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CONTROL AND INFORMATICS

# Stability and robustness analysis of discrete-time fractional-piecewise-constant-order PID controller

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**Abstract.** In the paper we propose a fractional-piecewise-constant-order PID controller and discuss the stability and robustness of a closed loop system. In stability analysis we use the transform method and include the Nyquist-like criteria. Simulations for designed controllers are performed for the second-order plant with a delay.

**Key words:** stability analysis; fractional calculus; control systems; digital control.

## 1. INTRODUCTION

Fractional-order calculus has recently attracted significant attention which is due to its usefulness in describing and solving the problems in multiple fields of science such as chaotic systems, thermodynamics, biophysics and many other, see e.g. [1, 2]. It can also be successfully applied in control engineering, where in many cases fractional calculus application allows us to create more accurate models of real life processes and helps with designing more accurate control procedures as shown e.g. in [3]. One of the first approaches to fractional-order control was an introduction of fractional-order PID controller by Igor Podlubny [4]. The difference between regular PID and fractional-order PID (FOPID) is that FOPID controller in addition to three standard tuning parameters $K_p$, $K_i$, $K_d$ (which correspond to proportional, integral and derivative gains) also has two order parameters often marked as $\lambda$ and $\mu$. In this case parameter $\lambda$ is an order of integral/summation, while $\mu$ is an order of derivative/difference term of PID controller. Having regard to the above, standard PID controller can be considered as a particular case of FOPID with both orders equal to 1. As it was presented in many papers, the introduction of two additional tuning parameters in many cases (depending on the nature of the control problem) may help to achieve more accurate and robust control processes [3–11]. An excellent review of fractional controllers is presented in [12]. The reason why in the work we focus on digital control is that nowadays the control of a system performance is mainly carried out with the aid of micro-controllers and microprocessors which work in a discrete time domain, see [13]. Additionally, many processes, described by analog systems, can be successfully controlled by digital systems.

The current paper focuses on a family of controllers which can be considered as a generalization of fractional-order PID and is called fractional-variable-order PID (FVOPID). The controllers of this type have constant orders of integral/summation and derivative/difference terms replaced by two non-constant order functions. It means that integral and derivative orders are in this case changed during the control process and can depend on e.g. time, or the current control error value as shown in [14–16]. An introduction to FVOPID controllers theory can be found in [17]. An example implementations of fractional-variable-order PID controllers are described in [18, 19]. Although the idea of FVOPID is not new and there are different approaches to FVOPID control design described in several papers, there are still no fully effective methods of tuning or establishing the stability of fractional-variable-order systems.

In this paper we introduce the method of determining the stability of closed loop system with a digital FVOPID controller. Additionally, the implementation and tuning method of FVOPID controller is presented. The paper is structured as follows. At the beginning the theoretical basis of fractional-variable-order Grünwald–Letnikov difference operator is introduced together with stability analysis of fractional-variable-order systems. Thereafter, the design details of FVOPID controllers are described followed by simulation results obtained by the implemented controllers. Additionally, the robustness of optimal controller is determined by checking its responses for the disturbed plant (modified value of the plant coefficients and delay) in frequency domain. There are also presented figures with Nyquist plots confirming stability of the closed loop systems.

## 2. PRELIMINARIES AND DESCRIPTIONS

### 2.1. Sampled-data systems

Computers used in control systems are connected to analog elements of the system with signal converters. The output of the computer is in this case processed by a digital-to-analog converter. We assume a constant sampling period $h > 0$. Sampled data (or a discrete signal), denoted in this work as $x(kh)$, are

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data obtained for the system variables only at discrete intervals. To obtain described sampled data a sampler device is used, which is basically a switch that closes every $h$ seconds for one instant time. If the input of such a sampler is a contours signal $r(t)$, then a sampled value at the moment $kh$ could be written as $r^*(t) = r(kh)\delta(t-kh)$, where $\delta$ is the impulse function. In this case digital-to-analog converter used in the system can be represented by a zero-order hold circuit. The zero-order hold takes the value $r(kh)$ and holds it constant for $kh \leq t < (k+1)h$. The transfer function, in Laplace transform, of the zero-order hold is

$$G_0(s) = \frac{1-e^{-sh}}{s}. \tag{1}$$

For discrete-time signals we use $\mathscr{Z}$-transform defined for the reference continuous signal by

$$\mathscr{Z}[r(t)](z) = \mathscr{Z}[r^*(t)](z) = \sum_{k=0}^{\infty} r(kh)z^{-k}. \tag{2}$$

Let us consider a closed-loop, sampled-data control system shown in Fig. 1. The transfer function of the system is described as

$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1+G(z)}, \tag{3}$$

where $G(z)$ is the $\mathscr{Z}$-transform of $G(s) = G_0(s)G_p(s)$ with $G_0(s)$ being the zero-order hold and $G_p(s)$ the process transfer function. Then $G(z)$ is described as

$$G(z) = \mathscr{Z}\left[\mathscr{L}^{-1}\left[G_0(s)G_p(s)\right](kh)\right](z). \tag{4}$$

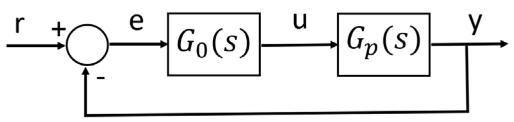


**Fig. 1.** Closed loop system with zero-order hold and process blocks

When to the presented system we add a digital controller with the $\mathscr{Z}$-transfer function $G_c(z)$, which is located between summation and zero-order hold $G_0(s)$ blocks, then the $\mathscr{Z}$-transfer function of the closed-loop system with the mentioned controller is given by

$$\frac{Y(z)}{R(z)} = \frac{G(z)G_c(z)}{1+G(z)G_c(z)}. \tag{5}$$

Equation (5) describes in this case the $\mathscr{Z}$-transfer function of the closed loop control system.

### 2.2. The Grünwald-Letnikov fractional-variable-order differences

In a digital fractional-variable-order PID controller we use the Grünwald-Letnikov summation and backward difference to calculate the output signal. Both operators: summation and difference use a sequence of values of oblivion function, which is presented in the following definition.

