Central European Journal of Economic Modelling and Econometrics

The Half-Logistic Odd Power Generalized Weibull-G Family of Distributions

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Submitted: 9.08.2020, Accepted: 30.08.2021

Abstract

We develop and study in detail a new family of distributions called Half-logistic Odd Power Generalized Weibull-G (HLOPGW-G) distribution, which is a linear combination of the exponentiated-G family of distributions. From the special cases considered, the model can fit heavy tailed data and has non-monotonic hazard rate functions. We further assess and demonstrate the performance of this family of distributions via simulation experiments. Real data examples are given to demonstrate the applicability of the proposed model compared to several other existing models.

Keywords: half logistic distribution, half logistic-G distribution, Weibull generalized-G distribution, maximum likelihood estimation

JEL Classification: C2, C4, C6, C8

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1 Introduction

Lately, more work has been done in developing new families by adding extra shape parameters to achieve better fits and more flexibility in modelling practical data. Such works include generalizations of the half logistic distribution that can fit skewed data and also model non-monotonic hazard function. Bagdonavicius and Nikulin (2002) developed an extension of the Weibull distribution, namely power generalized Weibull (PGW) distribution and also proposed a chi-square statistic for testing the validity of the PGW distribution and presented its application to censored survival times for cancer patients. Lai (2018) described the PGW as one of the extensions of Weibull distribution that can exhibit non-monotonic hazard rates. Nikulin and Haghighi (2009) obtained maximum likelihood estimates (MLEs) of the parameters and the importance of the model was illustrated using Efron's head-and-neck cancer clinical trial data, see Efron (1988). The PGW family can be used as a possible alternative to the Exponentiated Weibull family for modelling lifetime data. The other model that has been widely considered in modelling lifetime data is the Nadarajah-Haghighi (NH) distribution developed by Nadarajah and Haghighi (2011) and the distribution is a generalization of the exponential distribution.

Extensions of the half logistic distribution include work by Afify et al. (2017), Cordeiro et al. (2014, 2016), El-sayed and Mahmoud (2019), Torabi and Bagheri (2010), Sumeet et al. (2010), Balakrishnan and Aggarwala (1996), Kumar et al. (2015), Balakrishnan and Wong (1994), Balakrishnan and Chan (1992) and Muhammad (2017). The development of generalized-G (G-G) families led to the advent of flexible models that can handle heavy skewed and heavy tailed data well compared to some of the baseline distributions. The generalized models can also fit both non-monotonic and monotonic hazard rate functions. Some of the work on G-G families include work by Marshall and Olkin (1997), Eugene et al. (2002), Cordeiro and de Castro (2011) and Chipepa et al. (2019a, 2019b), to mention a few.

Cordeiro et al. (2016), developed the type I half-logistic family of distributions with the cumulative distribution function (cdf) given by

$$F(x;\lambda,\underline{\xi}) = \int_0^{-\ln(1-G(x;\underline{\xi}))} \frac{2\lambda\exp\{-\lambda x\}}{(1+\exp\{-\lambda x\})^2}dx = \\ = \frac{1-[1-G(x;\underline{\xi})]^{\lambda}}{1+[1-G(x;\underline{\xi})]^{\lambda}}, \tag{1}$$

where $G(x;\underline{\xi})$ is the cdf of the baseline distribution, $\lambda > 0$ is the shape parameter and $\underline{\xi}$ is the vector of parameters. If we set $\lambda = 1$ in Equation (1), then the type I half-logistic family of distributions reduces to half-logistic-G (HL-G) model, with cdf and pdf given by

$$F(x;\underline{\xi}) = \frac{G(x;\underline{\xi})}{1+\overline{G}(x;\underline{\xi})}, \tag{2}$$

and

$$f(x;\underline{\xi}) = \frac{2g(x;\underline{\xi})}{(1+\overline{G}(x;\underline{\xi}))^2}, \tag{3}$$

respectively.
Generalizations of the half logistic distribution produced very flexible distributions that fit well to data with varying levels of skewness and kurtosis. It is due to this motivation that we developed the Half Logistic Odd Power Generalized Weibull-G (HLOPGW-G) family of distributions which is versatile and more flexible in fitting data. The proposed model is considered under a situation where the baseline distribution has an extra shape parameter. The new distribution also has interesting tractability properties which can be traced back to the exponentiated-G (Exp-G) family of distributions.
The new family of distributions can model highly skewed and tailed data better than some of the baseline distributions. The flexibility of the new model applies to data sets with varying skewness and kurtosis including data that has non-monotonic hazard rate functions. The distribution can be expressed as an infinite linear combination of the exponentiated-G distribution, which easily allows for the derivation of its statistical properties. Furthermore, the new family of distributions contains several known and new sub-models and some of the existing sub-families include type I half-logistic and half-logistic-G families given by Equations (1) and (2), respectively. Finally, the development of this new family of distributions is necessitated by the need to model various forms of lifetime data to include, economics, engineering, survival analysis and finance with models that takes into consideration not only shape and scale but also skewness, kurtosis and tail variation. Consider a random variable X having the cdf given by $G(x;\underline{\xi})$, where $\underline{\xi}$, is a vector of parameters, then the survival function of X is given by $\bar{G}(x;\underline{\xi}) = 1 - G(x;\underline{\xi})$ with the pdf defined by $g(x;\underline{\xi}) = dG(x;\underline{\xi})/dx$. The odd power generalized Weibull-G (OPGW-G) family has its cdf and pdf defined by

$$F(x) = 1 - \exp\left\{1 - \left[1 + \left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha}\right]^{\beta}\right\} \tag{4}$$

and

$$f(x) = \alpha\beta\left[1 + \left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha}\right]^{\beta-1} \exp\left\{1 - \left[1 + \left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha}\right]^{\beta}\right\} \times$$
$$\times \left(G(x;\underline{\xi})\right)^{\alpha-1}\left(\overline{G}(x;\underline{\xi})\right)^{-(\alpha+1)} g(x;\underline{\xi}), \tag{5}$$

