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Application of spline functions in adjustment of rail tracks*

Spline functions play an important role in many technical applications. The paper presents some potential possibilities of their application in adjustment of railway tracks as a versatile tool of track axis approximation. Methods of determination of cubic interpolating and approximating splines are presented in detail. Splines are easily determined and they have satisfactory convergence to the desired curve.

INTRODUCTION

Spline functions have become quite popular in recent years as a perfect tool which could be used in many sciences and technology. They could be helpful in approximation with huge exactness of searched functions, which is appreciated in modelling of various physical processes, car bodies or aircraft profiles, designing and stock-taking of coating structures. Railroad engineering, and especially its branches connected with the process of rail track adjustment, may become the next domain in which spline functions will be used. A rail track is in fact a set of rectilinear and curvilinear sections, which in the point of connection preserve curvature continuity, so they are ideal to be represented by spline functions. Usefulness of application of these curves is usually reduced to comparison of periodical control measurement results before physical adjustment of rail tracks when one has a set of measurement points and has no information on the exact location of points which are the connection points for particular rail track sections. Parameters of functions which determine those sections are also unknown. In designing of rail routes, parameters of lines, connecting curves and circular arcs are known, so application of spline function can not be justified in such cases. Periodical control rail track measurements (GPS-RTK, electronic tacheometry) allow to determine a discrete set of points distributed not necessarily at

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permanent distances. A comparison of results of such measurements may cause some difficulties because observations performed periodically using the above mentioned methods will never relate to the same points on rail track axis. A way out of the situation is to transform the discrete model into a continuous one, assuming additionally continuity of curvature in each point of such a model. These requirements are fulfilled by spline functions and the representation of rail track axis obtained using them allows to determine differences between two periodical measurements in any its point, independent on the place of observation. The information obtained in such a way facilitates detection of possible deviations from track axis coming from the project or determined during the last adjustment and making a decision on necessity of another adjustment. There is also another possibility of spline curves application: this concerns the very adjustment process. The primary activity during it is recognition of initial points and final points of particular track sections (lines, connecting curves and circular arcs). Independent on the method of identification, its accuracy depends on density of measured points. During measurements of track axis, observations are made, usually every some, unnecessarily permanent distance, without knowing where exactly extreme *K* points of sections are situated. Generally, such points will lie between some measurement points *P* (Fig. 1). During identification of sections [9] such situation forces to substitute unknown *K* points with *P* points. Depending on which of *P* points is chosen, there will be different lengths of *L* sections. As one can estimate on the drawing, exactness of *L* length which is determined in this way will be limited on average to the length of one measurement interval (average distance between successive measurement points).

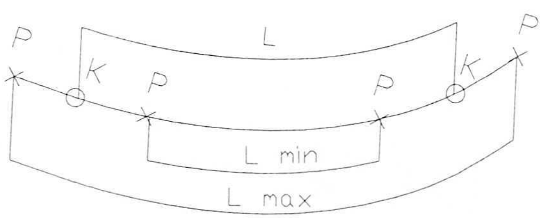

Fig.1

During observation carried out using GPS receivers applying a kinematic method (registration takes place at short time intervals during cart movement), any density of points may be achieved, e.g. one point per meter and in such situation inaccuracy of identification is of no importance. Nevertheless, as far as tacheometric observations or GPS observations using a stop-and-go method are considered (recording takes place only on cart stops), the measured points are situated every few or a dozen meters depending on the type of section. In such situation with unfavourable distribution of points, accuracy of section length recognition may go down even to circa 20 meters. New methods of rail tracks adjustment [9] are mainly based on execution of axis design for parameters determined on the basis of sections identified before. The design is compared with existing points and then changed till the moment when they satisfactorily fit. Such solution depends on precision of identification of sections, the higher accuracy, the closer to an optimum the initial parameters of adjustment are and by that a desired design may be easily determined. During preparation of a design, the length of connecting curve is the factor which allows to have greatest changes. Inaccuracy of determination of the length of curve section, as explained above, may be relatively high, which is unfavourable, especially, when the curve length is below 100 m. In such situation, when work is based on deformed information, it is difficult to prepare a design of adjustment well fitted to the points. Then one has to search, quite often many times, and by method of experiments find design parameters. Densification of the set of measurement points and identification of sections on data complemented in such way would be an ideal solution. When a continuous description of rail track is given, any densification of points is possible which favours application of spline curves.

