Q(x,t)$ [N/m] – a load moving uniformly along upper beam.
The load $Q(x,t)$ can be composed of various terms such as quasi-static stationary constant in time part generated e.g. by the weight of vehicle in real scenario, dynamic part arising from vertical irregularities of contact surface between upper beam and force (e.g. regular imperfections of rail head rolling surface) or regular changes of layers or beams stiffness. One can also consider random factors that cannot be described by regular functions [20, 21]. These are related to unpredictable geometrical and physical properties of the whole double-beam structure.
In this paper, the load is represented by a set of 3 identical forces moving at constant distance one from another. The forces are distributed on some interval and change harmonically in time with constant frequency:

(2.8) $$Q(x,t) = \sum_{k=1}^{3} Q_k(x,t)\frac{1}{r}\cos^2\left(\frac{\pi(x-Vt)}{2r}\right)H(r^2-(x-Vt)^2)\exp\left(i\Omega t\right)$$

where x [m] is a space variable along the beam, t [s] is a time variable, $H(.)$ is the Heaviside step function and $2r$ [m] is the span of each force. The set of forces moves with constant velocity V [m/s] and frequency $\Omega = 2\pi \cdot f_\Omega$ (the value of $f_\Omega = 10$ Hz is fixed for the parametrical analysis, except of examples in section 5).

The solution of this system is not a new result. It was solved recently but only some computational examples were provided in order to present possibilities of the developed method in the dynamical analysis of structures based on double-beam systems [12]. In these examples, it was assumed that nonlinear terms for both viscoelastic layers were identical. In reality, these parameters vary depending on local conditions. This observation was confirmed by experimental measurements in the case of railway track (i.e. for track bed and fastening system). Therefore detailed parametrical analysis of the theoretical system and its extensive investigations towards the evaluation of applicability domain must be done. Due to advanced technique of solution, involving several approximations supported by heuristic ideas, a short discussion regarding difficulties appearing in computations should be presented. Some crucial remarks are given in the next section.

3. Remarks on hybrid semi-analytical solution

Computational complexity of the analysed system requires an application of sophisticated tools different from classical ones based on closed-form solutions obtained by integration over contour or Fourier series. Because the main goal of this study remains an estimation of the model applicability, one must have possibility of efficient parametrical analysis with convergence conditions securing mathematically proper solution. The method applied in this paper combines a few approximation approaches, including mentioned before Adomian's decomposition applied to nonlinear terms of the layers' stiffness. The whole procedure can be divided into the following stages:

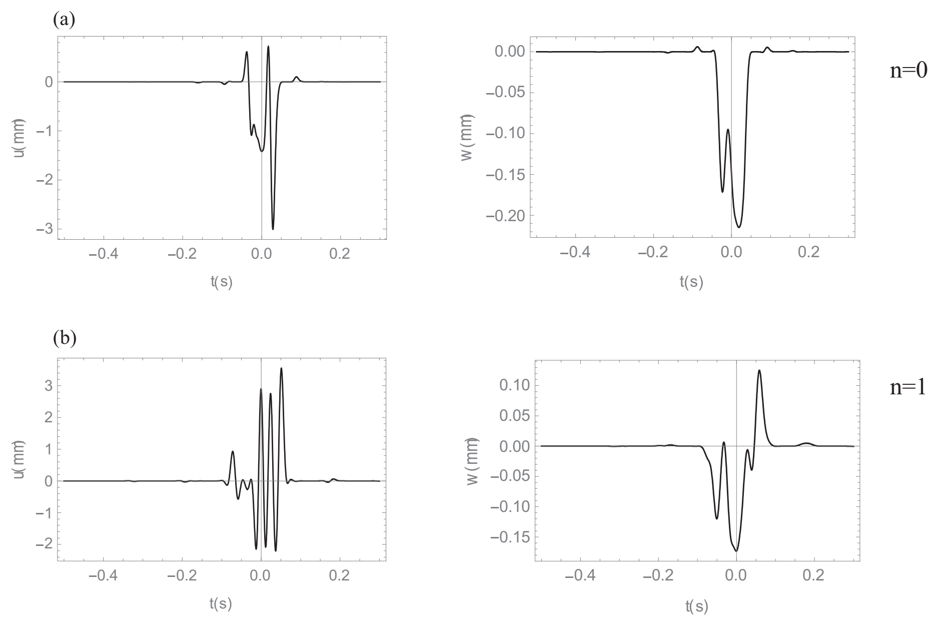
1. Application of the moving coordinate system.
2. Decoupling of the differential equations system.
3. Decomposition of nonlinear terms into Adomian series.
4. Applying the Fourier transform in its classical form, besides the Adomian polynomials which cannot be calculated directly.
5. According to 4., using a coiflet based approximation to calculate Adomian series.