respectively for $\alpha, \beta > 0$ and $\underline{\xi}$ as vector of parameters from the baseline distribution (see Moakofi et al. 2020).
In this article, we develop the new family of distributions namely the Half Logistic Odd Power Generalized Weibull-G (HLOPGW-G) family of distributions. The organization

of this paper is as follows: Section 2 presents the new generalized family of distributions with series expansion of the density, quantile and hazard rate functions. Some of the special cases of the HLOPGW-G family of distributions are presented in Section 3. Structural properties including the distribution of order statistics, Rényi entropy, moments, probability weighted moments and generating functions are presented in Section 4. Section 5 presents the maximum likelihood estimates. Monte Carlo simulations are performed to study the behaviour of maximum likelihood estimators for some selected parameters under Section 6 We run some applications of this model to real datasets under Section 7 and finally give concluding remarks in Section 8.

2 Half-logistic odd power generalized Weibull-G model

A new family of distributions, namely, the Half Logistic odd Power Generalized Weibull-G (HLOPGW-G) distributions is developed and studied in this section. Series representation of this new family of distributions is also presented. We combine and extend the results by Cordeiro et al. (2016) (see Equations (2) and (3)) and Equations (4) and (5), to derive the HLOPGW-G family of distributions with cdf and pdf given by

$$F(x;\alpha,\beta,\underline{\xi}) = \frac{1-\exp\left\{1-\left[1+\left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha}\right]^{\beta}\right\}}{1+\exp\left\{1-\left[1+\left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha}\right]^{\beta}\right\}} \tag{6}$$

and

$$\begin{aligned} f(x;\alpha,\beta,\underline{\xi}) = 2\alpha\beta &\left[1+\left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha}\right]^{\beta-1}\exp\left\{1-\left[1+\left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha}\right]^{\beta}\right\}\times \\ &\times\left(G(x;\underline{\xi})\right)^{\alpha-1}\left(\overline{G}(x;\underline{\xi})\right)^{-(\alpha+1)}\times \\ &\times\left(1+\exp\left\{1-\left[1+\left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha}\right]^{\beta}\right\}\right)^{-2} g(x;\underline{\xi}), \end{aligned} \tag{7}$$

respectively, for $\alpha,\beta>0$ and parameter vector $\underline{\xi}$. Note that by fixing some of the parameters, we obtain new sub-families of the HLOPGW-G family of distributions given by Table 1.

Table 1: New Sub-families of HLOPGW-G family of Distributions

Distribution	α	β
Half-Logistic Odd Weibull-G (HLOW-G)	-	1
Half-Logistic Odd Nadarajah Haghighi-G (HLONH-G)	1	-
Half-Logistic Odd Exponential-G (HLOE-G)	1	1
Half-Logistic Odd Rayleigh-G (HLOR-G)	2	1

2.1 The hazard rate and quantile functions

This subsection presents the hazard rate function (hrf) and quantile function (qf) for the HLOPGW-G family of distributions. The hrf for the HLOPGW-G family of distributions is given by

$$h(x;\alpha,\beta,\underline{\xi})=\frac{\alpha\beta\left[1+\left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha}\right]^{\beta-1}\left(G(x;\underline{\xi})\right)^{\alpha-1}g(x;\underline{\xi})}{\left(\overline{G}(x;\underline{\xi})\right)^{\alpha+1}\left(1+\exp\left\{1-\left[1+\left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha}\right]^{\beta}\right\}\right)},$$

for $\alpha,\beta>0$ and parameter vector $\underline{\xi}$. The qf is derived by inverting the cdf given by Equation (6). We invert the function

$$\frac{1-\exp\left\{1-\left[1+\left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha}\right]^{\beta}\right\}}{1+\exp\left\{1-\left[1+\left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha}\right]^{\beta}\right\}}=u,$$

for $0\leq u\leq 1$, which simplifies to

$$1-u=(1+u)\exp\left\{1-\left[1+\left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha}\right]^{\beta}\right\},$$

and can be written as

$$\left([1+\ln(1+u)-\ln(1-u)]^{1/\beta}-1\right)^{1/\alpha}=\left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right),$$

which simplifies to

$$G(x;\underline{\xi})=\left(\left[(1+\ln(1+u)-\ln(1-u))\right]^{1/\beta}-1\right]^{-1/\alpha}+1\right)^{-1}.$$

The quantiles of the HLOPGW-G family of distributions may be determined by solving the equation

$$x(u) = G^{-1}\left[\left(\left[(1+\ln(1+u)-\ln(1-u))\right]^{1/\beta}-1\right]^{\frac{-1}{\alpha}}+1\right)^{-1}\right], \tag{8}$$

using iterative methods.

2.2 Expansion of the density function

Series expansion of the HLOPGW-G family of distributions is presented under this subsection. By applying Equation (7) and using series representation $(1-x)^{-2}=\sum_{n=1}^{\infty} nx^{n-1}$, for $|x|<1$, we obtain

$$\left(1+\exp\left\{1-\left[1+\left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha}\right]^{\beta}\right\}\right)^{-2} = \\ = \sum_{n=1}^{\infty}(-1)^{n-1}n\left[\exp\left\{1-\left[1+\left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha}\right]^{\beta}\right\}\right]^{n-1}$$

so that

$$f(x;\alpha,\beta,\underline{\xi}) = 2\alpha\beta\sum_{n=1}^{\infty}(-1)^{n-1}n\left[1+\left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha}\right]^{\beta-1}\times \\ \times\exp\left\{n\left(1-\left[1+\left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha}\right]^{\beta}\right)\right\}\times \\ \times\left(G(x;\underline{\xi})\right)^{\alpha-1}\left(\overline{G}(x;\underline{\xi})\right)^{-(\alpha+1)}g(x;\underline{\xi}).$$