During approximation of track axis with spline functions one may make use of two models: interpolation and approximation models. Interpolation forces to draw the spline curve exactly through all measurement points, which may cause axis waving, if there are some measurement errors and there is high density of points, and in effect the picture might be not too clear. Approximation provides more possibilities: when the size of measurement errors is assumed and a parameter of adjustment to measurement data is established at a given level, a compromise between the smoothest curve and the most adjusted one can be reached. The methods of determination of interpolating and approximating curves are explained below.

1. *Methods of determining one-dimensional cubic spline curves*

1.1. Basic assumptions

Track axis should be presented as a continuous line which additionally has a continuous curvature. Line continuity is identical with (spline) function continuity and the curvature is determined by first two derivatives of the function:

$$K = \frac{f''}{\left[1 + (f')^2\right]^{\frac{3}{2}}} \tag{1}$$

Therefore, it is necessary to use function in class C^2, that is, a double continuously differentiable function, in order to meet the assumed requirements. Such requirements are met by cubic spline curves which consist of sections of third degree polynominals. This degree is sufficient to preserve curvature continuity and it also does not cause undesirable waving of function typical of high degree polynominals. If one operates on a given interval $[a, b]$ and determines its division: $\Delta = \{a = x_0 < x_1 < ... < x_n = b\}$, a third degree spline function could be determined$S:[a, b] \rightarrow R$ with the following properties:

1) function S belongs to C^2 class, that is, it is double continuously differentiable on $[a, b]$
2) function S is a third degree polynominal on every subinterval $[x_i, x_{i+1}]$, $i = 0,1...n$

If function S assumes in nodes $\{x_0, x_1...x_n\}$ values $S(x_i) = y_i$, $\{y_0, y_1...y_n\}$ this is a case of interpolation, on the other hand, if it assumes values only approximate to these $S(x_i) = \overline{y}_i$, $\{\overline{y}_0, \overline{y}_1...\overline{y}_n\}$ it is a case of approximation. Equations which allow to determine spline functions are set making use of the fact that function value with their two first derivatives must be equal in nodes x_i. In order to define spline functions unanimously it is necessary to determine two degrees of freedom which requires additional conditions out of which equating of second derivatives on section ends to zero: $S''(a) = S''(b) = 0$, is the simplest and a very universal one.

1.2. Interpolation using one-dimensional cubic spline curves

Interpolating spline functions [1,10], as was indicated before, are functions which in $\{x_0, x_1..x_n\}$ achieve some required values $\{y_0, y_1...y_n\}$. Assuming as: $\Delta = \{x_i \mid i = 0, 1...n\}$ – a determined division of interval $[a,b]$ with nodes $a = x_0 < x_1 < ... < x_n = b$, $Y = \{y_i \mid i = 0, 1...n\}$ – a set $(n + 1)$ of required real numbers and also defining:

$$M_i = S''(x_i) \qquad i = 0, 1...n \tag{2}$$

as M moments of function S, that is the values of second derivatives of searched function in nodes x_i, and:

$$h_{i+1} = x_{i+1} - x_i \qquad i = 0, 1...n - 1 \tag{3}$$

one may begin determining parameters of a spline function. Second derivative $S''(x)$ is a continuous function in the interval $[a,b]$ and a linear one in its every subinterval $[x_i, x_{i+1}]$, so one may present it as follows:

$$S''(x) = M_i \frac{x_{i+1} - x}{h_{i+1}} + M_{i+1} \frac{x - x_i}{h_{i+1}} \qquad x \in [x_i, x_{i+1}] \tag{4}$$