6. Re-transformation of the obtained solution in the transform domain. Once again, the wavelet based semi-analytical method is applied, leading to results in the physical domain.

All these approximations make the procedure relatively complex and due to several overlapping approximations make possible an unacceptable error level highly probable. Convergence conditions for the solution of the investigated model were formulated and discussed in previously published papers [12-14]. However, the main criterion, i.e. the solution stabilization with regard to a set of parameters used in calculations, must be checked separately, being individual feature for particular cases.

The developed hybrid method [9] is already recognized by researchers as an efficient approach giving new solutions in subjects of nonlinear and stochastic modelling [22, 23].

Figure 2 shows subsequent approximations of beams vertical vibrations for 3 forces moving along the upper beam (Eqs. 2.6-2.7) with constant velocity $V = 100\ \mathrm{m/s}$ and the distance between them equal to 10 m, with an assumption of identical nonlinear characteristics of both layers: between beams and supporting the structure.



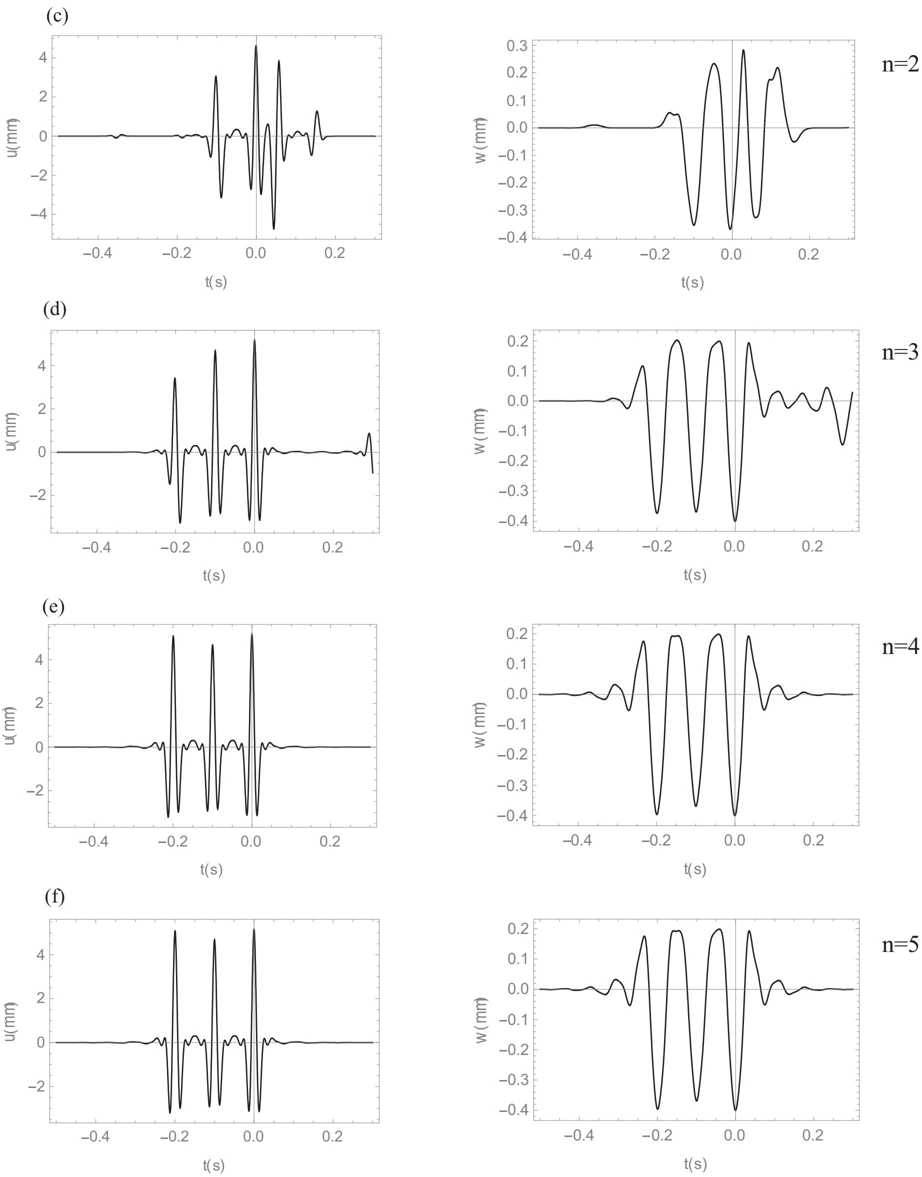


Fig. 2. Successive wavelet approximations (2.1-2.2) of double-beam system vertical vibrations generated by a set of 3 forces acting on upper beam.

The system of other parameters is taken close to previously considered in published papers [11-14]:

$P = 5 \cdot 10^5$ N/m, $k_{Nu} = 10^{14}$N/m^4, $k_{Nw} = 10^{14}$N/m^4, $EI_u = 10^7$Nm^2, $m_u = 100$ kg/m, $k_u = 4 \cdot 10^7$N/m^2, $c_u = 0.06 \cdot \sqrt{k_u \cdot m_u}$, $EI_w = 1.5 \cdot 10^9$Nm^2, $m_w = 3500$ kg/m, $k_w = 5 \cdot 10^7$N/m^2, $c_w = 0.06 \cdot \sqrt{k_w \cdot m_w}$, $f_\Omega = 10$ Hz. As before, this is done on purpose, to better compare various cases of similar models with nonlinearities, i.e. the system with nonlinear foundation and linear stiffness of beams connection, the system with linear foundation