**Definition 1.** [20] Let $k, i \in \mathbb{Z}$ and $\mu \colon h\mathbb{Z} \to \mathbb{R}$ be an order function. Then, an *oblivion function* is defined as

$$a^{\mu(kh)}(i) = \begin{cases} 0, & i < 0 \\ 1, & i = 0 \\ (-1)^i \frac{\mu(kh)(\mu(kh)-1)\cdots(\mu(kh)-i+1)}{i!}, & i > 0 \end{cases}. \tag{6}$$

Note that $\dfrac{\mu(kh)(\mu(kh)-1)\cdots(\mu(kh)-i+1)}{i!} = \dbinom{\mu(kh)}{i}$.

Hence the sequence $\left(a^{\mu(kh)}(i)\right)_{i\in\mathbb{N}}$ can be rewritten as $a^{\mu(kh)}(i) = (-1)^i\binom{\mu(kh)}{i}$. Presented oblivion function can be used to define the Grünwald-Letnikov difference operator as described below.

**Definition 2.** [21] Let $h > 0$ and $x \colon h\mathbb{Z} \to \mathbb{R}$ be a bounded function. The Grünwald–Letnikov fractional-variable-order difference operator with step $h > 0$ of function $x$ with an order function $\mu \colon h\mathbb{Z} \to \mathbb{R}$ is defined as a finite sum

$$\Delta^{\mu(\cdot)}x(kh) = \sum_{i=0}^{k} h^{-\mu(ih)}a^{\mu(ih)}(i)x(kh-ih)$$

$$= \begin{bmatrix} \frac{1}{h^{\mu(0)}} & \frac{a^{\mu(h)}(1)}{h^{\mu(h)}} & \dots & \frac{a^{\mu(kh)}(kh)}{h^{\mu(kh)}} \end{bmatrix} \begin{bmatrix} x(kh) \\ x(kh-h) \\ \dots \\ x(h) \\ x(0) \end{bmatrix}. \tag{7}$$

For negative values of an order function $\mu(\cdot)$ Definition 2 can be treated as a discrete-time version of fractional-variable-order integration that is defined by (7) with the limit of the sample step $h$ going to zero. For positive values of an order function $\mu(\cdot)$ it becomes a backward difference with memory. Additionally in our calculations we assumed that $\mu(0) = \mu(h)$.

Observe that having convolution operator we can write that

$$\mathscr{Z}\left[\Delta^{\mu(\cdot)}x\right](z) = X(z)\mathscr{A}_\mu(z), \tag{8}$$

where $X(z) := \mathscr{Z}[\bar{x}](z)$ and

$$\mathscr{A}_\mu(z) := \sum_{k=0}^{\infty}(-1)^k\binom{\mu(kh)}{k}z^{-k}h^{-\mu(kh)}. \tag{9}$$

It is known that for the linear discrete-time system to be stable, the closed loop poles must lie within the unit circle in $z$-plane. Otherwise the linear system would be unstable. Hence we are interested in the image of complex numbers from the unit circle, i.e. complex numbers that belong to $\{z \in \mathbb{C} \colon |z| = 1\}$.

Let $z = \exp(j\omega h)$ and

$$\mathscr{A}_\mu(e^{j\omega h}) = a_\mu(\omega) + jb_\mu(\omega). \tag{10}$$

Then,

$$a_\mu(\omega) = \sum_{k=0}^{\infty}(-1)^k\binom{\mu(kh)}{k}\frac{\cos(k\omega h)}{h^{\mu(kh)}} \tag{11}$$

and

$$b_\mu(\omega) = -\sum_{k=0}^{\infty}(-1)^k \binom{\mu(kh)}{k} \frac{\sin(k\omega h)}{h^{\mu(kh)}}. \tag{12}$$

It is possible to use equations (8)–(12) only if all the series are convergent.

## 3. A DIGITAL FRACTIONAL-VARIABLE-ORDER PID CONTROLLER DESIGN AND STABILITY ANALYSIS

### 3.1. Discretisation of the plant and construction of Nyquist contour

The methods of the stability analysis in discrete-time domain are very much the same as for continuous-time systems [22]. One of the most popular is frequency domain criterion developed by H. Nyquist in 1932 which remains a fundamental approach to the investigation of the stability of linear control systems, see [23, 24]. The Nyquist criterion is a graphical technique which involves the creation of the Nyquist contour. Generated contour must encircle the entire unstable region of the complex plane. tability is determined in this case by looking at the number of encirclements of the point at $(-1,\ 0)$. Equation $q\left(e^{j\omega h}\right) = 0$ is one for which the Nyquist stability criterion is derived and the Argument Variation Principle is used. Apart from stability the Nyquist method can be utilized to analyze the frequency characteristics, phase of the input signals and the time-shift of the system. We follow the ideas given in [25] and in the presented work we draw the Nyquist contour, the graph of points $(q_x(\omega),\ q_y(\omega))$ of one $\omega$, to confirm stability of the closed loop systems with the designed controllers.

For our research we use a second-order plant with the delay. Let us consider the general version of the mentioned plant with the transfer function in a form of

$$G(s) = \overline{G}(s)e^{-s}, \tag{13}$$

where $\overline{G}$ is a transfer function of some second-order plant. Then, from [25] we have the following discretization formula:

$$\overline{H}(z) = (1-z^{-1})\mathscr{Z}\left[\mathscr{L}^{-1}\left\{\frac{\overline{G}(s)}{s}\right\}\bigg|_{t=kh}\right](z).$$

Then considering the delay in $\tau$ steps we receive

$$\begin{aligned} H(e^{j\omega h}) &= z^{-\tau}\overline{H}(z)\big|_{z=e^{j\omega h}} \\ &= (\cos(\tau\omega h) - j\sin(\tau\omega h))\overline{H}(e^{j\omega h}). \end{aligned} \tag{14}$$

