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INTL JOURNAL OF ELECTRONICS AND TELECOMMUNICATIONS, 2021, VOL. 67, NO. 3, PP. 395-401
Manuscript received July 26, 2021; revised August, 2021 DOI: 10.24425/ijet.2021.137825

Extended Definitions of Spectrum of a Sampled Signal

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***Abstract*—It is shown that a number of equivalent choices for the calculation of the spectrum of a sampled signal are possible. Two such choices are presented in this paper. It is illustrated that the proposed calculations are more physically relevant than the definition currently in use.**

***Keywords*—spectrum of sample signal, ideal versus non-ideal sampling, modelling of signal sampling process**

I. INTRODUCTION

FREQUENCY domain representation of a continuous time signal is called its frequency spectrum. Because in this paper we restrict ourselves to consideration of well-behaved, energy, and bandlimited signals, the spectrum in this case is, without any doubt, the Fourier transform of a signal considered. In other words, the usual Fourier transform transforms a signal from the continuous time domain to the continuous frequency domain, providing us with the frequency spectrum of this signal. Further, when such the signal is sampled, its sampled version is created – it is called here a sampled signal. And, it can be represented in the continuous time domain by the signal samples occurring periodically on the time axis. Furthermore, when the sampling operation is an ideal one, then these signal samples occur at strictly discrete time instants. So, this signal (function) ceases then to be a well-behaved one (in the sense used in this paper) because it becomes a not integrable function – in this case. As a consequence, its Fourier transform does not exist. However, does it mean at the same time that its frequency spectrum does not exit, too? The people believe that this is not the case. On the contrary, they believe that the sampled signals do possess the spectra, and they use them in their analyses and projects.

How is it possible? How did it happen? This simply becomes possible by redefining the notion of the signal spectrum. And, the redefinition is done by adjusting, in some way, the not integrable signal mentioned above to the needs of the Fourier transform. For example, by making it, say, “a quasi integrable function”. How?

A choice made in the theory of signals and in signal processing relies upon connecting the above problem with the theory of distributions, in which “not differentiable functions are moved to differentiable objects”. Or, by analogy – equivalently here, “the not integrable functions are moved to integrable objects”. And, these objects are the distributions like, for example, the Dirac delta (Dirac impulse) – the most exploited distribution in the theory of signals and in signal processing. (By the way, note that the distributions are not usual functions – but, sometimes, they are also called “generalized functions”.) Exactly the same is done in the today's theory of signals and in signal processing with not well-behaved functions. This is illustrated in Fig. 1.

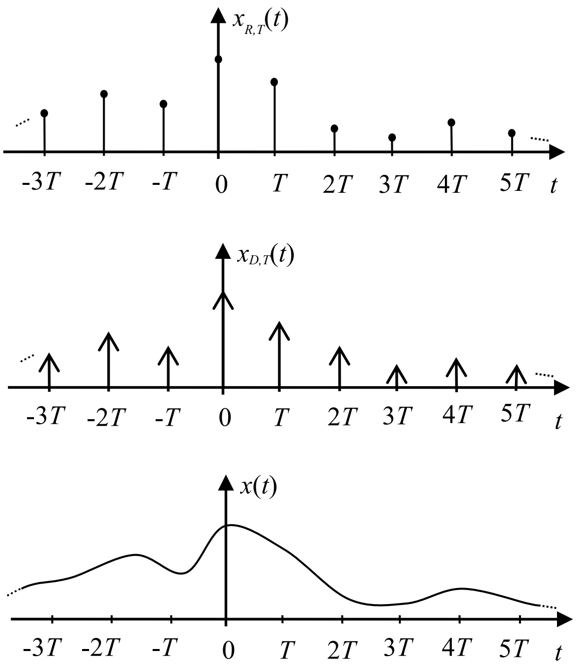

Fig. 1. Graphical illustration of two possibilities of modelling a sampled signal that is sampled ideally. The upper curve shows a representation of an unsampled signal that is visible in the bottom of this figure - in form of a series of time-dependent signal samples occurring uniformly on the continuous time axis in the distance of T from each other. Whereas the middle curve presents its another representation - in form of a series of weighted Dirac deltas occurring uniformly on the continuous time axis in the distance of T from each other. Moreover, we note here that this figure exploits the same signal, which was also discussed in [1].