Applying the exponential series expansion $e^{x}=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ and the binomial expansion $(x+y)^n=\sum_{m=0}^{\infty}\binom{n}{m}x^m y^{n-m}$, $n\geq 0$, an integer or $|x/y|<1$, we can write

$$\exp\left\{n\left(1-\left[1+\left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha}\right]^{\beta}\right)\right\} = \sum_{q,k=0}^{\infty}\frac{n^q}{q!}\binom{q}{k}(-1)^k\left[1+\left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha}\right]^{\beta k},$$

so that

$$f(x;\alpha,\beta,\underline{\xi}) = 2\alpha\beta\sum_{n=1}^{\infty}\sum_{q,k=0}^{\infty}\frac{n^{q+1}}{q!}\binom{q}{k}(-1)^{k+n-1}\left[1+\left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha}\right]^{\beta(k+1)-1}\times$$
$$\times\left(G(x;\underline{\xi})\right)^{\alpha-1}\left(\overline{G}(x;\underline{\xi})\right)^{-(\alpha+1)}g(x;\underline{\xi}).$$

Considering the generalized binomial expansion, we get

$$\left[1+\left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha}\right]^{\beta(k+1)-1} = \sum_{m=0}^{\infty}\binom{\beta(k+1)-1}{m}\left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha m},$$

for $\beta(k+1)>1$ and $\beta>1$ such that

$$f(x;\alpha,\beta,\underline{\xi}) = 2\alpha\beta\sum_{n=1}^{\infty}\sum_{q,k,m=0}^{\infty}\binom{\beta(k+1)-1}{m}\frac{n^{q+1}}{q!}\binom{q}{k}(-1)^{k+n-1}\times$$
$$\times\left(G(x;\underline{\xi})\right)^{\alpha(m+1)-1}\left(\overline{G}(x;\underline{\xi})\right)^{-(\alpha(m+1)+1)}g(x;\underline{\xi}).$$

Now applying the series expansion

$$[\overline{G}(x;\underline{\xi})]^{-(\alpha(m+1)+1)} = \sum_{l=0}^{\infty}\binom{-\alpha((m+1)+1)}{l}(-1)^{l}G^{l}(x;\underline{\xi}),$$

we have

$$f(x;\alpha,\beta,\underline{\xi}) = 2\alpha\beta\sum_{n=1}^{\infty}\sum_{q,k,m,l=0}^{\infty}\binom{-\alpha((m+1)+1)}{l}\binom{\beta(k+1)-1}{m}\times$$
$$\times\frac{n^{q+1}}{q!}\binom{q}{k}(-1)^{l+k+n-1}\left(G(x;\underline{\xi})\right)^{l+\alpha(m+1)-1}g(x;\underline{\xi}).$$

We can therefore write the series expansion of Equation (7) as

$$f(x;\alpha,\beta,\underline{\xi}) = \sum_{p=0}^{\infty}v_p g_p(x;\underline{\xi}), \tag{9}$$

where

$$v_p = 2\alpha\beta\sum_{n=1}^{\infty}\sum_{q,k,m=0}^{\infty}\binom{-\alpha((m+1)+1)}{l}\binom{\beta(k+1)-1}{m}\frac{n^{q+1}}{q!}\binom{q}{k}\frac{(-1)^{l+k+n-1}}{p+\alpha(m+1)} \tag{10}$$

for $\beta(k+1) > 1$, $\beta > 1$ and $g_p(x;\underline{\xi}) = pg(x;\underline{\xi})[G(x;\underline{\xi})]^{p-1}$ is an Exp-G with power parameter p. The HLOPGW-G distribution is a linear combination of Exp-G densities and the mathematical properties of the HLOPGW-G family of distributions can be readily obtained directly from the Exp-G family of distributions.

3 Some special cases

This section presents some special cases of the HLOPGW-G family of distributions. The baseline distributions considered are restricted to at most two parameter models to avoid the problem of over-parametrization.

3.1 Half logistic odd power generalized Weibull-uniform (HLOPGW-U) distribution

Let the uniform distribution be the baseline distribution with pdf and cdf given by $g(x) = 1/\lambda$ and $G(x;\lambda) = x/\lambda$, respectively, for $0 < x < \lambda$. We obtain the cdf, pdf and hrf of the HLOPGW-U distribution which are given by

$$F(x;\alpha,\beta,\lambda) = \frac{1-\exp\left\{1-\left[1+\left(\frac{x}{\lambda-x}\right)^{\alpha}\right]^{\beta}\right\}}{1+\exp\left\{1-\left[1+\left(\frac{x}{\lambda-x}\right)^{\alpha}\right]^{\beta}\right\}}, \tag{11}$$

$$\begin{aligned} f(x;\alpha,\beta,\lambda) &= \frac{2\alpha\beta}{\lambda}\left[1+\left(\frac{x}{\lambda-x}\right)^{\alpha}\right]^{\beta-1}\exp\left\{1-\left[1+\left(\frac{x}{\lambda-x}\right)^{\alpha}\right]^{\beta}\right\}\left(\frac{x}{\lambda}\right)^{\alpha-1}\times \\ &\times\left(1-\frac{x}{\lambda}\right)^{-(\alpha+1)}\left(1+\exp\left\{1-\left[1+\left(\frac{x}{\lambda-x}\right)^{\alpha}\right]^{\beta}\right\}\right)^{-2} \end{aligned} \tag{12}$$

and

$$h(x;\alpha,\beta,\lambda) = \frac{\alpha\beta\left[1+\left(\frac{x}{\lambda-x}\right)^{\alpha}\right]^{\beta-1}\left(\frac{x}{\lambda}\right)^{\alpha-1}}{\lambda\left(1-\frac{x}{\lambda}\right)^{\alpha+1}\left(1+\exp\left\{1-\left[1+\left(\frac{x}{\lambda-x}\right)^{\alpha}\right]^{\beta}\right\}\right)},$$

respectively for $\alpha,\beta,\lambda > 0$. The pdf and hrf plots for the HLOPGW-U distribution are given in Figures 1 and 2.
Figures 1 and 2 demonstrate the flexible nature of the HLOPGW-U distribution for some parameter values. The pdfs of the HLOPGW-U distribution can take various shapes that include reverse-J, uni-modal, left or right skewed shapes. Furthermore, the HLOPGW-U distribution exhibit decreasing, increasing, upside down bathtub, bathtub followed by upside down bathtub shapes for the hrf.