In our simulations we consider the control object described by the following transfer function:

$$G(s) = \frac{2e^{-s}}{(0.21s+1)(4s+1)}. \tag{15}$$

The presented transfer function is a plant from the engine control system which is described in more details in [23]. Moreover, this is a model of second-order plus dead time process for which we do not have oscillations. More details about the behavior of this class of objects under PID classical control one can find in [26]. Equation (15) can be written in the following general form:

$$G(s) = \frac{A}{s+a}e^{-s} + \frac{B}{s+b}e^{-s} \tag{16}$$

with $A = 0.539868588$, $a = 4.711074549$, $B = -A$, $b = 0.300830213$. For our calculations we choose sampling step $h = 0.02$. In this case the system delay of 1 [s] gives the number of steps $\tau = 50$. The dicretization in $\mathscr{Z}$-transform of the presented plant function is given by

$$H(z) = z^{-\tau}\left(\frac{\frac{A}{a}(1-e^{-ah})}{z-e^{-ah}} + \frac{\frac{B}{b}\left(1-e^{-bh}\right)}{z-e^{-bh}}\right). \tag{17}$$

### 3.2. FVOPID controller design and mathematical description

A digital fractional-variable-order PID controller, which we use in our simulations, can be described in a similar way as a standard first-order PID. Let $K_p$, $K_i$, $K_d$ be proportional, summation and difference gains and $\mu_i(\cdot)$, $\mu_d(\cdot)$ are summation and difference order functions. In this case the controller output signal is generated according to the following equation:

$$u(kh) = K_p e(kh) + K_i\Delta^{\mu_i(\cdot)}e(kh) + K_d\Delta^{\mu_d(\cdot)}e(kh). \tag{18}$$

In equation (18), $k$ is a number of sample, $h > 0$ is a sampling step, $e(kh)$ is an input and $u(kh)$ is an output of the controller.

In the paper we discuss the situation, when the order functions are variable along time; however we put constant values for some intervals of time. Additionally, (as a second example of FVOPID controller design) we are also analysing the case when, in the final stage of the control the difference order value is set as 1 and summation order value is assigned with $-1$, which means that our FVOPID controller starts to act as a regular first-order PID. Hence, we consider order functions in the following general form:

$$\mu_i(kh) = \begin{cases} \alpha_1, & \text{for } kh < t_1 \\ \alpha_2, & \text{for } t_1 \le kh < t_2 \\ \vdots & \\ \alpha_n, & \text{for } t_{n-1} \le kh < t_n \\ -1, & \text{for } kh \ge t_n \end{cases}, \tag{19}$$

$$\mu_d(kh) = \begin{cases} \beta_1, & \text{for } kh < t_1 \\ \beta_2, & \text{for } t_1 \le kh < t_2 \\ \vdots & \\ \beta_n, & \text{for } t_{n-1} \le kh < t_n \\ 1, & \text{for } kh \ge t_n \end{cases}, \tag{20}$$

where $n \in \mathbb{N}$, $0 \le t_1 < t_2 < \ldots t_{n-1} < t_n$. Moreover, values $\alpha_i, \beta_i$ can be both positive and negative.

Since stability is related to the location of closed loop poles in $z$-plane and the unit circle is the curve that divides the $z$-plane into stable and unstable parts, we take into account

$z = \exp(j\omega h)$ and check how the image of some points from the unit circle looks like with respect to the $\mathscr{Z}$-transform of the oblivion function for order functions given by (19) and (20). Then for the integral operator we get (based on the equations (8)–(12)):

$$\mathscr{A}_{\mu_i}(z) = \sum_{k=0}^{\frac{t_1}{h}-1} (-1)^k \binom{\alpha_1}{k} z^{-k} h^{-\alpha_1} + \ldots + \sum_{k=\frac{t_{n-1}}{h}}^{\frac{t_n}{h}-1} (-1)^k \binom{\alpha_n}{k} z^{-k} h^{-\alpha_n} + \frac{zh}{z-1} z^{-\frac{t_n}{h}} \tag{21}$$

and

$$a_{\mu_i}(\omega) = \sum_{k=0}^{\frac{t_1}{h}-1} (-1)^k \binom{\alpha_1}{k} \cos(k\omega h) h^{-\alpha_1} + \ldots + \sum_{k=\frac{t_{n-1}}{h}}^{\frac{t_n}{h}-1} (-1)^k \binom{\alpha_n}{k} \cos(k\omega h) h^{-\alpha_n} + \frac{h}{2} \frac{\sin(\omega(0.5h - t_n))}{\sin(\omega h/2)}, \tag{22}$$

$$b_{\mu_i}(\omega) = \sum_{k=0}^{\frac{t_1}{h}-1} (-1)^k \binom{\alpha_1}{k} \sin(k\omega h) h^{-\alpha_1} + \ldots + \sum_{k=\frac{t_{n-1}}{h}}^{\frac{t_n}{h}-1} (-1)^k \binom{\alpha_n}{k} \sin(k\omega h) h^{-\alpha_n} - \frac{h}{2} \frac{\cos(\omega(0.5h - t_n))}{\sin(\omega h/2)}. \tag{23}$$

We have similar expressions for the part of difference operator:

$$\mathscr{A}_{\mu_d}(z) = \sum_{k=0}^{\frac{t_1}{h}-1} (-1)^k \binom{\beta_1}{k} z^{-k} h^{-\beta_1} + \ldots + \sum_{k=\frac{t_{n-1}}{h}}^{\frac{t_n}{h}-1} (-1)^k \binom{\beta_n}{k} z^{-k} h^{-\beta_n}, \tag{24}$$

and

$$a_{\mu_d}(\omega) = \sum_{k=0}^{\frac{t_1}{h}-1} (-1)^k \binom{\beta_1}{k} \cos(k\omega h) h^{-\beta_1} + \ldots + \sum_{k=\frac{t_{n-1}}{h}}^{\frac{t_n}{h}-1} (-1)^k \binom{\beta_n}{k} \cos(k\omega h) h^{-\beta_n}, \tag{25}$$

$$b_{\mu_d}(\omega) = \sum_{k=0}^{\frac{t_1}{h}-1} (-1)^k \binom{\beta_1}{k} \sin(k\omega h) h^{-\beta_1} + \ldots + \sum_{k=\frac{t_{n-1}}{h}}^{\frac{t_n}{h}-1} (-1)^k \binom{\beta_n}{k} \sin(k\omega h) h^{-\beta_n}. \tag{26}$$

Considering all the presented equations the $\mathscr{Z}$-transform of the described fractional-variable-order PID controller is given by

$$G_c(z) = \frac{U(z)}{E(z)} = K_p + K_i \mathscr{A}_{\mu_i}(z) + K_d \mathscr{A}_{\mu_d}(z), \tag{27}$$

where $E(z) = \mathscr{Z}[e(kh)](z)$.