stiffness and nonlinear layer connecting beams, and the entirely linear model. One can see that the desired order of wavelet approximation is relatively high. The sixth order of approximation is used in further calculations. One can observe specific nature of this approximation which can be treated as a shape estimation, instead of point convergence [9]. The point convergence is obtained at the final stage, when the shape of function is already stabilized according to the “stabilization condition”.

4. Parametrical analysis

In this section, a load composed of 3 forces varying in time with the same frequency of 10 Hz is assumed. Each of these forces is distributed on some interval with a length *2r* = 0.02 m and with density defined by Eq. 2.8. The load moves with constant velocity along upper beam.

Figure 3 shows the vertical vibrations of double-beam in the case of stronger nonlinearity appearing in the layer between beams, compared to foundation nonlinear stiffness. One can observe strong influence of nonlinearity on behaviour of upper beam. The nonlinear amplitude is much higher than for linear solution. At the same time, nonlinear solution for a lower beam vibrations only slightly differs from linear one. Because nonlinear solution is difficult to interpret, very often its complex modulus is analysed leading to the “system sensitivity” investigation in terms of its “maximal response”. This kind of visualisation gives more precise conclusions regarding the effect of various parameters on the system reaction (or the model stability – solution convergence) but also about behaviour of real structures when applied in practical case. This feature can be confirmed by Figs. 3b, 3d and 4b, 4d.

Because the distance between consecutive forces is relatively big and their speed is quite low, the obtained results are easy to analyse, e.g. the response for 3 separated forces is clearly visible, which might be different when shorter distance between forces or higher velocity of moving load are assumed.