Integration by sides of this expression will result in:

$$S'(x) = M_i \frac{(x_{i+1} - x)^2}{2h_{i+1}} + M_{i+1} \frac{(x - x_i)^2}{2h_{i+1}} + A_i \tag{5}$$

$$S(x) = M_i \frac{(x_{i+1} - x)^3}{6h_{i+1}} + M_{i+1} \frac{(x - x_i)^3}{6h_{i+1}} + A_i(x - x_i) + B_i \tag{6}$$

If one imposes on (5) and (6) interpolation conditions:

$$S(x_i) = y_i = M_i \frac{h^2_{i+1}}{6} + B_i$$

$$S(x_{i+1}) = y_{i+1} = M_{i+1} \frac{h^2_{i+1}}{6} + A_i h_{i+1} + B_i$$

integration constants A_i, B_i determined:

$$B_i = y_i - M_i \frac{h^2_{i+1}}{6} \tag{7}$$

$$A_i = \frac{y_{i+1} - y_i}{h_{i+1}} - (M_{i+1} - M_i) \frac{h_{i+1}}{6} \tag{8}$$

These constants are used to determine parameters of function S in every subinterval $[x_i, x_{i+1}]$ in the following equation [10]:

$$S(x) = \alpha_i + \beta_i (x - x_i) + \gamma_i (x - x_i)^2 + \delta_i (x - x_i)^3 \tag{9}$$

where:

$$\alpha_i = y_i$$

$$\beta_i = S'(x_i) = \frac{y_{i+1} - y}{h_{i+1}} - \frac{2M_i + M_{i+1}}{6} h_{i+1} \tag{10}$$

$$\gamma_i = \frac{M_i}{2}$$

$$\delta_i = \frac{S''(x_i^+)}{6} = \frac{M_{i+1} - M_i}{6h_{i+1}}$$

Moments M_i could be determined imposing on function S a condition that its first derivative is continuous in interval $[a, b]$ in nodes x_i. Using (5) and (8) one-side limits are calculated in points:

$$S'(x_i^-) = \frac{y_i - y_{i-1}}{h_i} + \frac{h_i}{3} M_i + \frac{h_i}{6} M_{i-1}$$

$$S'(x_i^+) = \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{h_{i+1}}{3} M_i - \frac{h_{i+1}}{6} M_{i+1}$$

and in accordance with continuity condition they are equated:

$$\frac{h_i}{6} M_{i-1} + \frac{h_i + h_{i+1}}{3} M_i + \frac{h_{i+1}}{6} M_{i+1} = \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i} \tag{11}$$

which results in a system of $(n - 1)$ equations for $(n + 1)$ moments.

Such equations may be expressed in the following way:

$$\mu_i M_{i-1} + 2M_i + \lambda_i M_{i+1} = d_i \qquad i = 0, 1 \ldots n - 1 \tag{12}$$

where:

$$\lambda_i = \frac{h_{i+1}}{h_i + h_{i+1}}$$

$$\mu_i = \frac{h_{i+1}}{h_i + h_{i+1}} \qquad i = 0, 1 \ldots n - 1 \tag{13}$$

$$d_i = \frac{6}{h_i + h_{i+1}} \left(\frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i} \right)$$

In order to determine the other two equations one should use one of exemplary conditions:

$$S''(a) = S''(b) = 0 \tag{14a}$$

$$S'(a) = y_0', \; S'(b) = y_n' \qquad \text{for known } y_0' \text{ i } y_n' \tag{14b}$$

$$S''(a) = y_0'', \; S''(b) = y_n'' \qquad \text{for known } y_0'' \text{ i } y_n'' \tag{14c}$$

Each of these conditions ensures a unanimous determination of all parameters of a spline function. Equations which result from the above conditions may be expressed as (12), and in particular for $S''(a) = S''(b) = 0$ they will be expressed as follows:

$$2M_0 + \lambda_0 M_1 = d_0$$
$$\mu_n M_{n-1} + 2M_n = d_n \tag{15}$$

assuming that: $\lambda_0 = 0$, $d_0 = 0$, $\mu_n = 0$, $d_n = 0$

Having a full system of linear equations which determines coefficients of interpolating spline function, it may be presented as a matrix form:

$$\begin{bmatrix} 2 & \lambda_0 & 0 & . & . & 0 \\ \mu_1 & 2 & \lambda_1 & 0 & . & 0 \\ 0 & \mu_2 & 2 & \lambda_2 & . & 0 \\ . & . & . & . & . & . \\ 0 & . & . & \mu_{n-1} & 2 & \lambda_{n-1} \\ 0 & . & . & . & \mu_n & 2 \end{bmatrix} \times \begin{bmatrix} M_0 \\ M_1 \\ . \\ . \\ . \\ M_n \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ . \\ . \\ . \\ d_n \end{bmatrix} \tag{16}$$