To draw the Nyquist contour for the presented controller, which is the graph of points $(q_x(\omega),\ q_y(\omega))$ we have to perform the following calculations.

Let

$$q(e^{j\omega h}) = 1 + H(e^{j\omega h}) G_c(e^{j\omega h}) = q_x(\omega h) + j q_y(\omega h),$$

then

$$q_x(\omega) = 1 + \operatorname{Re} H(e^{j\omega h}) \left[K_p + K_i a_{\mu_i} + K_d a_{\mu_d}\right] - \operatorname{Im} H(e^{j\omega h}) \left[K_i b_{\mu_i} + K_d b_{\mu_d}\right], \tag{28}$$

$$q_y(\omega) = \operatorname{Re} H(e^{j\omega h}) \left[K_i b_{\mu_i} + K_d b_{\mu_d}\right] + \operatorname{Im} H(e^{j\omega h}) \left[K_p + K_i a_{\mu_i} + K_d a_{\mu_d}\right], \tag{29}$$

where $a_{\mu_i}$, $b_{\mu_i}$, $a_{\mu_d}$, $b_{\mu_d}$ are described by equations (22), (23), (25), (26).

### 3.3. FVOPID controller parameters searching process

For presented in this paper FVOPID controller design it is assumed that the controller has four possible order values for summation and four order values for difference gain which depend on time. In this case the control process can be divided into four phases which are represented by numbers 1-4, where phase 1 can be considered as the initial phase and 4 as the final phase of the control process. In every phase the controller has different values of summation and difference orders which can be marked respectively as:

- $\alpha_1$, $\beta_1$ – integral and derivative order values for phase 1,
- $\alpha_2$, $\beta_2$ – integral and derivative order values for phase 2,
- $\alpha_3$, $\beta_3$ – integral and derivative order values for phase 3,
- $\alpha_4$, $\beta_4$ – integral and derivative order values for phase 4.

In this work we use two variants of FVOPID tuning algorithm which was also presented e.g. in [16]. In the first variant the algorithm used to find FVOPID controller parameters can be described by the following steps:

1. Finding parameters $K_p$, $K_i$ and $K_d$ of the first-order PID controller using some selected method (could be e.g. Ziegler-Nichols or any other tuning approach).
2. Using Nelder-Mead optimisation to find new parameters $K_p$, $K_i$ and $K_d$ of first-order PID controller (as the starting point for the optimisation $K_p$, $K_i$ and $K_d$ values calculated in step 1 should be used).
3. Using Nelder-Mead optimisation to find fractional-order PID (FOPID) controller parameters (as the starting point for the optimisation $K_p$, $K_i$ and $K_d$ values calculated in step 2 should be used). In this step apart from $K_p$, $K_i$, $K_d$ parameters, additionally the optimisation algorithm searches for

optimal constant order values $\alpha$ (summation) and $\beta$ (difference).

4. Using Nelder-Mead optimisation to find new parameters $K_p$, $K_i$ and $K_d$ and order values $\alpha_1 - \alpha_4$, $\beta_1 - \beta_4$ of fractional-variable-order PID controller (as the starting point for the optimisation $K_p$, $K_i$, $K_d$, $\alpha$, $\beta$ values calculated in step 3 should be used).

The described algorithm as a starting point for FVOPID controller parameters searching procedure takes the parameters of previously found optimal FOPID controller. In other words first it searches for FOPID controller parameters, which minimise given error criteria and then use found optimal parameters as a base for searching optimal FVOPID controller parameters. This is the reason why we are referring to the algorithm as FOPID based. What is worth noticing is that instead of Nelder-Mead any other method of optimisation could be used including biologically inspired algorithms like e.g.: particle swarm, genetic algorithm, grey wolf optimisation and many others.

The second version of the algorithm is similar to FOPID based algorithm. The difference is that FVOPID parameters are searched starting from the parameters of optimal PID controller instead of FOPID controller. Because of this feature the algorithm is in this work referred as PID based. Note that by optimal PID controller we understand the PID controller designed with the Nelder-Mead optimisation to minimise given error criteria. Furthermore, it is assumed that $\alpha_4$ is set to $-1$ (first-order summation) and $\beta_4$ is set to 1 (first-order difference). The reason of setting the order values to $-1$ and 1 for the final phase of the control is to allow stability analysis of the system, which otherwise cannot be evaluated. Additionally, setting the order of integral gain to $-1$ (which makes it first-order integral) could potentially help to prevent steady state error which is a common issue of FOPID controllers. The PID based algorithm can be described by the following steps [16]:

1. Finding parameters $K_p$, $K_i$ and $K_d$ of the first-order PID controller using some selected method (could be e.g. Ziegler-Nichols or any other tuning approach).
2. Using Nelder-Mead optimisation to find new parameters $K_p$, $K_i$ and $K_d$ of the first-order PID controller (as the starting point for the optimisation $K_p$, $K_i$ and $K_d$ values calculated in step 1 should be used).
3. Using Nelder-Mead optimisation to find new parameters $K_p$, $K_i$ and $K_d$ and order values $\alpha_1 - \alpha_3$, $\beta_1 - \beta_3$ of fractional-variable-order PID controller (as the starting point for the optimisation values $K_p$, $K_i$, $K_d$ calculated in step 2 should be used).

The controllers presented in this work were designed to minimise the objective function given by:

$$OF = w_1 \sum_{k=0}^{N} |e(kh)|kh^2 + w_2 OS + w_3|E_{ss}| + w_4 t_s\,, \qquad (30)$$

where $e(kh)$ is a control error, $OS$ is an overshoot, $E_{ss}$ is steady-state error, $t_s$ is a settling time and $N$ is a total number of steps. Similar objective function was used for continuous-time controllers e.g. in [27]. Symbols $w_1$–$w_4$ are weighting coefficients whose values were set according to [27] as: $w_1 = 1$, $w_2 = 0.02$, $w_3 = 1$, $w_4 = 5$.