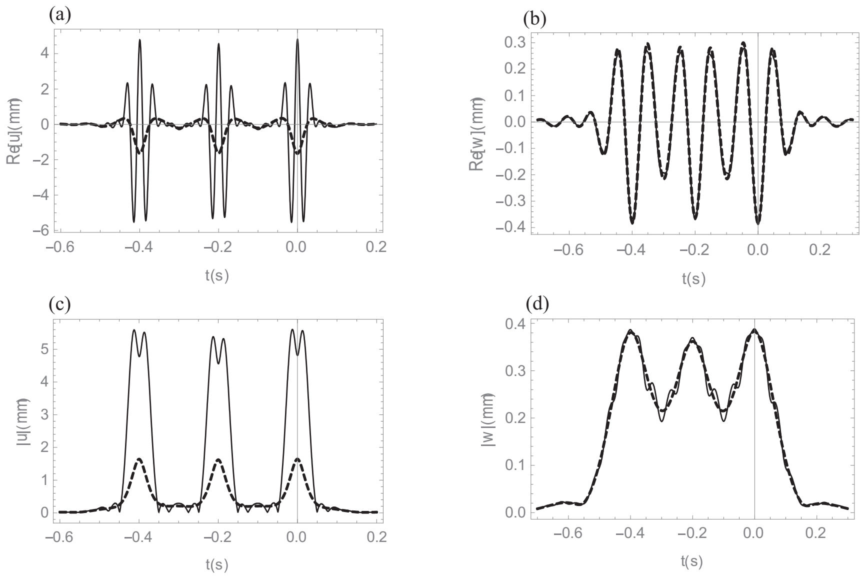


Fig. 3. Vertical vibrations of double-beam system in the case of $V = 50$ m/s, $k_{Nu} = 10^{14}$N/m^4 and $k_{Nw} = 4 \cdot 10^{13}$N/m^4: linear – dashed, nonlinear – solid.

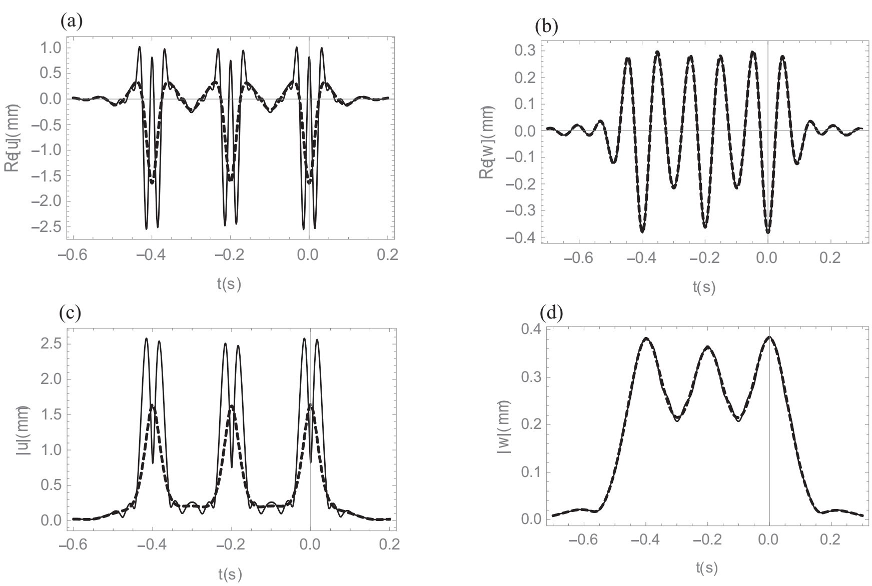


Fig. 4. Vertical vibrations of double-beam system in the case of $V = 50$ m/s, $k_{Nu} = 4 \cdot 10^{13}$N/m^4 and $k_{Nw} = 10^{14}$N/m^4: linear – dashed, nonlinear – solid.