The notation used in this paper is as follows: the time and frequency variables are denoted by t and f, respectively. Further, T is used to denote the signal sampling period but f_s the sampling frequency. Moreover, the following: $T = 1/f_s$ holds.

The upper curve in Fig. 1 shows a sampled version of an analogue signal sketched in the bottom of this figure; it is denoted here by $x_{R,T}(t)$ for the reasons which are explained below. This sampled signal was obtained at the output of a

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sampler about which we assume that it works in an ideal way. Therefore, this signal is modelled here as a series of perpendicular line segments of different lengths, which are proportional to the values of the signal samples at the corresponding time instants. And, all these values are finite numbers. So, on the other words, this image of a sampled signal can be viewed as the most natural one when it is sampled ideally.

Further, because of the reasons explained above, let us call this image of a signal sampled ideally, as shown by the upper curve of Fig. 1, a reference sampled signal visualization in the case of performing the sampling operation ideally. And, denote this reference image by $x_{R,T}(t)$, where the first index, *R*, in it, stands for the word "reference", but the second one, *T*, means the sampling period.

The biggest drawback of the reference representation $x_{R,T}(t)$ is, as we know, the fact that it does not possess the spectrum (that is an equivalent image in the frequency domain) in the usual sense. And, as we also know, this follows from the fact that it is not an integrable function in the sense of Riemann or – when it is integrated in the sense of Lebesgue – the result is equal identically to zero. (In this paper, we call such the functions not well-behaved.) If interested in more details and explanations on this topic, the reader is referred to [1].

II. In Search of an Alternative Definition for the Spectrum of the Reference Representation

In the situation sketched above, but having a strong will of possessing an image of $x_{R,T}(t)$ in the frequency domain, the people decided to use its well-conceived and properly designed equivalent. How it was done is explained in what follows below. Simply, the people said that the best way seems to be the one, in which the not-integrable function $x_{R,T}(t)$ is transferred to an integrable object created by multiplying an analogue (i.e. un-sampled) signal $x(t)$ by a Dirac comb [2-49] – as visualized by the middle curve in Fig. 1. So, this sampled signal representation can be expressed analytically as

$$x_{D,T}(t) = \delta_T(t) \cdot x(t) \ , \tag{1}$$

where the Dirac comb $\delta_T(t)$ is defined as

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT) \ , \tag{2}$$

where $\delta(t-kT)$, $k = .,-1,0,1,.,$ mean the time-shifted Dirac deltas. Furthermore, note that the new sampled signal representation, obtained in (1), differs from $x_{R,T}(t)$. (A comparison of the upper curve with the middle one in Fig. 1 demonstrates this clearly.) So, for this reason, it is denoted here otherwise – by $x_{D,T}(t)$, where now the first index, *D*, stands for the name of Dirac.

As well known [2-49], the object given by (1) possesses a Fourier transform. That is, in other words, it has a well-defined spectrum (in the usual sense). Moreover, this spectrum can be expressed as

$$X_{D,T}(f) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(f - k/T) \ , \tag{3}$$

where $X_{D,T}(f)$ and $X(f)$ mean the Fourier transforms of the signals: $x_{D,T}(t)$ and $x(t)$, respectively.

Now, let us realize what the people have done in the next step. They simply tied up the non-existing Fourier transform of $x_{R,T}(t)$ with $X_{D,T}(f)$, and said that this must be a spectrum of $x_{R,T}(t)$ (that is a frequency domain image of this signal).