The initial values of searched parameters for Nelder-Mead optimisation were selected by MATLAB® *pidtune* function. Obtained by the mentioned function initial PID parameters (which will be referred to as initial PID controller) are: $K_p = 1.06$, $K_i = 0.252$, $K_d = 0.172$.

Step response of the closed loop system with initial PID (tuned by *pidtune* function), optimal PID and FOPID controllers (by optimal we mean controllers which minimise the objective function $OF$) are shown in Fig. 2. Parameter values found with the qualitative criteria which are: rise time, overshoot ($OS$), settling time ($t_s$), steady-state error ($E_{ss}$) and the objective function value (OF value), are presented in Table 1. What is worth noticing is that the rise time and the settling time used in the objective function definition were calculated with MATLAB® *stepinfo* function. In this case the rise time is defined as time it takes for the response to rise from 10% to 90% of steady-state response. By the settling time we define a time it takes for the error to fall within 2% of the final system response [28].

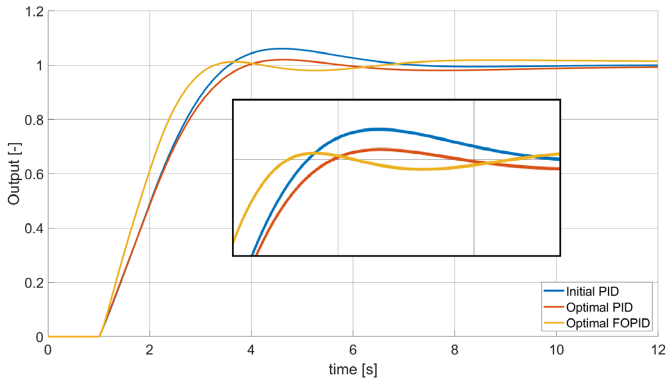


**Fig. 2.** Step response of the closed loop system with PID controllers

**Table 1**
Obtained constant order controllers results

| | Initial PID | Optimal PID | Optimal FOPID |
|---|---|---|---|
| $K_p$ | 1.06 | 1.048256 | 1.018652 |
| $K_i$ | 0.252 | 0.225139 | 0.277876 |
| $K_d$ | 0.172 | 0.183689 | 0.468006 |
| $\alpha$ | $-1$ | $-1$ | $-1.009685$ |
| $\beta$ | 1 | 1 | 0.685430 |
| Rise time [s] | 1.8261 | 1.9486 | 1.4869 |
| $OS$ [%] | 6.0740 | 1.9989 | 1.8432 |
| $t_s$ [s] | 6.2440 | 3.7113 | 3.0932 |
| $E_{ss}$ | 0.000296 | 0.0070 | $-0.0144$ |
| OF value | 34.363102 | 21.828096 | 18.290935 |

When it comes to FVOPID controller design we have to decide how to set up the time intervals of controller's orders, or in other words determine the moments when order values are changed.

When used in real life control systems, optimal time intervals for every control phase should be individually selected for the plant type, to optimise given control quality criteria. What is worth mentioning is that when it comes to controller parameters searching process another possibility, apart from searching only for the values of orders, would be looking at the same for both: the order values and the time intervals during which given orders should be used. The separated aspect is also a selection of the optimisation method which can provide satisfying solution for this kind of optimisation problem. Because this paper mainly focuses on the stability criteria of variable order systems, the time intervals for control phases were selected arbitrarily and are based on initial (obtained before optimisation process) step responses of the system with constant order controller. In this case three different sets of the control phases (three different order functions setup) were created and used to design three different FVOPID controllers. Described sets of the control phases are presented below:

- Order function I setup
  – phase 1 for $t < 1.4$ [s],
  – phase 2 for $1.4$ [s] $\leq t < 1.8$ [s],
  – phase 3 for $1.8$ [s] $\leq t < 2.2$ [s],
  – phase 4 for $t \geq 2.2$ [s].
- Order function II setup
  – phase 1 for $t < 1.9$ [s],
  – phase 2 for $1.9$ [s] $\leq t < 2.8$ [s],
  – phase 3 for $2.8$ [s] $\leq t < 3.7$ [s],
  – phase 4 for $t \geq 3.7$ [s].
- Order function III setup
  – phase 1 for $t < 3.0$ [s],
  – phase 2 for $3.0$ [s] $\leq t < 5.0$ [s],
  – phase 3 for $5.0$ [s] $\leq t < 7.0$ [s],
  – phase 4 for $t \geq 7.0$ [s].

For order function I all the phases are located in the initial stage of the control process, where the output signal of the closed loop system is rising, but not yet reaching the set-point. After that for time $t = 2.2$ [s] the controller orders are set to their final values – $\alpha_4$, $\beta_4$. In order function II all the phases are longer and FVOPID controller orders change to their final values for $t = 3.7$ [s], which corresponds to the time when closed loop output signal is close to its highest/peak value. Order function III contains the longest phases which means that FVOPID controller orders are set with their final values after $t = 7$ [s] from the beginning of the simulation. In this case the orders are also changed after the process reaches the set-point value.

When it comes to the implementation details of the FVOPIDs presented in the paper, the output of the controllers was calculated strictly according to Definition 2. The moments of switching/changing the order values were determined by the simulation time, which was calculated by multiplying the sampling time by the current number of the input signal sample. In other words during the simulation process the controller, based on the current simulation time, was changing the order values in accordance with the previously presented rules. What is important to mention is that in real-life applications the order changes should be triggered by some external events like e.g. changes in the control error, or modified process setup. But as it was mentioned before in the presented work we are mainly focusing on the stability of variable-order systems and this can be evaluated when we represent orders as a function of time.

## 4. SIMULATION RESULTS

All the FVOPID controllers presented in this section were tuned by a previously described algorithm which utilise Nelder-Mead optimization method. As a result of the tuning process for every controller the values of $K_p$, $K_i$, $K_d$, $\alpha_1 - \alpha_4$, $\beta_1 - \beta_4$ parameters were obtained which were later used in the simulations whose results are described in the following subsections.