Figure 2 presents the course of function S on $[a, b]$, which is determined on the basis of the above method with particular subfunctions S_i determined on subintervals $[x_i, x_{i+1}]$ and node points.

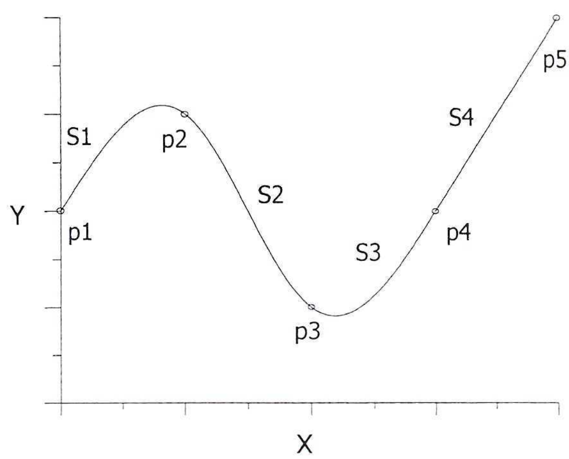

Fig. 2. Interpolating cubic spline curve

Ability to fit best to searched function and minimisation of total curvature are very important features characteristic of spline functions. Considering sets $K^m[a, b]$ of real functions f projecting $[a, b] \rightarrow R$, which have absolutely continuous derivatives f^{m-1} on $[a, b]$ to the order $(m - 1)$ inclusive and integrable with square on $[a, b]$: $f^m \in L^2[a, b]$ and determining $m = 2$, one can introduce the following norms:

$$\| f \|^2 = \int_a^b | f''(x) |^2 dx \tag{17}$$

It is connected with total curvature of function f in $[a, b]$, which is defined by a formula (1). When f' is small in comparison with the unity, the curvature will be approximately f'', so $\| f \|$ will determine the size of total curvature. Based on the basic Holladay's identity: If $f \in K^2(a, b)$ and $\Delta = \{a = x_0 < x_1 < ... < x_n = b\}$ is a division of interval $[a, b]$ and S is a spline function with nodes x_i then:

$$\| f - S \|^2 = \| f \|^2 - \| S \|^2 - 2 \left[\left(f'(x) - S'(x)\right) S''(x) \Big|_b^a - \sum_{i=1}^{n} \left(f(x) - S(x)\right) S''(x) \Big|_{x^+_{i-1}}^{x^-_i} \right] \tag{18}$$

which with given values $Y = \{y_0, y_1..y_n\}$ and when $f(x_1) = y_i$, and when one of conditions (14) is fulfilled, is reduced to the following form:

$$\| f - S(Y) \|^2 = \| f \|^2 - \| S(Y) \|^2 \geq 0 \tag{19}$$

This is the so called first integral relation which has the property of minimum norm. With assumptions from Holladay's theorem and additionally condition (14a) one may conclude that it is the spline function $S(Y)$ that minimises an integral (17) and for this very condition (14a), $S(Y)$ is called a natural spline function. Such function is the smoothest of functions which interpolate a given set of points. From the point of view of usefulness of approximation of a searched function f, convergence of a spline function S to it is an interesting issue. It is particularly important for a function with a quickly changeable curvature which has additionally local extremes. The convergence problem was precisely discussed in studies [2, 10], here it is only reminded that it is essential that conditions (14) are always precisely fulfilled, otherwise, the convergence may be very weak or there might be no convergence at all. If there are sudden changes of function curvature, instead of cubic curves it is better to use rational spline curves in the following form:

$$r_{m,n}(x) = \frac{p_m(x)}{q_n(x)} \tag{20}$$

where $p_m(x)$, $q_n(x)$ are relatively prime polynominals of m and $n$degree, that is, they are not divisible by the same polynominal of a positive degree.