Note that we can interpret the above way of solving the problem with calculation of the spectrum of our not well-behaved function $x_{R,T}(t)$ as a one, which needs an extension of the usual definition of the signal spectrum. This extension can be explained in a descriptive way as follows: If a signal possesses a Fourier transform, its spectrum is equal to this transform. However, when this is not the case, a signal (being an integrable function or an integrable object, as, for example, a distribution), which is "highly related" (in some sense) with the one considered, must be first constructed. (Note that in our case $x_{D,T}(t)$ is "highly related" with $x_{R,T}(t)$ in the sense that, say, both of them seem graphically to be identical – to see this, compare the upper curve with the middle one in Fig. 1.) And, when such a signal is already found, its Fourier transform is calculated. Next, it is said that this Fourier transform is, at the same time, the spectrum of the signal considered – obviously in this extended sense, just formulated.

Let us also express the sketched above procedure in a compact form, analytically; this can be done in the following way:

$$x_{D,T}(t) = \sum_{k=-\infty}^{\infty} F_k\left(x_{R,T}(t)\right)\delta(t-kT) = D\left(x_{R,T}(t)\right) \tag{4}$$

and

$$\mathrm{SPECT}\left(x_{R,T}(t)\right) = \mathcal{F}\left(x_{D,T}(t)\right) = \mathcal{F}\left(D\left(x_{R,T}(t)\right)\right) \ , \tag{5}$$

where $F_k\left(x_{R,T}(t)\right)$ denotes an operator that filters out the *k*-th nonzero value from the signal $x_{R,T}(t)$. The next operator in (4), *D*, means a composite operator (consisting of a sum of filter operators $F_k(\cdot)$ and Dirac deltas $\delta(\cdot)$) that maps $x_{R,T}(t)$ into $x_{D,T}(t)$. Moreover, the operator, SPECT (in (5)), stands for the spectrum when a given signal does not have a Fourier transform. Furthermore, $\mathcal{F}(\cdot)$ denotes the Fourier transform of an object or a function indicated.

Finally, note that connecting (5) with (3) results in

$$\mathrm{SPECT}\left(x_{R,T}\left(t\right)\right)=\frac{1}{T}\sum_{k=-\infty}^{\infty}X\left(f-k/T\right), \tag{6}$$

what permits the current theory [2-49] to say that there occur aliasing and folding effects in the spectra of signals sampled ideally.

The author of this paper does not know who is the "father" of the above scheme of identification of the non-existing (in the usual sense) spectrum of the signal $x_{R,T}\left(t\right)$ with the Fourier transform of the "highly related" signal $x_{D,T}\left(t\right)$ – and "responsible" for applying the Dirac deltas for just this purpose. Probably, it happened in the times of appearance of the theory of distributions (generalized functions) of Laurent Schwartz [50], and because of the great fascination with this theory among the researchers, then. And, probably, the "father" of the above concept was not even aware what he in fact did. It seems that he simply said "any sampled signal must be modelled as an $x_{D,T}\left(t\right)$ – and, that is all". And, the whole signal processing world took over this point of view.

III. New Alternative Definition for the Spectrum of the Reference Representation $x_{R,T}\left(t\right)$

Observe that the choice of the operator D in (4) as the one which transforms the reference signal $x_{R,T}\left(t\right)$ into an integrable function "highly related" with this signal – by exploiting the Dirac deltas – is quite arbitrary. Hence, one can conclude that another choice is also possible and admissible. And, maybe, it can be even better than the one with the Dirac deltas. In what follows, we just present one of such possible solutions; at least in our opinion, it is a better one. However, before starting with this, let us first describe the reference signal $x_{R,T}\left(t\right)$ analytically. To this end, we use here an excellent tool to perform this task, the time-shifted Kronecker time functions [1]. And, here, we denote them by a generic symbol $\delta_{i,t/T}\left(t\right)$.