### 4.1. Step response results comparison

All the simulations have been carried out for continuous time plant described by equation (15) and discrete controllers with the sampling time set to $h = 0.02$ [s]. Step response of designed FOPID-based FVOPID controllers (the controllers designed with FOPID-based algorithm) and reference optimal FOPID controller (which was also a starting point for FVOPID parameters searching process) are presented in Fig. 3. Table 2 contains values of designed controllers parameters, obtained step response, rise time, overshoot, settling time, steady state error and the value of minimised objective function. When it comes to the naming convention FVOPID controllers with the previously presented order functions setup I, II and III were named respectively as FVOPID I, FVOPID II and FVOPID III.

**Table 2**
FOPID based FVOPID controllers parameters

| | Optimal FOPID | FVOPID I | FVOPID II | FVOPID III |
|---|---|---|---|---|
| $K_p$ | 1.018652 | 1.129751 | 1.055163 | 1.061935 |
| $K_i$ | 0.277876 | 0.281097 | 0.309236 | 0.310343 |
| $K_d$ | 0.468006 | 0.483578 | 0.637718 | 0.759846 |
| $\alpha_1$ | −1.009685 | −0.248836 | −1.043217 | −1.230921 |
| $\alpha_2$ | −1.009685 | −0.293109 | −1.023013 | −0.965040 |
| $\alpha_3$ | −1.009685 | −1.945047 | −1.082247 | −1.011194 |
| $\alpha_4$ | −1.009685 | −1.009975 | −0.981024 | −0.987937 |
| $\beta_1$ | 0.685430 | 0.705681 | 0.710755 | 0.698173 |
| $\beta_2$ | 0.685430 | 1.533361 | 0.924323 | 0.556207 |
| $\beta_3$ | 0.685430 | −0.280578 | 0.350890 | 0.621507 |
| $\beta_4$ | 0.685430 | 1.054745 | 0.719598 | 0.704262 |
| Rise time [s] | 1.4869 | 1.2114 | 1.2643 | 1.1600 |
| $OS$ [%] | 1.8432 | 1.9986 | 1.9795 | 1.9995 |
| $t_s$ [s] | 3.0932 | 2.6543 | 2.7330 | 2.5836 |
| $E_{ss}$ | −0.0144 | −0.000121 | 0.000274 | 0.000051 |
| OF value | 18.290935 | 15.246906 | 15.674142 | 14.97931 |

Figure 4 presents step response of the designed PID-based FVOPID controllers. The parameters and qualitative criteria of the mentioned controllers can be found in Table 3.

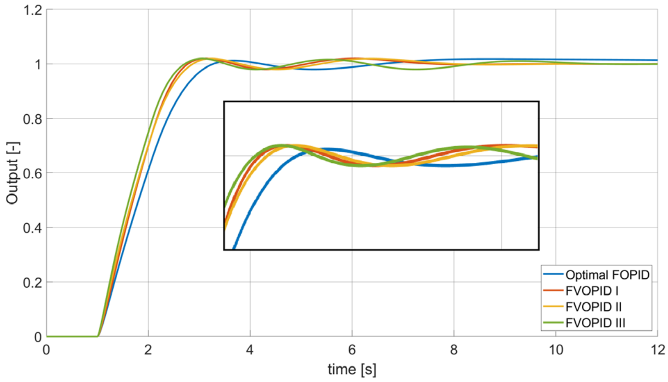


**Fig. 3.** Comparison of step response of closed loop system with different FOPID-based VFOPID controller order functions

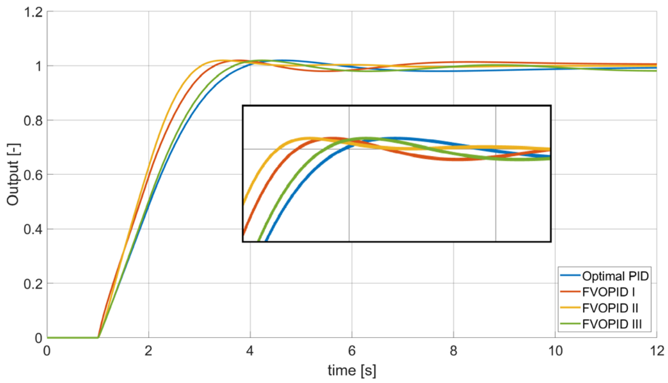


**Fig. 4.** Comparison of step response of closed loop system with different PID-based VFOPID controller order functions

**Table 3**
PID based FVOPID controllers parameters

| | Optimal PID | FVOPID I | FVOPID II | FVOPID III |
|---|---|---|---|---|
| $K_p$ | 1.048256 | 1.206551 | 1.123921 | 1.097999 |
| $K_i$ | 0.225139 | 0.271086 | 0.272832 | 0.225264 |
| $K_d$ | 0.183689 | 0.281758 | 0.374317 | 0.175048 |
| $\alpha_1$ | $-1$ | $-0.811067$ | $-0.580050$ | $-0.938975$ |
| $\alpha_2$ | $-1$ | $-0.747638$ | $-1.392419$ | $-1.031497$ |
| $\alpha_3$ | $-1$ | $-0.481482$ | $-0.912055$ | $-1.029915$ |
| $\alpha_4$ | $-1$ | $-1$ | $-1$ | $-1$ |
| $\beta_1$ | 1 | 1.088148 | 0.749115 | 0.995860 |
| $\beta_2$ | 1 | 1.114045 | 0.189669 | 1.055026 |
| $\beta_3$ | 1 | 1.206448 | 0.926764 | 1.119200 |
| $\beta_4$ | 1 | 1 | 1 | 1 |
| Rise time [s] | 1.9486 | 1.6123 | 1.3802 | 1.8105 |
| $OS$ [%] | 1.9989 | 1.9968 | 1.9995 | 1.9999 |
| $t_s$ [s] | 3.7113 | 3.1276 | 2.8917 | 3.4801 |
| $E_{ss}$ | 0.0070 | $-0.0060$ | 0.000038 | 0.0186 |
| OF value | 21.971737 | 18.320159 | 16.459438 | 20.479537 |

As we can see in all the cases systems with fractional-variable-order controllers obtained lower values of minimised objective function in comparison to corresponding closed loop systems with constant order control. It means that FOPID-based FVOPID controllers gave better results than the optimal FOPID, while PID-based FVOPID controllers achieve smaller values of minimised objective function than the systems with optimal first-order PID controller. What we can also observe is that the systems with FVOPID controllers are characterised by very low rise time at the same time with overshoot similar to constant order controllers. Characteristics obtained by FVOPID controllers highly depend on the approach to searching parameters. In general FVOPID controllers whose parameters were searched taking as a starting point FOPID controller parameters achieved lower values of minimised objective function (which in this case also means lower values of overshoot, rise time and settling time). On the other hand PID-based controllers whose order values in the final phase of the control are set to $-1$ and 1 allow us to analyse mathematically the stability of the system which is described in detail in the next subsection.