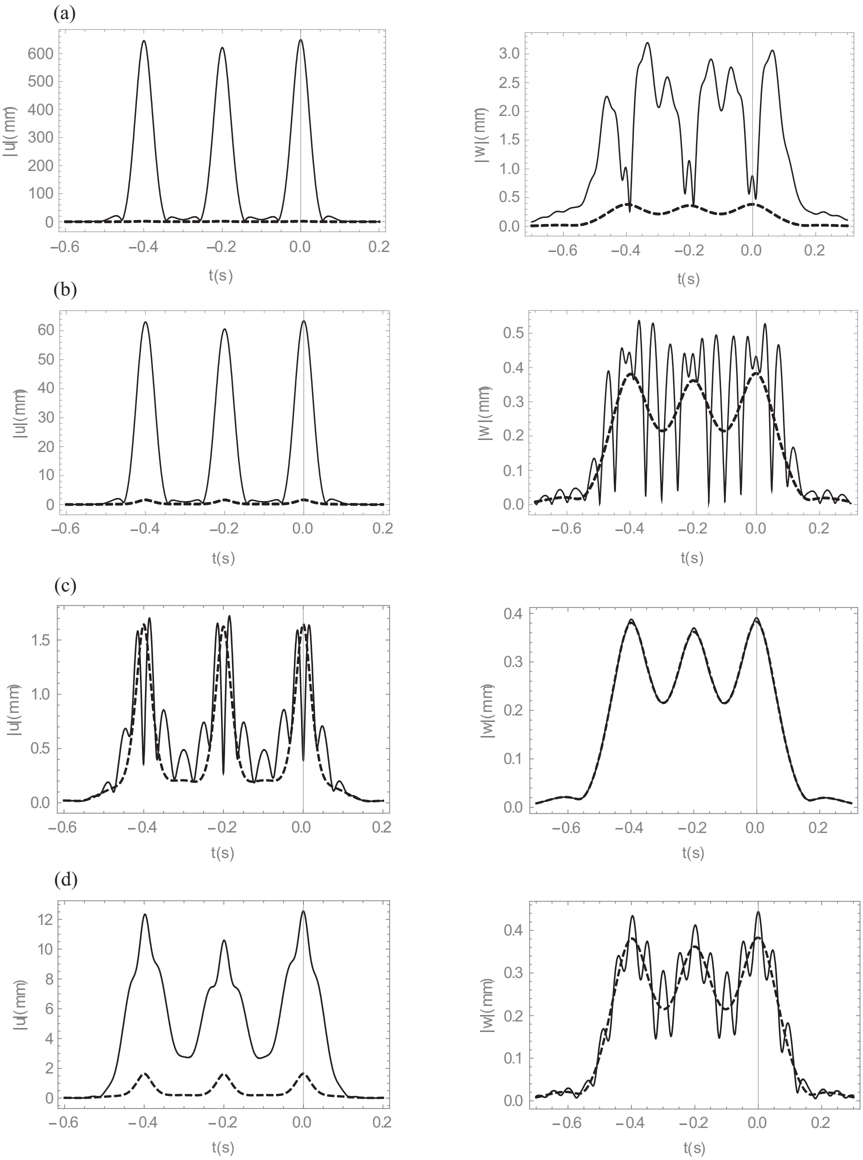


Fig. 5. System sensitivity (linear – dashed, nonlinear – solid): (a) $k_{Nu} = 10^{16}\mathrm{N/m^4}$, $k_{Nw} = 4 \cdot 10^{13}\mathrm{N/m^4}$; (b) $k_{Nu} = 10^{15}\mathrm{N/m^4}$, $k_{Nw} = 4 \cdot 10^{13}\mathrm{N/m^4}$; (c) $k_{Nu} = 4 \cdot 10^{13}\mathrm{N/m^4}$, $k_{Nw} = 10^{15}\mathrm{N/m^4}$; (d) $k_{Nu} = 4 \cdot 10^{13}\mathrm{N/m^4}$, $k_{Nw} = 10^{16}\mathrm{N/m^4}$.