Such curves assure smaller maximum errors of approximation and quicker convergence to function f. If applied in rail tracks approximation, application of ordinary cubic splines is fully sufficient. In such cases curvature changes have such a mild character that one could try to apply polynominal spline curves of a higher degree (4,5), which at once interpolate a bigger set of points, that is, when every section of a 3rd degree curve is contained between every pair of points, a section of a 4th degree curve can be drawn through three points, and a section of a 5th degree curve – through 4 points. In such cases it is sufficient to provide conditions of continuity of function and of its two first derivatives only in extreme points which are connectors of successive sections of a spline curve. A negative influence of oscillation of higher degree polynominals on section extremes should not be noticeable when this method is applied; additionally, it will allow to save considerably on the total number l of coefficients needed for determination of all sections of spline curve on interval $[a, b]$. These numbers for n measurement points are as follows:

– for 3rd degree $l_3 = (n - 1)$ curves $\times$ 4 coefficients, $l_3 = 4(n - 1)$ coefficients

– for 4th degree $l_4 = \dfrac{(n-1)}{2}$ curves $\times$ 5 coefficients, $l_4 = 2.5(n - 1)$ coefficients

– for 5th degree $l_5 = \dfrac{(n-1)}{3}$ curves $\times$ 6 coefficients, $l_5 = 2(n - 1)$ coefficients

1.3. Approximation using one-dimensional cubic spline curves

Modern measurement points allow to determine location of rail track axis with high degree of precision, nevertheless, every observation is burdened with an error which makes it impossible to locate points precisely. Having some, most probable co-ordinates calculated on the basis of measurement results and related mean errors σ, a spline function may be drawn not necessarily through given points $\{y_0, y_1 ... y_n\}$ but through their correspondent points $\{\bar{y}_0, \bar{y}_1 ... \bar{y}_n\}$ which are determined in such a way that they meet conditions of lowest squares method and make it possible to determine a possibly smooth function which minimises the following expression [2, 3]:

$$p \sum_{i=1}^{n} \left(\frac{y_i - S(x_i)}{\sigma_{y_i}} \right)^2 + (1 - p) \int_{x_1}^{x_n} (S''(t))^2 dt \tag{21}$$

Its first member is responsible for mean square approximation and the second member is the above presented value of the standard $\| f \|^2$ which determines minimum curvature of function S. When the required value of parameter p, $0 \le p \le 1$ is chosen, the result is a possibly smooth curve alternately with a curve which is best fitted in a set of points. The size of error σ_{y_i} has also an impact on function course; the higher the impact, the bigger a relative value of the error.

Now, an algorithm which allows to determine approximating spline function S which in nodes $\{x_0, x_1 ... x_n\}$ assumes values $S(x_i) = \bar{y}_i$, $\{\bar{y}_0, \bar{y}_1 ... \bar{y}_n\}$ with superimposition of the condition $S''(a) = S''(b) = 0$ characteristic of natural spline function, will be presented. Using for this purpose a system of equations (11) which defines an interpolating spline function and remembering to substitute value y_i with $\overline{y}_i$, one can write:

$$M_0 = 0$$

$$h_i M_{i-1} + 2(h_i + h_{i+1}) M_i + h_{i+1} M_{i+1} = 6\left(\frac{\overline{y}_{i+1} - \overline{y}_i}{h_{i+1}} - \frac{\overline{y}_i - \overline{y}_{i-1}}{h_i}\right) \quad i = 0, 1 \ldots n-1 \quad (22)$$

$$M_n = 0$$

In a matrix form the following expression will result:

$$\mathbf{RM} = 6\mathbf{Q}^T \overline{\mathbf{Y}} \quad (23)$$

where: $\mathbf{R}$ is a symmetric tridiagonal matrix with size $(n-2) \times (n-2)$ which in every row has expressions: h_i, $2(h_i + h_{i+1})$, h_{i+1} $\mathbf{M}$ is a vector with size $(n-2)$ which clusters searched moments M_i, $\mathbf{Q}$ is a tridiagonal matrix with size $(n-2) \times n$ and comprises rows: $\frac{1}{h_i}, -\frac{1}{h_i} -\frac{1}{h_{i+1}}, \frac{1}{h_{i+1}}$, $\overline{\mathbf{Y}}$ is a vector with size n which contains elements $\overline{y}_i$.