Figure 1: pdf plots for the HLOPGW-U distribution

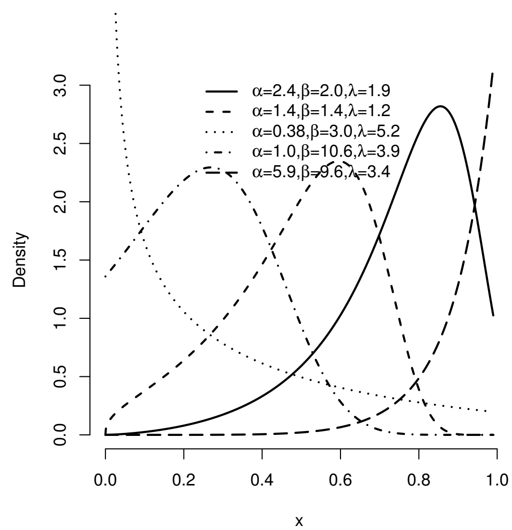
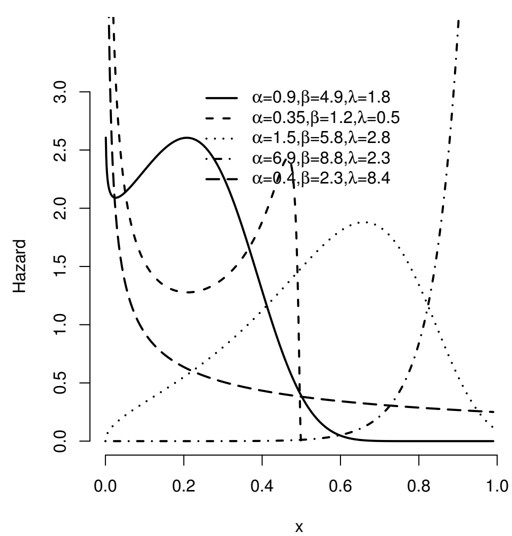


Figure 2: hrf plots for the HLOPGW-U distribution



The corresponding quantile function (qf) for the half logistic odd power generalized Weibull-Uniform (HLOPGW-U) distribution is derived by inverting the cdf given by Equation (11) as follows:

$$F(x;\alpha,\beta,\lambda)=\frac{1-\exp\left\{1-\left[1+\left(\frac{x}{\lambda-x}\right)^{\alpha}\right]^{\beta}\right\}}{1+\exp\left\{1-\left[1+\left(\frac{x}{\lambda-x}\right)^{\alpha}\right]^{\beta}\right\}}=u,$$

for $0\leq u\leq 1$, which implies

$$1-u=(1+u)\exp\left\{1-\left[1+\left(\frac{x}{\lambda-x}\right)^{\alpha}\right]^{\beta}\right\},$$

and we can write

$$\left(\frac{1-u}{1+u}\right)=\exp\left\{1-\left[1+\left(\frac{x}{\lambda-x}\right)^{\alpha}\right]^{\beta}\right\},$$

such that

$$\ln\left(\frac{1-u}{1+u}\right)=\left(1-\left[1+\left(\frac{x}{\lambda-x}\right)^{\alpha}\right]^{\beta}\right).$$

Consequently, after some few algebra, we can write the quantile function for HLOPGW-U distribution as

$$x(u)=\lambda\left\{\left[\left(\left[1-\ln\left(\frac{1-u}{1+u}\right)\right]^{\frac{1}{\beta}}-1\right)^{\frac{1}{\alpha}}\right]^{-1}+1\right\}^{-1},$$

and may be determined using iterative methods. The quantile values for the HLOPGW-U distribution are given in Table 2.

Table 2: Table of Quantiles for Selected Parameters of the HLOPGW-U Distribution

u	(0.4,0.6,1.0)	(0.5,0.6,1.0)	(0.2,0.8,1.0)	(0.5,0.5,1.0)	(0.9,0.6,1.0)
0.1	0.0704	0.1126	0.0010	0.1632	0.2411
0.2	0.3374	0.3682	0.0402	0.4875	0.4256
0.3	0.6276	0.6029	0.2780	0.7244	0.5577
0.4	0.8089	0.7603	0.6713	0.8533	0.6550
0.5	0.9029	0.8561	3.8921	0.9205	0.7292
0.6	0.9505	0.9140	0.9670	0.9565	0.7881
0.7	0.9752	0.9497	0.9902	0.9767	0.8365
0.8	0.9884	0.9724	0.9973	0.9883	0.8786
0.9	0.9957	0.9873	0.9994	0.9953	0.9185