After [1], the function $\delta_{i,t/T}\left(t\right)$ is defined as

$$\delta_{i,r}\left(t\right)=\delta_{i,t/T}\left(t\right)=\begin{cases}1 & \text{if} \quad i=r=t/T \\ 0 & \text{otherwise}\end{cases}, \tag{7}$$

with $r=t/T$ meaning a real number. (That is we assume here that r belongs to the set $\mathbb{R}$ of real numbers.) Thus, $\delta_{i,r}$ in (7) stands for a slightly modified standard Kronecker delta symbol in which now the second index r is a real-valued one and changes with time. And only when it becomes an integer equal to i, the function $\delta_{i,r}\left(t\right)=\delta_{i,t/T}\left(t\right)$ differs from zero (assumes the value 1).

The function $\delta_{i,t/T}\left(t\right)$ for $i=1$ is illustrated in Fig. 2.

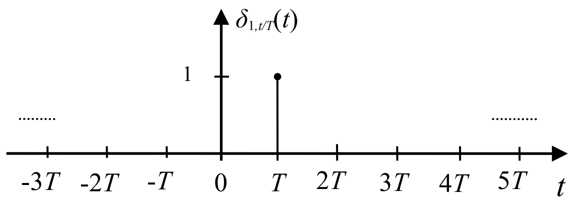

Fig. 2. Illustration of the function $\delta_{i,t/T}\left(t\right)$ for the parameter $i=1$. Figure taken from [1].

Having defined $\delta_{i,t/T}\left(t\right)$, we can define the so-called Kronecker comb, $\delta_{K,T}\left(t\right)$ (so named in [1] because it is in fact a counterpart of the Dirac comb in the description presented now). Its defining equation has the following form:

$$\delta_{K,T}\left(t\right)=\sum_{k=-\infty}^{\infty}\delta_{k,t/T}\left(t\right), \tag{8}$$

where the first index K at $\delta_{K,T}\left(t\right)$ stands for the name of Kronecker, but the second one, T, means a repetition period (sampling period).

So, observe now that with the help of (7) and (8) we can express the reference signal $x_{R,T}\left(t\right)$ analytically as

$$x_{R,T}\left(t\right)=\sum_{k=-\infty}^{\infty}x\left(kT\right)\delta_{k,t/T}\left(t\right)=\delta_{K,T}\left(t\right)\cdot x\left(t\right). \tag{9}$$

Now, we remind that we said at the beginning of this paper that we restrict ourselves here to consideration of the well-behaved, energy, and bandlimited signals. To such signals, when they are sampled with the sampling frequency f_s greater than twice the Nyquist frequency [2-49], applies the following reconstruction (recovery) formula [2-49]:

$$x\left(t\right)=\sum_{k=-\infty}^{\infty}F_k\left(x_{R,T}\left(t\right)\right)\mathrm{sinc}\left(t/T-k\right)=\mathrm{REC}\left(x_{R,T}\left(t\right)\right), \tag{10}$$

where $\mathrm{sinc}\left(\cdot\right)$ stands for the standard sinc function [2-49], but the operator $\mathrm{REC}\left(x_{R,T}\left(t\right)\right)$ maps $x_{R,T}\left(t\right)$ into $x\left(t\right)$ (according to the rule indicated in (10)).

Note now that the operator $\mathrm{REC}\left(\cdot\right)$ in (10) plays the same role as the operator $D\left(\cdot\right)$ in (4), and the resulting function $x\left(t\right)$ is a "well-behaved, integrable" one. Therefore, it can be used to construct another alternative definition of the spectrum of the reference representation $x_{R,T}\left(t\right)$. So, applying this function in the second equation defining an extended spectrum of the sampled signal, (5), that is replacing $x_{D,T}\left(t\right)$ with $x\left(t\right)$ there, we get

$$\mathrm{SPECT1}\left(x_{R,T}\left(t\right)\right)=\mathrm{F}\left(x\left(t\right)\right)=\mathrm{F}\left(\mathrm{REC}\left(x_{R,T}\left(t\right)\right)\right), \tag{11}$$

where the notation $\mathrm{SPECT1}(\cdot)$ is used to distinguish it from $\mathrm{SPECT}(\cdot)$ given by (5).