Much weaker influence of nonlinearities can be observed when the nonlinear factor in supporting layer is stronger than in the layer connecting two beams (Fig. 4). This observation is valid for a wide set of parameters and confirms the necessity of fastening systems analysis under assumptions of their nonlinear characteristics. These nonlinear properties of fastening systems, being in use in operational railway tracks, are confirmed experimentally, while their influence on the rail track dynamics was not studied so far, mainly due to a lack of reliable analytical models and solution possibilities. The model developed and analysed in present paper can be used in such an analysis with a modification of a lower layer, by assumption of a zero bending stiffness of a lower beam. This modification makes it a rigid body which can characterize sleepers in the case of conventional railway track [1-3].
Additional examples of the system sensitivity characterization (the maximal response) can be found in Fig. 5. They confirm conclusions made above. In addition, one should underline that in the case of higher values of parameter k_{Nw}, compared to k_{Nu}, stability of the lower beam can be lost only for a relatively strong nonlinearity of foundation stiffness. Therefore, the analysis of the upper beam and properties of the layer connecting beams seems to be crucial for stability and convergence study, and, consequently, for the model applicability when applied to railway engineering problems. In that case, characteristics of rails and fastening systems must be carefully checked.

5. Forces with varying frequencies

In reality, each axle of train vibrates with different frequency. Even more, these frequencies are varying during a passage of vehicle. Frequencies coming from some regular imperfections can be modelled, more or less, as periodical phenomena, e.g. corrugations or stiffness changes generated by sleepers spacing. Others are difficult to describe, e.g. those coming from vehicle structure.
The model described in this section is a first attempt to the analysis of real rail track behaviour. It is assumed that two boundary forces in the considered sequence forming a moving load vibrate with frequency different than the middle force. As a further work, an assumption of frequency changing in time is considered. This, however, as well as other details making the model possible to apply in railway engineering, needs additional studies yet.
Figure 6 shows the system response (vertical vibrations) to a load consisting of 3 forces: the first one and the third one are varying with frequency $f_\Omega = 10$ Hz, while the middle one with frequency $f_\Omega = 2$ Hz.

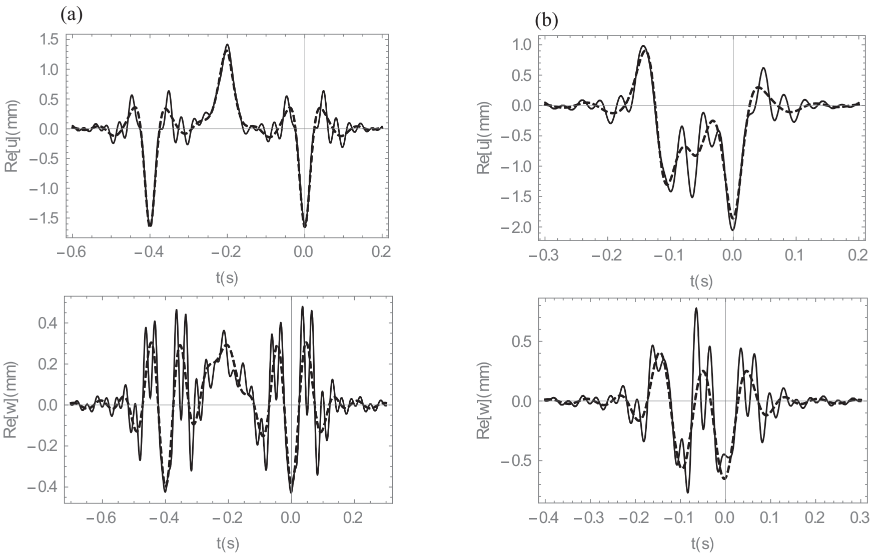


Fig. 6. Vertical vibrations of double-beam system (linear – dashed, nonlinear – solid) in the case of $V = 50\ \mathrm{m/s}$, $k_{Nu} = 10^{15}\mathrm{N/m^4}$, $k_{Nw} = 4 \cdot 10^{13}\mathrm{N/m^4}$ and different distances between forces: (a) 10 m; (b) 3 m.