3.2 Half logistic odd power generalized Weibull-Lomax (HLOPGW-Lx) distribution

Consider the Lomax distribution as the baseline distribution having shape parameters $a > 0, b > 0$ having cdf and pdf $G(x) = [1 + \frac{x}{b}]^{-a}$ and $g(x) = (\frac{a}{b})[1 + \frac{x}{b}]^{-a-1}$, respectively. The cdf, pdf and hrf of the HLOPGW-Lx distribution are given by

$$F(x;\alpha,\beta,a,b) = \frac{1-\exp\left\{1-\left[1+\left(\frac{\left[1+\frac{x}{b}\right]^{-a}}{1-\left[1+\frac{x}{b}\right]^{-a}}\right)^{\alpha}\right]^{\beta}\right\}}{1+\exp\left\{1-\left[1+\left(\frac{\left[1+\frac{x}{b}\right]^{-a}}{1-\left[1+\frac{x}{b}\right]^{-a}}\right)^{\alpha}\right]^{\beta}\right\}}, \tag{13}$$

$$\begin{aligned} f(x;\alpha,\beta,a,b) = {} & \frac{2\alpha\beta\left[\left(1+\frac{x}{b}\right)^{-a}\right]^{\alpha-1}\left[1+\left(\frac{\left[1+\frac{x}{b}\right]^{-a}}{1-\left[1+\frac{x}{b}\right]^{-a}}\right)^{\alpha}\right]^{\beta-1}}{\left(1+\exp\left\{1-\left[1+\left(\frac{\left[1+\frac{x}{b}\right]^{-a}}{1-\left[1+\frac{x}{b}\right]^{-a}}\right)^{\alpha}\right]^{\beta}\right\}\right)^{2}} \times \\ & \times \exp\left\{1-\left[1+\left(\frac{\left[1+\frac{x}{b}\right]^{-a}}{1-\left[1+\frac{x}{b}\right]^{-a}}\right)^{\alpha}\right]^{\beta}\right\} \times \\ & \times \left[1-\left(1+\frac{x}{b}\right)^{-a}\right]^{-(\alpha+1)}\left(\frac{a}{b}\right)\left(1+\frac{x}{b}\right)^{-a-1} \end{aligned}$$

$$\begin{aligned} h(x;\alpha,\beta,a,b) = {} & \frac{\alpha\beta\left(\frac{a}{b}\right)\left[\left(1+\frac{x}{b}\right)^{-a}\right]^{\alpha-1}\left[1+\left(\frac{\left[1+\frac{x}{b}\right]^{-a}}{1-\left[1+\frac{x}{b}\right]^{-a}}\right)^{\alpha}\right]^{\beta-1}}{\left[1-\left(1+\frac{x}{b}\right)^{-a}\right]^{\alpha+1}\left(1+\frac{x}{b}\right)^{a+1}} \times \\ & \times \left(1+\exp\left\{1-\left[1+\left(\frac{\left[1+\frac{x}{b}\right]^{-a}}{1-\left[1+\frac{x}{b}\right]^{-a}}\right)^{\alpha}\right]^{\beta}\right\}\right)^{-1}, \end{aligned}$$

respectively for $\alpha, \beta, a, b > 0$. The pdf and hrf plots for the HLOPGW-Lx distribution are given in Figures 3 and 4.
Figures 3 and 4 illustrate the flexible nature of the HLOPGW-Lx distribution for some parameter values. The pdfs of the HLOPGW-Lx distribution can take various shapes that include reverse-J, uni-modal, left or right skewed shapes. Furthermore, the HLOPGW-Lx distribution exhibit reverse-J, decreasing, increasing, bathtub and upside down bathtub shapes for the hazard rate function.

Figure 3: pdf plots for the HLOPGW-Lx distribution

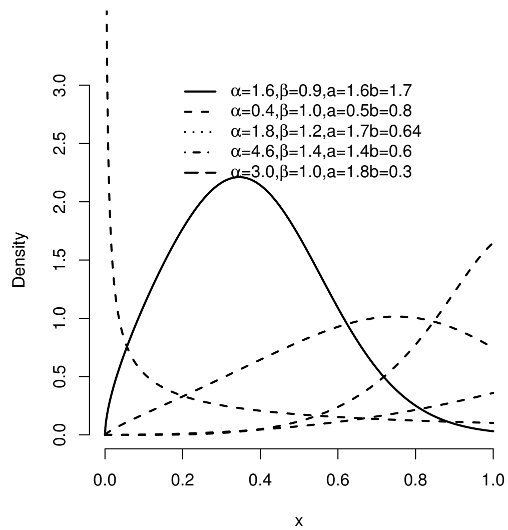
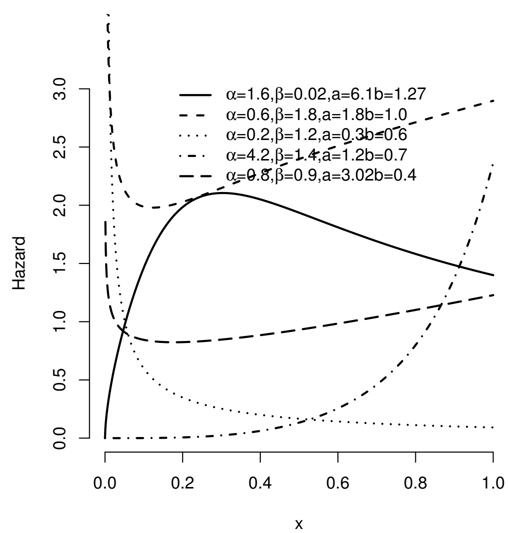


Figure 4: hrf plots for the HLOPGW-Lx distribution



Similarly, we derive the quantile function for the HLOPGW-Lx distribution by inverting Equation (13) as follows:

$$F(x;\alpha,\beta,a,b)=\frac{1-\exp\left\{1-\left[1+\left(\frac{\left[1+\frac{x}{b}\right]^{-a}}{1-\left[1+\frac{x}{b}\right]^{-a}}\right)^{\alpha}\right]^{\beta}\right\}}{1+\exp\left\{1-\left[1+\left(\frac{\left[1+\frac{x}{b}\right]^{-a}}{1-\left[1+\frac{x}{b}\right]^{-a}}\right)^{\alpha}\right]^{\beta}\right\}}=u,$$

for $0\le u\le 1$ and

$$1-u=(1+u)\exp\left\{1-\left[1+\left(\frac{\left[1+\frac{x}{b}\right]^{-a}}{1-\left[1+\frac{x}{b}\right]^{-a}}\right)^{\alpha}\right]^{\beta}\right\},$$

such that

$$\left[1+\left(\frac{\left[1+\frac{x}{b}\right]^{-a}}{1-\left[1+\frac{x}{b}\right]^{-a}}\right)^{\alpha}\right]=\left[1+\ln(1+u)-\ln(1-u)\right]^{1/\beta}.$$