Now, note that the following:

$$\mathrm{SPECT1}\left(x_{R,T}(t)\right) = X(f) \neq \frac{1}{T}\sum_{k=-\infty}^{\infty} X(f-k/T) = \\ =\mathrm{SPECT}\left(x_{R,T}(t)\right) \tag{12}$$

holds. So, the two definitions of the extended spectrum of the sampled signal – considered above – are not equivalent. And, both of them are arbitrary – as noted before. Therefore, it is legitimate to ask which of them is more useful for engineers. In what follows, we give two relevant arguments for the second definition.

To explain the first argument, consider each of the above definitions as a kind of "entanglement", in the frequency domain, of two spectra we suspect that must be related with the sampled signal $x_{R,T}(t)$: the spectrum $X(f)$ and the next associated with the periodic signal having a Fourier series representation with its all coefficients equal identically to zero. In the first definition, it is allowed "the zero Fourier series coefficients mentioned above to combine with $X(f)$" to result in (6). Unlike this, the second definition does not allow for the above effect. As a result, we get in this case a "pure" $X(f)$.

Let us also illustrate the above in the following way:

$$\underset{\text{under definition 1}}{En\left(X(f), \text{zero Fourier series coeff.}\right)} \rightarrow \underset{\text{given by (6)}}{\frac{1}{T}\sum_{k=-\infty}^{\infty} X(f-k/T)} \tag{13}$$

or

$$\underset{\text{under definition 2}}{En\left(X(f), \text{zero Fourier series coeff.}\right)} \rightarrow \underset{\text{expressed in (11)}}{X(f)} \quad , \tag{14}$$

where $En(\cdot)$ stands for an entanglement of two spectra as mentioned.

In opinion of the author of this paper, the entanglement illustrated by (14), i.e. carried out under the second definition is physically more reasonable.

To present the second argument, let us note first that we can write the following:

$$\delta_T(t)\cdot x_{R,T}(t) = \delta_T(t)\cdot x(t) \ . \tag{15}$$

In the next step, substituting $x_{R,T}(t)$ given by (9) in (15) results in

$$\delta_T(t)\cdot \delta_{K,T}(t)\cdot x(t) = \delta_T(t)\cdot x(t) \ . \tag{16}$$

So, the function $\delta_{K,T}(t)$ in (16) plays a role of an identity operator. But, on the other hand, it is a multiplier of the function $x(t)$ in (9) in building up the reference signal $x_{R,T}(t)$. This suggests a relation "one to one" between these two signals – of course, not in the time domain – but maybe in another domain. And, we have such a relation in the frequency domain under the second definition. So, this makes an argument for the latter definition.

IV. An Alternative Definition of the Spectrum of the Sampled Signal Based on a Model of a Non-ideal Sampling Process

Note that both the extended definitions of the spectrum of a sampled signal refer to the sampling operation performed in an ideal way. And just then, as we have seen, a need for an extension of the usual definition of the signal spectrum arises. But, obviously, the spectrum of a sampled signal can be also determined on the basis of a real process of sampling that takes some time for providing a signal sample.

Note that the latter effect can be called "a smearing of an ideal signal sample" – as, for example, in [51]. Denoting then the smeared sampled signal by $x_{S,T}(t)$, we can formulate one more proposal for the extended spectrum definition of the reference signal $x_{R,T}(t)$ according to the same scheme as before. That is as

$$x_{S,T}(t) = S\left(x_{R,T}(t)\right) \tag{17}$$

and

$$\mathrm{SPECT2}\left(x_{R,T}(t)\right) = \mathrm{F}\left(x_{S,T}(t)\right) = X_{S,T}(f) \ , \tag{18}$$

where the symbol S stands for the sample smearing operation analyzed in detail in [51]. Moreover, the notation $\mathrm{SPECT2}(\cdot)$ is used to distinguish this extended spectrum of $x_{R,T}(t)$ from the previous two alternative ones, and $X_{S,T}(f)$ means the Fourier transform of $x_{S,T}(t)$.