### 4.2. Robustness and stability analysis

The mathematical theory presented in the previous sections facilitates stability analysis only for fractional-variable-order controllers whose order values in the final phase of the control are set to $-1/1$ (first-order summation/difference). It means that for the robustness and stability analysis we have to chose one of the designed PID-based FVOPID controllers. The controller which obtained the lowest values of minimised objective function and whose stability is checked in the current section is in this case FVOPID with Order Function II. The parameters of the mentioned controller are presented in Table 4.

**Table 4**
PID-based FVOPID controller parameters with Order Function II

| Parameter name | Parameter value | Simulation time |
|---|---|---|
| $K_p$ | 1.123921 | All |
| $K_i$ | 0.272832 | All |
| $K_d$ | 0.374317 | All |
| $\alpha_1$ | $-0.580050$ | $t < 1.9$ [s] |
| $\alpha_2$ | $-1.392419$ | $1.9$ [s] $\leq t < 2.8$ [s] |
| $\alpha_3$ | $-0.912055$ | $2.8$ [s] $\leq t < 3.7$ [s] |
| $\alpha_4$ | $-1$ | $t \geq 3.7$ [s] |
| $\beta_1$ | 0.749115 | $t < 1.9$ [s] |
| $\beta_2$ | 0.189669 | $1.9$ [s] $\leq t < 2.8$ [s] |
| $\beta_3$ | 0.926764 | $2.8$ [s] $\leq t < 3.\ 7$[s] |
| $\beta_4$ | 1 | $t \geq 3.7$ [s] |

The purpose of robustness analysis is to ensure stability and high quality of the step response in the case of changed parameters values of control object or some disturbances occurrence.

The plant that we analyse is described by a second-order transfer function which can be given by a general expression:

$$G(s) = \frac{2e^{-s\tau}}{as^2 + bs + 1}. \tag{31}$$

The original (initial) plant parameters are $\tau = 1$, $a = 0.84$ and $b = 4.21$. Images 5, 7, 9 show the step response of the designed FVOPID controller for modified plant parameters. Figure 5 presents the system step response for the plant object with modified $a$ parameter. In Figs. 7 and 9 we can see the system response for modified $b$ and $\tau$ (delay) parameters, respectively. Presented figures show that the system with the designed controller should remain stable even when the parameters of the controlled process are slightly changed.

The method of stability analysis utilized in this work can be easily applied even for the systems with delays or the systems described by non-rational transfer functions, which may be problematic using the other methods. Stability is determined by looking at the number of encirclements of the point $(-1,\ 0)$. The range of gains over which the system remains stable can be determined by looking at crossings of the real axis. Figures 6, 8, 10 present Nyquist contours of the closed loop system with designed FVOPID controller and with modified values of plant parameters.

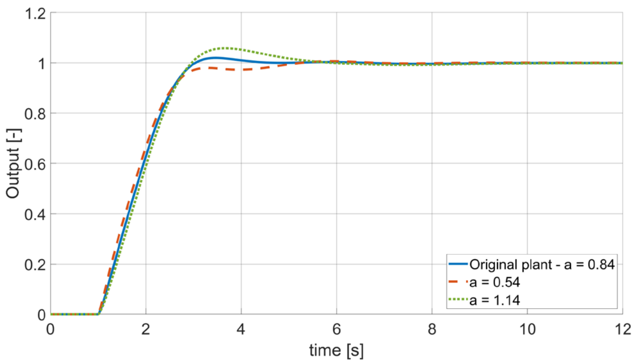


**Fig. 5.** Step response of closed loop system with PID-based FVOPID controller for different values of $a$ parameter

In Fig. 6 we see that the Nyquist contours are confirming that stability is preserved for modified plant $a$ coefficient where $a \in \{0.54,\ 0.84,\ 1.14\}$. Figure 8 presents the Nyquist contours for modified $b$ coefficient of the plant where $b \in \{3.71,\ 4.21,\ 4{,}71\}$. In Fig. 10 we see that the Nyquist contours are confirming that stability is preserved when we slightly change the value of delays: $\tau \in \{0.8s,\ 1s,\ 1.2s\}$. We see from Figs. 6, 8, 10 that our diagrams cross the real axis on the right side of point $(-1,\ 0)$ and do not encircle it. It means that considered closed loop systems are stable, see for example [29].

In the last stage of the research we have checked the system behavior when a White Noise is added to the control signal generated by the controller (control disturbance) or to measured feedback signal (measurement noise). Power spectral density (PSD) value of control disturbance was in this case set to 0.0001, while for measurement noise PSD = 0.00001. The

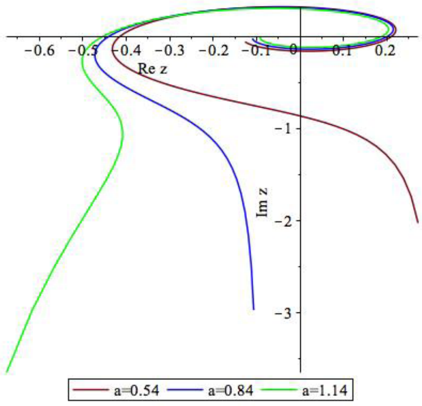


**Fig. 6.** The Nyquist contour for stable closed loop system with various $a$ coefficients from the plant transform, $a \in \{0.54,\ 0.84,\ 1.14\}$ with coefficients of FVOPID controller from Table 4