Furthermore, we can write

$$\frac{\left[1+\frac{x}{b}\right]^{-a}}{1-\left[1+\frac{x}{b}\right]^{-a}}=\left(\left[1+\ln(1+u)-\ln(1-u)\right]^{\frac{1}{\beta}}-1\right)^{1/\alpha},$$

that is to say

$$\left[1+\frac{x}{b}\right]^{a}=\left(\left[1+\ln(1+u)-\ln(1-u)\right]^{\frac{1}{\beta}}-1\right)^{-1/\alpha}+1.$$

Finally, the quantile values for the HLOPGW-Lx distribution are obtained by solving the equation

$$x(u)=b\left\{\left[\left(\left[1+\ln(1+u)-\ln(1-u)\right]^{\frac{1}{\beta}}-1\right)^{-1/\alpha}+1\right]^{1/a}-1\right\},$$

using iterative methods.

3.3 Half logistic odd power generalized Weibull-Beta (HLOPGW-B) distribution

Let the Beta distribution be the baseline distribution having shape parameters $a,b>0$ having the cdf and pdf $G(x)=\mathrm{I}_x(a,b)=(1/B(a,b))\int_0^x t^{a-1}(1-t)^{b-1}dt$ and $g(x)=(1/B(a,b))x^{a-1}(1-x)^{b-1}$, for $0<x<1$, respectively. The cdf, pdf and hrf of HLOPGW-B distribution are given by

$$F(x;\alpha,\beta,a,b)=\frac{1-\exp\left\{1-\left[1+\left(\frac{\mathrm{I}_x(a,b)}{1-\mathrm{I}_x(a,b)}\right)^{\alpha}\right]^{\beta}\right\}}{1+\exp\left\{1-\left[1+\left(\frac{\mathrm{I}_x(a,b)}{1-\mathrm{I}_x(a,b)}\right)^{\alpha}\right]^{\beta}\right\}}, \tag{14}$$

$$f(x;\alpha,\beta,a,b)=\frac{2\alpha\beta\left[\mathrm{I}_x(a,b)\right]^{\alpha-1}\left[1+\left(\frac{\mathrm{I}_x(a,b)}{1-\mathrm{I}_x(a,b)}\right)^{\alpha}\right]^{\beta-1}x^{a-1}(1-x)^{b-1}}{B(a,b)\left(1+\exp\left\{1-\left[1+\left(\frac{\mathrm{I}_x(a,b)}{1-\mathrm{I}_x(a,b)}\right)^{\alpha}\right]^{\beta}\right\}\right)^{2}}\times$$
$$\times\exp\left\{1-\left[1+\left(\frac{\mathrm{I}_x(a,b)}{1-\mathrm{I}_x(a,b)}\right)^{\alpha}\right]^{\beta}\right\}(1-\mathrm{I}_x(a,b))^{-(\alpha+1)}$$

and

$$h(x;\alpha,\beta,a,b)=\frac{\alpha\beta\left[\mathrm{I}_x(a,b)\right]^{\alpha-1}\left[1+\left(\frac{\mathrm{I}_x(a,b)}{1-\mathrm{I}_x(a,b)}\right)^{\alpha}\right]^{\beta-1}x^{a-1}(1-x)^{b-1}}{B(a,b)\left[1-\mathrm{I}_x(a,b)\right]^{\alpha+1}\left(1+\exp\left\{1-\left[1+\left(\frac{\mathrm{I}_x(a,b)}{1-\mathrm{I}_x(a,b)}\right)^{\alpha}\right]^{\beta}\right\}\right)},$$

respectively for $\alpha,\beta,a,b>0$. The pdf and hrf plots for the HLOPGW-B distribution are given in Figures 5 and 6.
Figures 5 and 6 present good illustration on the flexibility of the HLOPGW-B distribution for selected parameter values. The pdfs of the HLOPGW-B distribution exhibits various shapes that include reverse-J, uni-modal, left or right skewed shapes. The shapes of HLOPGW-B hrf are decreasing, reverse-J, bathtub, bathtub followed by upside down bathtub shape.
The quantile function for the HLOPGW-B distribution is derived by inverting Equation (14), such that

$$F(x;\alpha,\beta,a,b)=u,\quad 0\le u\le 1,$$

and therefore we can write

$$1-u=(1+u)\exp\left\{1-\left[1+\left(\frac{\mathrm{I}_x(a,b)}{1-\mathrm{I}_x(a,b)}\right)^{\alpha}\right]^{\beta}\right\},$$

Figure 5: pdf plots for the HLOPGW-B distribution

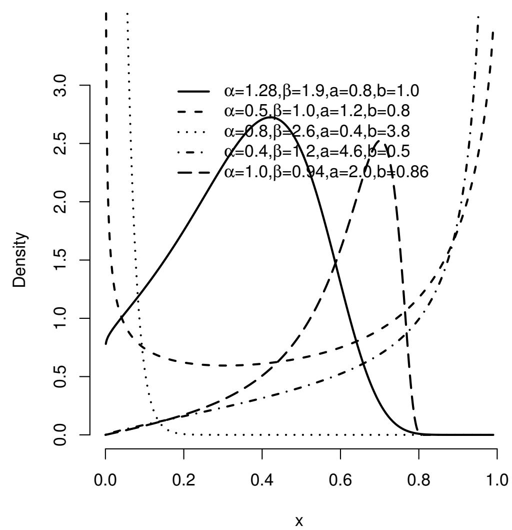
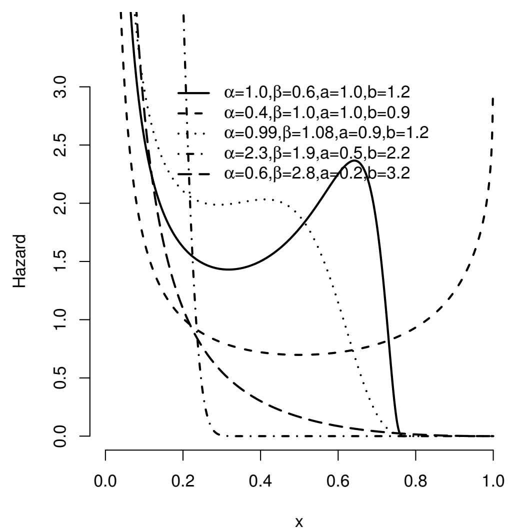