V. Spectrum Inconsistencies in the Vetterli's Model of Sampling Operation

In [14], Vetterli et al. have demonstrated a model of signal sampling operation that takes into account non-idealities occurring in performing it by real sampling devices (real A/D converters). It is shown here on Fig. 3, after [14, see Fig. 1].

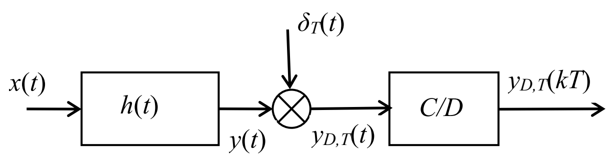

Fig. 3. Graphical representation of the Vetterli's model of signal sampling operation that reflects formula (1) complemented with the preceding and following operations (averaging of sampled signal and picking up samples, respectively).

In Fig. 3, the whole non-ideal behavior of a real A/D converter is "put into" the first block named "$h(t)$". And, it is modelled by a linear filter possessing an impulse response $h(t)$, which represents a local signal averaging process or any other appropriate one [3]. Hence, we can express the signal $y(t)$ in Fig. 3 as a result of a convolution of the continuous time signal $x(t)$, which is applied to the input of the A/D converter, with $h(t)$. The resulting signal is then sampled ideally as foreseen by formula (1), where obviously $\delta_T(t)$ in Fig. 3 (as in (1)) means the Dirac comb. (The symbol $\otimes$ in Fig. 3 means a multiplication.) In effect, we get the signal $y_{D,T}(t)$ (replacing now the signal $x_{D,T}(t)$ standing on the left-hand side of (1)). Finally, $y_{D,T}(kT)'$ s on the right-hand side of Fig. 3 stand for the samples of the signal $y(t)$. They are picked up from the signal $y_{D,T}(t)$ in the block named „C/D" (in fact, picking up the samples from $y_{D,T}(t)$ is the only task of this processing unit).

Note now that in the current theory of signal sampling the spectra of the signals $y_{D,T}(t)$ and $y_{D,T}(kT)$ occurring at the input and output of the C/D block, respectively, are the same.

This is so because we have

$$\begin{aligned} &\mathrm{F}\left(y_{D,T}(t)\right)=\mathrm{F}\left(\delta_T(t)\right)*\mathrm{F}\left(x(t)\right)= \\ &=\left(\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(2\pi f-k\frac{2\pi}{T}\right)\right)*X(f)\ , \end{aligned} \tag{19}$$

where the symbol $*$ stands for the convolution operation. Or, alternatively, $\mathrm{F}\left(y_{D,T}(t)\right)$ can be calculated as

$$\begin{aligned} &\mathrm{F}\left(y_{D,T}(t)\right)=\mathrm{F}\left(\sum_{k=-\infty}^{\infty}y_{D,T}(kT)\delta(t-kT)\right)= \\ &=\sum_{k=-\infty}^{\infty}y_{D,T}(kT)\exp(-j2\pi fkT)\ . \end{aligned} \tag{20}$$

Moreover, observe that the right-hand side of (20) is in fact nothing else than (per definition; see, for example, [3]) the Discrete Time Fourier Transform (DTFT) of the sequence of $y_{D,T}(kT)'$ s. Hence, we can write

$$\begin{aligned} &\mathrm{F}\left(y_{D,T}(t)\right)=\sum_{k=-\infty}^{\infty}y_{D,T}(kT)\exp(-j2\pi fkT)= \\ &=\mathrm{DTFT}\left(y_{D,T}(kT)\right)\ . \end{aligned} \tag{21}$$

(Note that all the derivations presented in (19), (20), and (21) are standard. They use standard properties of Fourier transforms and Dirac deltas referenced to in textbooks; see, for instance, [2-8].)