Linearity of second derivative of function S allows to put a part of the equation (21) which is responsible for curvature minimising in the following form [3]:

$$\int_{x_1}^{x_n} (S''(t))^2 dt = \frac{1}{3} \sum_{i=1}^{n-1} h_{i+1} (M_i^2 + M_i M_{i+1} + M_{i+1}^2) \quad (24)$$

and the whole equation (21):

$$p \sum_{i=1}^{n} \left[\frac{y_i - \overline{y}_i}{\sigma_{y_i}}\right]^2 + \frac{1}{3}(1-p) \sum_{i=1}^{n-1} h_{i+1} (M_i^2 + M_i M_{i+1} + M_{i+1}^2) \quad (25)$$

presenting that in a matrix form:

$$p\mathbf{V}^T \mathbf{P} \mathbf{V} + \frac{1}{6}(1-p) \mathbf{M}^T \mathbf{R} \mathbf{M} \quad (26)$$

where $\mathbf{V} = \mathbf{Y} - \overline{\mathbf{Y}}$, and $\mathbf{P} = \mathrm{diag}\left[\frac{1}{\sigma_{y_{i_1}}^2}, \ldots, \frac{1}{\sigma_{y_n}^2}\right]$

Determining **M** from (23) and substituting it in (26), one will obtain an expression which depends only on variables $\overline{y}_i$:

$$p(\mathbf{Y} - \overline{\mathbf{Y}}^T\mathbf{P}(\mathbf{Y} - \overline{\mathbf{Y}}) + 6(1 - p)(\mathbf{R}^{-1}\mathbf{Q}^T\overline{\mathbf{Y}})^T\mathbf{R}(\mathbf{R}^{-1}\mathbf{Q}^T\overline{\mathbf{Y}}) \tag{27}$$

which allows to minimise it using smallest squares' method, which results in a system of equations:

$$-2p\mathbf{P}(\mathbf{Y} - \overline{\mathbf{Y}}) + 12(1 - p)(\mathbf{R}^{-1}\mathbf{Q}^T)^T\mathbf{R}(\mathbf{R}^{-1}\mathbf{Q}^T)\overline{\mathbf{Y}} = 0 \tag{28}$$

which after some development and another change of variables from $\overline{y}_i$ to M_i using (23) will have the final form:

$$(6(1 - p)\mathbf{Q}^T\mathbf{P}^{-1}\mathbf{Q} + p\mathbf{R})\mathbf{M} = 6p\mathbf{Q}^T\mathbf{Y} \tag{29}$$

or if one puts it in an abbreviated form:

$$\mathbf{AM} = \mathbf{L} \tag{30}$$

for $\mathbf{A} = (6(1 - p)\mathbf{Q}^T\mathbf{P}^{-1}\mathbf{Q} + p\mathbf{R})$ and $\mathbf{L} = 6p\mathbf{Q}^T\mathbf{Y}$

Solution of such system has a known form:

$$\mathbf{M} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{L} \tag{31}$$

Determined moments of function S make it possible to determine points $\overline{y}_i$, through which an approximating curve is drawn:

$$\overline{\mathbf{Y}} = \mathbf{Y} - \frac{1 - p}{p}\mathbf{P}^{-1}\mathbf{QM} \tag{32}$$

The value of total distance B of spline function from approximated function could be a handy piece of information, which may be determined on the basis of the formula:

$$B = (\mathbf{Y} - \overline{\mathbf{Y}})^T\mathbf{P}(\mathbf{Y} - \overline{\mathbf{Y}}) \tag{33}$$

Approximating spline curves provide high flexibility in the process of searching for the most suitable approximation of measured rail track axis. Parameter p is of crucial importance in modelling of such curves, nevertheless, as measurement errors grow, their role also becomes important. This situation is explained on the following Fig. 3.