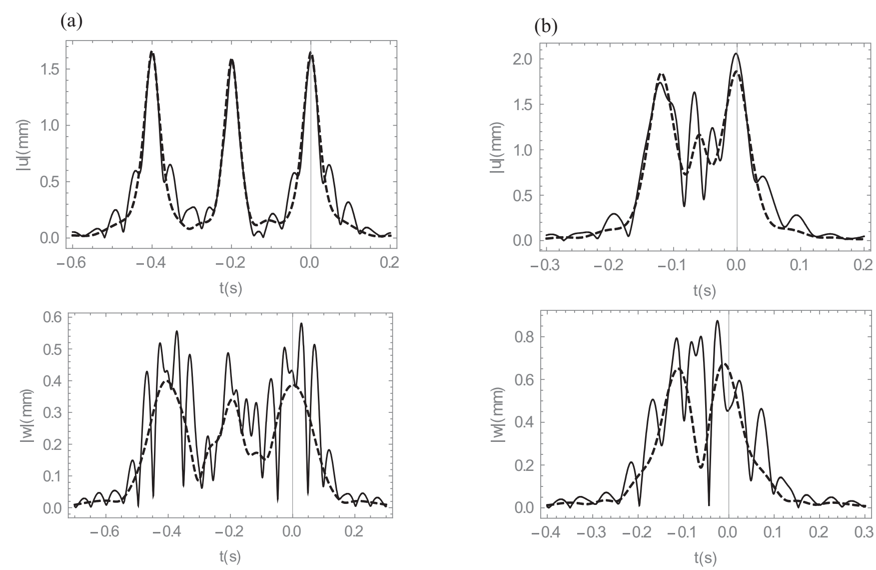


Fig. 7. Double-beam system sensitivity (linear – dashed, nonlinear – solid) in the case of $V = 50\ \mathrm{m/s}$, $k_{Nu} = 10^{15}\mathrm{N/m^4}$, $k_{Nw} = 4 \cdot 10^{13}\mathrm{N/m^4}$ and different distances between forces: (a) 10 m; (b) 3 m.

Figure 7 presents the maximal response in the same situation. One can see that the decreasing distance between forces leads to accumulation of nonlinear effects (stronger response in terms of vibrations amplitude and the maximal response), in the case of stronger nonlinearity between beams. The considered form of the load reflects stronger effect on the system behaviour compared to uniform forces, so it can have essential influence on solutions and system stability. This situation appears when the layer between beams possesses stronger nonlinearity. When the nonlinearity in foundation becomes stronger in relation to the layer between beams, the nonlinear part of solution is relatively small and the influence of nonlinearities can be practically neglected. Figure 8 shows the second terms of Adomian series (Eq. 2.3), which are responsible for nonlinear solution of the system 2.6-2.7 (see also Fig. 5c). One can see their marginal values in the considered cases. They can be of course higher for other systems of parameters and this issue is left for further analysis.

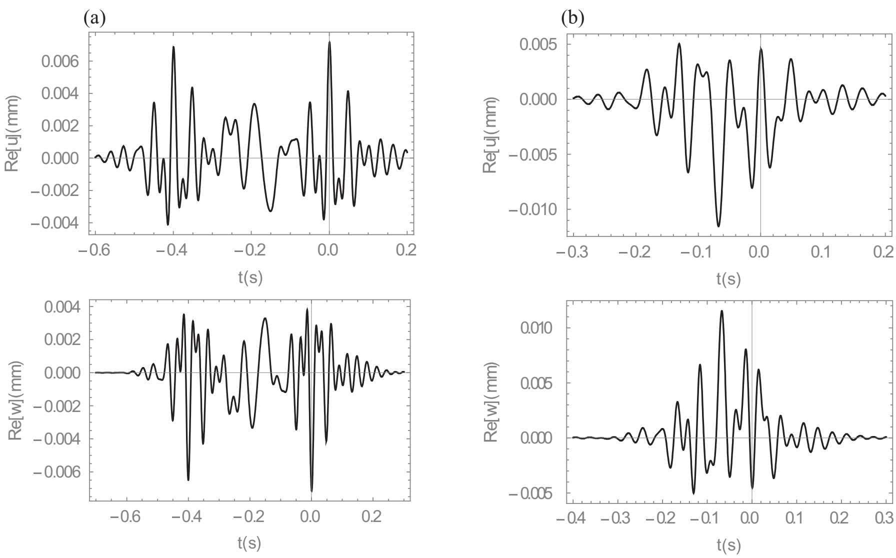


Fig. 8. Second term of Adomian series (Eq. 2.3) ($V = 50$ m/s, $k_{Nu} = 4 \cdot 10^{13}$N/m^4, $k_{Nw} = 10^{15}$N/m^4) for different distances between forces: (a) 10 m; (b) 3 m.