Furthermore, the spectrum of the sequence of discrete values, as these ones $y_{D,T}(kT)'$ s at the output of the C/D processing unit in Fig. 3, is well defined in the literature. It is just the $\mathrm{DTFT}\left(y_{D,T}(kT)\right)$ in this case.

Let us now check whether or not the C/D processing unit of Fig. 3 represents a linear system (circuit); to carry out this task, we need to verify whether or not, in this case, the superposition principle is satisfied. Therefore, to start, let us assume that the input signal $y_{D,T}(t)$ consists of two signals of the form

$$y_{D,T}^{(i)}(t)=\sum_{k=-\infty}^{\infty}y_{D,T}^{(i)}(kT)\delta(t-kT),\ i=1,\ 2\ . \tag{22}$$

Further, denote the operator describing the relation between input and output signals of the processing unit C/D simply as C/D. So, using this and the description of what it does, given beneath Fig. 3, we can write

$$\begin{aligned} &C/D\left(\alpha\cdot y_{D,T}^{(1)}(t)+\beta\cdot y_{D,T}^{(2)}(t)\right)=C/D\left(\alpha\sum_{k=-\infty}^{\infty}y_{D,T}^{(1)}(kT)\cdot\right. \\ &\left.\cdot\delta(t-kT)+\beta\sum_{k=-\infty}^{\infty}y_{D,T}^{(2)}(kT)\delta(t-kT)\right)= \\ &=C/D\left(\sum_{k=-\infty}^{\infty}\left(\alpha\cdot y_{D,T}^{(1)}(kT)+\beta\cdot y_{D,T}^{(2)}(kT)\right)\delta(t-kT)\right)= \\ &=\mathrm{sequence}\left(\alpha\cdot y_{D,T}^{(1)}(kT)+\beta\cdot y_{D,T}^{(2)}(kT)\right)= \\ &=\mathrm{sequence}\left(\alpha\cdot y_{D,T}^{(1)}(kT)\right)+\mathrm{sequence}\left(\beta\cdot y_{D,T}^{(2)}(kT)\right)= \\ &=\alpha\cdot\mathrm{sequence}\left(y_{D,T}^{(1)}(kT)\right)+\beta\cdot\mathrm{sequence}\left(y_{D,T}^{(2)}(kT)\right)\ , \end{aligned} \tag{23}$$

where α and β are some reals. Moreover, the symbol $\mathrm{sequence}(\cdot)$ stands for a sequence of indexed reals indicated in the brackets of this operator, with $k=..,-2,-1,0,1,2,..$ playing here a role of an index.

The final result in (23) shows that the operator C/D fulfills the superposition principle. So, we can say that it is a linear operator (that is describes a linear circuit (device)).

Many engineers believe that all linear systems (circuits) possess representations in form of convolution integrals (or convolution sums) accompanying with well-defined impulse responses. Obviously, this is not true – as shown in the literature; see, for example, [52-55].

It follows from the description of the operator C/D given beneath Fig. 3 that this operator does not possess a representation in form of a convolution integral as well as an accompanying impulse response. Hence, it does not have a spectrum (which is the Fourier transform of its impulse response).

So, in view of what was said above, the C/D processing unit in the Vetterli's model of Fig. 3 is a rather quirky one. It represents a linear device that possesses no frequency characteristic.

And, obviously, this hinders – in the frequency domain – explanation of what happens on the way between the input and output of the C/D processing unit. Note that at the input and output of this device we have signals possessing the same spectra, but evidently different time characteristics.

VI. Conclusion

In this paper, it has been shown that the formula used in the literature for the spectrum of ideally sampled signals is arbitrary, and there exist at least two possible relevant choices. Very strong arguments for their use have been presented.

Finally, it has been also pointed out that there exist spectrum inconsistencies in the Vetterli's model [14] of sampling operation.

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