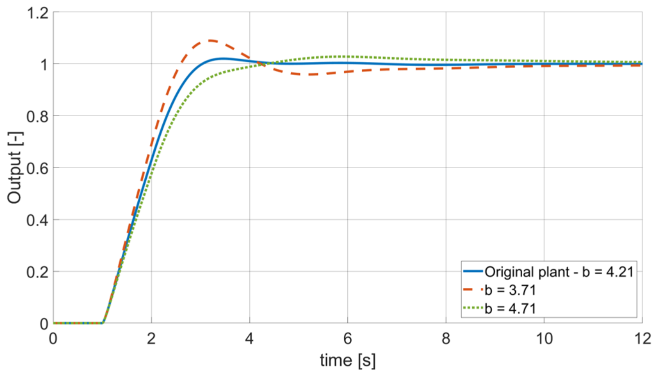


**Fig. 7.** Step response of closed loop system with PID-based VFOPID controller for different values of $b$ parameter

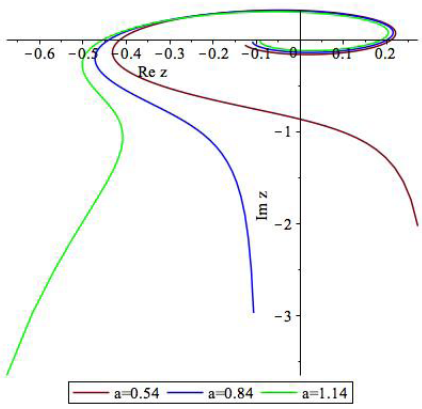


**Fig. 8.** The Nyquist contour for stable closed loop system with various $b$ coefficients from the plant transform, $b \in \{3.71,\ 4.21,\ 4{,}71\}$ with coefficients of FVOPID controller from Table 4

way mentioned disturbance values were included in the closed loop schema is presented in Fig. 11. Note that in the presented simulations it is assumed that the Sensor block has a transfer function equal to 1.

The results of the system behavior with described disturbances added to the closed loop system are presented in Fig. 12.

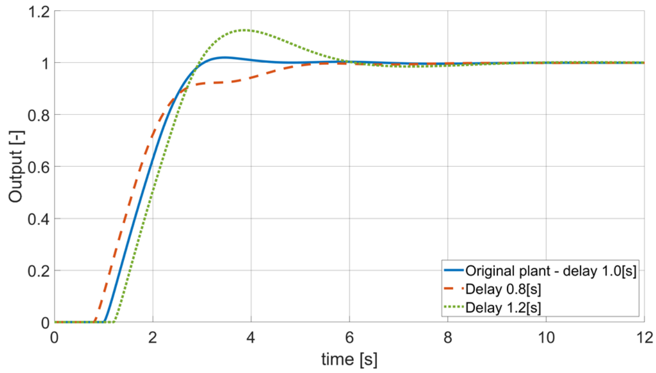


**Fig. 9.** Step response of closed loop system with PID-based VFOPID controller for different values of $\tau$ (delay) parameter

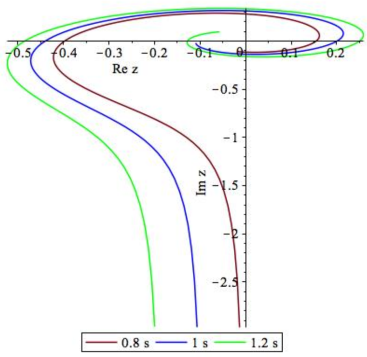


**Fig. 10.** The Nyquist contour for stable closed loop system with different delays $\tau \in \{0.8s,\ 1s,\ 1.2s\}$ with coefficients of FVOPID controller from Table 4

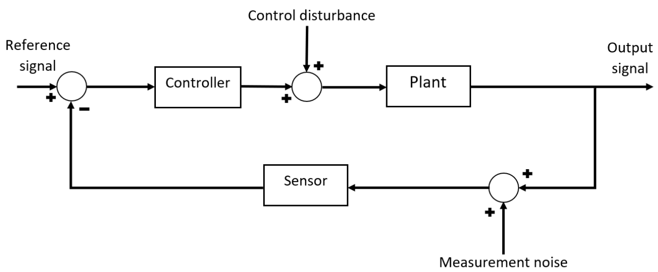


**Fig. 11.** Closed loop control schema with added disturbances

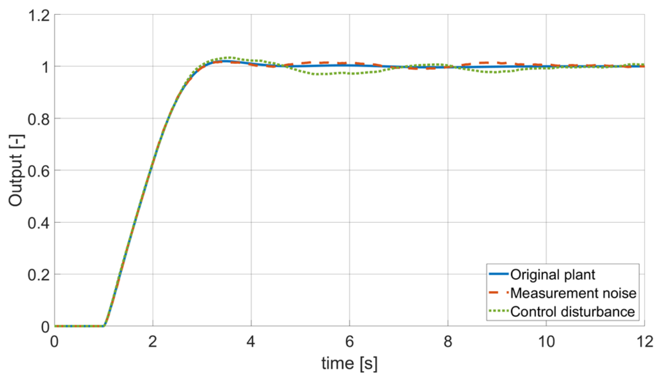


**Fig. 12.** Step response of closed loop system with PID-based VFOPID controller with added disturbances

As we can see also in case of disturbed control and measurement signals (with selected PSD values), obtained results of the step responses show that the system persists stable.

## 5. CONCLUSIONS AND FUTURE WORKS

Closed loop systems with fractional-variable-order controllers presented in the work obtained low values of minimised objective function and very short rise time in comparison to the systems with constant order controllers. What is more the stability analysis of FVOPID controller with the order values converging (in the final phase of the control) to –1 and 1 can be performed with well known Nyquist criterion, which paves the way for further research and mathematical analysis of this kind of systems. As the future work we plan to extend the stability analysis to the systems which use different definitions of Grünwald-Letnikov backward difference. Additionally, we want to make it applicable also for the controllers with the order functions which depend not directly on time but also on the current controller error value. Simultaneously with stability research, new optimal methods of finding the values of FVOPID controller order functions should be developed and evaluated.

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