Figure 6: hrf plots for the HLOPGW-B distribution



such that

$$\frac{\mathrm{I}_x(a,b)}{1-\mathrm{I}_x(a,b)} = \left([1+\ln(1+u)-\ln(1-u)]^{\frac{1}{\beta}}-1\right)^{\frac{1}{\alpha}},$$

which simplifies to

$$\mathrm{I}_x(a,b) = \left\{\left[(1+\ln(1+u)-\ln(1-u))^{\frac{1}{\beta}}-1\right]^{-1/\alpha}+1\right\}^{-1}.$$

Finally, we can write the quantile function for the HLOPGW-B distribution as

$$x(u) = Q^{-1}\left\{a, b\left[\left([1+\ln(1+u)-\ln(1-u)]^{\frac{1}{\beta}}-1\right)^{-1/\alpha}+1\right]^{-1}\right\},$$

which can be solved using iterative methods.

3.4 Identifiability of some special cases

In this subsection, we prove the identifiability property for all the special cases considered in this study. Identifiability is a statistical property that a model must satisfy for precise inference to be possible. To prove the identifiability property for special cases studied in this work, we will show that the transformation which maps the prediction function is one-to-one for all sets of parameters. A model that fails to be identifiable is said to be non-identifiable if two or more parameters are observationally equivalent.

i) The HLOPGW-Uniform (HLOPGW-U) case

Let the random variable $X \sim HLOPGW-U(\underline{\xi})$, where $\underline{\xi} = (\alpha, \beta, \lambda)^{\mathbf{T}}$ is a vector of model parameters. Consider $\underline{\xi}_i = (\alpha_i, \beta_i, \lambda_i)^T$ and $\underline{\xi}_j = (\alpha_j, \beta_j, \lambda_j)^T$ as the two vectors of model parameters having cdfs given by $G(x;\alpha_i,\beta_i,\lambda_i)$ and $G(x;\alpha_j,\beta_j,\lambda_j)$, respectively, for all $\underline{\xi}_i = \underline{\xi}_j$, with $i=j$ and $i,j \geq 1$, then by defination of identifiability, we have

$$G(x;\underline{\xi}_i) = G(x;\underline{\xi}_j).$$

Note that by the above definition and using Equation (11), we can write

$$\frac{1-\exp\left\{1-\left[1+\left(\frac{x}{\lambda_i-x}\right)^{\alpha_i}\right]^{\beta_i}\right\}}{1+\exp\left\{1-\left[1+\left(\frac{x}{\lambda_i-x}\right)^{\alpha_i}\right]^{\beta_i}\right\}} = \frac{1-\exp\left\{1-\left[1+\left(\frac{x}{\lambda_j-x}\right)^{\alpha_j}\right]^{\beta_j}\right\}}{1+\exp\left\{1-\left[1+\left(\frac{x}{\lambda_j-x}\right)^{\alpha_j}\right]^{\beta_j}\right\}},$$

that is to say

$$F(x;\underline{\xi}_i) = F(x;\underline{\xi}_j),$$

for $\underline{\xi}_i = \underline{\xi}_j$, and therefore the HLOPGW-U distribution is identifiable.

ii) The HLOPGW-Lomax (HLOPGW-Lx) case

Applying the same approach used under the HLOPGW-Uniform case, we can show that if $X \sim HLOPGW - Lx(\alpha, \beta, a, b)$ with $\underline{\psi}_1 = (\alpha_1, \beta_1, a_1, b_1)^T$ and $\underline{\psi}_2 = (\alpha_2, \beta_2, a_2, b_2)^T$ as vectors of model parameters having cdfs given by $G(x; \alpha_1, \beta_1, a_1, b_1)$ and $G(x; \alpha_2, \beta_2, a_2, b_2)$, respectively. Let $\underline{\psi}_1 = \underline{\psi}_2$, then by Equation (13), we can write

$$F(x;\underline{\psi}_1) = F(x;\underline{\psi}_2) = \frac{1-\exp\left\{1-\left[1+\left(\frac{\left[1+\frac{x}{b_1}\right]^{-a_1}}{1-\left[1+\frac{x}{b_1}\right]^{-a_1}}\right)^{\alpha_1}\right]^{\beta_1}\right\}}{1+\exp\left\{1-\left[1+\left(\frac{\left[1+\frac{x}{b_1}\right]^{-a_1}}{1-\left[1+\frac{x}{b_1}\right]^{-a_1}}\right)^{\alpha_1}\right]^{\beta_1}\right\}} = \frac{1-\exp\left\{1-\left[1+\left(\frac{\left[1+\frac{x}{b_2}\right]^{-a_2}}{1-\left[1+\frac{x}{b_2}\right]^{-a_2}}\right)^{\alpha_2}\right]^{\beta_2}\right\}}{1+\exp\left\{1-\left[1+\left(\frac{\left[1+\frac{x}{b_2}\right]^{-a_2}}{1-\left[1+\frac{x}{b_2}\right]^{-a_2}}\right)^{\alpha_2}\right]^{\beta_2}\right\}},$$

and finally we can conclude that the HLOPGW-Lx distribution is identifiable.