Figure 7 shows also a good example of better representation of the nonlinear system response by the complex modulus. One can clearly see 3 separated forces acting on upper beam (comp. Fig. 6b and Fig. 7b). This effect vanishes for a lower beam and appears again along with increasing distance between forces (Fig. 7a).

6. Conclusions

Theoretical continuous infinitely long double-beam system with two nonlinearities included in the layer connecting beams and foundation is considered. Semi-analytical solution based on Adomian's decomposition and wavelet based approximation is used for parametrical study of the developed model. Vertical vibrations and complex modulus (the maximal response) are analysed for several cases and a wide range of parameters. Particularly, an influence of differences between values of parameters describing nonlinear stiffness of two layers (the foundation layer and the layer between beams) on the system response is investigated. Additional assumption about varying frequencies of several forces forming the moving load is introduced making the model closer to practical applications in railway engineering. The study done in this paper makes another step towards evaluation of the model applicability supported by stability and solution convergence analysis.

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Dynamika nieliniowego układu belki podwójnej

Słowa kluczowe: belka podwójna, dynamika nieliniowa, rozwiązanie semi-analityczne, metoda hybrydowa, inżynieria kolejowa

Streszczenie:

Model belki podwójnej jest często wykorzystywany w budowie układów wielowarstwowych opisujących zjawiska związane z ruchomymi obciążeniami. Nieliniowe i stochastyczne właściwości tych układów wpływają znacząco na ich dynamiczne zachowania, co zostało potwierdzone zarówno eksperymentalnie, jak i w wyniku badań teoretycznych. Dlatego wskazana jest szczegółowa analiza czułości rozważanych modeli na różne parametry, przed ich zastosowaniem do badania rzeczywistych konstrukcji.

W artykule rozważany jest problem odpowiedzi układu belki podwójnej na system sił poruszających się wzdłuż górnej belki ze stałą prędkością i różnymi częstotliwościami, przy założeniu różnych wartości parametrów opisujących nieliniową sztywność dwóch warstw: podłoża i warstwy łączącej belki. Otrzymane semi-analityczne rozwiązanie dla opisanego przypadku jest znaczącym rozszerzeniem poprzednio opublikowanych rezultatów. Założenie dotyczące występowania dwóch nieliniowości prowadzi do konieczności zastosowania szeregu analitycznych aproksymacji, włączając dekompozycję Adomiana i falkową estymację szukanego rozwiązania. Te nakładające się przybliżenia czynią całą procedurę bardziej skomplikowaną i dlatego kontrolowanie zbieżności rozwiązań jest trudniejsze niż w przypadku poprzednio rozważanych uproszczonych modeli.

Analiza parametryczna, wykonana dla szerokiego zakresu różnych parametrów, pokazuje, że nieliniowość warstwy pomiędzy belkami wpływa bardziej znacząco na zachowanie układu, w porównaniu do podobnego założenia dotyczącego podłoża. Analiza przedstawiona w artykule może być traktowana jako kolejny etap określenia zakresu stosowalności modelu i zbadania możliwości jego zastosowania w inżynierii kolejowej, w odniesieniu do dwuwarstwowego modelu

toru kolejowego, opartego na układzie belki podwójnej, w którym pierwsza warstwa opisuje szyny, a druga modeluje warstwę podkładów. Takie podejście do modelowania drogi szynowej zostało już wcześniej poddane walidacji razem z hybrydową semi-analityczną metodą rozwiązania opartą o aproksymacje wykorzystujące filtry falkowe typu „coiflet". Rozszerzenie tego modelu o wprowadzenie dodatkowej nieliniowości, stanowi ważny element badań w zakresie analizy dynamiki dróg szynowych.

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