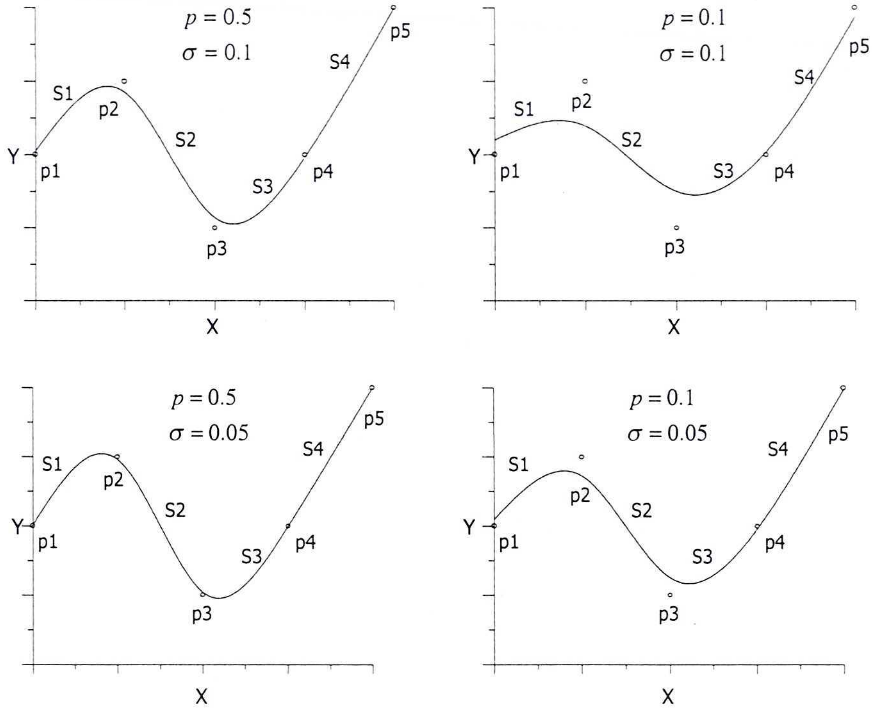

Fig. 3. Dependence of approximation on choice of parameter p and the size of error σ

It should be emphasised that during comparison of two periodical track measurements it is necessary to make approximations with the same determined parameter p. In both measurements one should also assume similar values of measurement errors. If these rules are not followed, effects of comparisons might be unreliable.

Cubic interpolating and approximating spline curves described in the study are a useful tool which has many applications. Issues related to adjustment of rail track axis should be added soon to this group. Relatively simple numerical methods necessary to determine them and good effects of approximation using such curves are in favour of such application. Additionally, track axis has an uncomplicated course, therefore, there is no need to use more refined splines in order to get a better convergence.

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Zastosowania funkcji sklejanych w procesie regulacji osi torów kolejowych

S t r e s z c z e n i e

Funkcje sklejane znalazły poczesne miejsce w wielu dziedzinach techniki. Artykuł ukazuje kilka potencjalnych możliwości ich wykorzystania w procesie regulacji torów kolejowych, jako elastycznego narzędzia przybliżania funkcji opisujących oś toru. Przedstawione zostały także dokładnie metody wyznaczania interpolacyjnych i aproksymacyjnych kubicznych krzywych sklejanych, cechujących się łatwością wyznaczenia i wystarczającą w takich zastosowaniach zbieżnością do szukanej krzywej.

Гжегош Ленда

Применение склеиванных функций в процессе регуляции осей железнодорожных путей

Р е з ю м е

Склеиванные функции нашли почётное место в многих областях техники. В статье представлены некоторые потенциальные возможности их использования в процессе регуляции железнодорожных путей как гибкого инструмента приближения функций описывающих ось пути. Представлены тоже точные методы определения интерполяционных и аппроксимативных склеиванных кубических кривых, характеризующихся простотой определения и достаточной сходимостью с определяемой кривой.