iii) The HLOPGW-Beta (HLOPGW-B) case

Let $\mathbf{\Omega}_1 = (\alpha_1, \beta_1, a_1, b_1)^T$ and $\mathbf{\Omega}_2 = (\alpha_2, \beta_2, a_2, b_2)^T$ be the two vector of parameters from the HLOPGW-B distribution with cdfs given by $G(x; \alpha_1, \beta_1, a_1, b_1)$ and $G(x; \alpha_2, \beta_2, a_2, b_2)$, respectively, then by definition of identifiability, we have

$$G(x; \alpha_1, \beta_1, a_1, b_1) = G(x; \alpha_2, \beta_2, a_2, b_2),$$

such that using Equation (14), we have

$$\frac{1-\exp\left\{1-\left[1+\left(\frac{\mathrm{I}_x(a_1,b_1)}{1-\mathrm{I}_x(a_1,b_1)}\right)^{\alpha_1}\right]^{\beta_1}\right\}}{1+\exp\left\{1-\left[1+\left(\frac{\mathrm{I}_x(a_1,b_1)}{1-\mathrm{I}_x(a_1,b_1)}\right)^{\alpha_1}\right]^{\beta_1}\right\}} = \frac{1-\exp\left\{1-\left[1+\left(\frac{\mathrm{I}_x(a_2,b_2)}{1-\mathrm{I}_x(a_2,b_2)}\right)^{\alpha_2}\right]^{\beta_2}\right\}}{1+\exp\left\{1-\left[1+\left(\frac{\mathrm{I}_x(a_2,b_2)}{1-\mathrm{I}_x(a_2,b_2)}\right)^{\alpha_2}\right]^{\beta_2}\right\}}.$$

Note since we have $\mathbf{\Omega}_1 = \mathbf{\Omega}_2$, then the HLOPGW-B distribution is identifiable.

4 Mathematical and statistical properties

In this section, some statistical properties of the HLOPGW-G family of distributions, namely: order statistics, entropy, moments, incomplete moments, probability weighted moments (PWMs) and generating functions are derived.

4.1 Distribution of order statistics

The use of order statistics is critical in the study of probability and statistical inference. It has wider applications in various studies, namely: hypothesis testing, theory of estimation, reliability, and statistical quality control. The pdf of the i^{th} order statistics can be obtained using Equation (15):

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-j} \binom{n-i}{j} F(x)^{j+i-1}, \tag{15}$$

where $B(\cdot, \cdot)$ is the beta function. Substituting Equations (6) and (7) into Equation (15), and applying the generalized binomial series expansion used under the density expansion, we can write the pdf of the i^{th} order statistic from the HLOPGW-G family of distributions as

$$\begin{aligned} f_{i:n}(x) = & \frac{2\alpha\beta}{B(i,n-i+1)} \binom{n-i}{j} \sum_{m,n,p,k,q,w=0}^{\infty} \binom{-(\alpha(q+1)+1)}{w} (-1)^w \times \\ & \times \binom{\beta(k+1)-1}{q} \frac{(m+n+1)^p(-1)^{k+n}}{p!} \binom{p}{k} \binom{j+i-1}{n} \binom{-(j+i+1)}{m} \times \\ & \times \left(G(x;\underline{\xi})\right)^{w+\alpha(q+1)-1} g(x;\underline{\xi}) = \\ = & \sum_{w,q=0}^{\infty} v^*_{w,q} g_{w,q}(x;\underline{\xi}), \end{aligned} \tag{16}$$

after some algebra, where $g_{w,q}(x;\underline{\xi}) = (w+\alpha(q+1))g(x;\underline{\xi})[G(x;\underline{\xi})]^{w+\alpha(q+1)-1}$ is an Exp-G distribution with power parameter $w+\alpha(q+1)$ and

$$\begin{aligned} v^*_{w,q} = & \frac{2\alpha\beta(-1)^w}{B(i,n-i+1)} \binom{n-i}{j} \sum_{m,n,p,k=0}^{\infty} \binom{-(\alpha(q+1)+1)}{w} \binom{p}{k} \times \\ & \times \binom{\beta(k+1)-1}{q} \frac{(m+n+1)^p(-1)^{k+n}}{p!(w+\alpha(q+1))} \binom{j+i-1}{n} \binom{-(j+i+1)}{m}. \end{aligned}$$

Finally, the pdf of the i^{th} order statistics from the HLOPGW-G family of distributions can be expressed as a linear combination of Exp-G densities.

4.2 Entropy

An entropy is a measure of variation for uncertainty on a random variable X with the probability distribution $f(x)$. There are two common types of entropy, namely, Rényi entropy (Rényi 1960) and Shannon entropy (Shannon 1951). Shannon entropy is a special case of Rényi entropy. In this paper, we derive the Rényi entropy ($I_R(\nu)$) of the HLOPGW-G family of distributions as follows

$$I_R(\nu) = (1-\nu)^{-1} \log\left[\int_0^\infty f^\nu(x)dx\right], v \neq 1, v > 0. \tag{17}$$

Using Equation (7), $f^\nu(x)$ can be written as

$$f^\nu(x) = (2\alpha\beta)^\nu \left[1 + \left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^\alpha\right]^{\nu(\beta-1)} \exp\left\{\nu\left(1 - \left[1 + \left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^\alpha\right]^\beta\right)\right\} \times$$
$$\times \left(\overline{G}(x;\underline{\xi})\right)^{-\nu(\alpha+1)} \left(1 + \exp\left\{1 - \left[1 + \left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^\alpha\right]^\beta\right\}\right)^{-2\nu} \times$$
$$\times \left(G(x;\underline{\xi})\right)^{\nu(\alpha-1)} g(x;\underline{\xi})^\nu.$$

Applying the generalized binomial expansion, used in Section 2.2, the Rényi entropy for the HLOPGW-G family of distributions can be written as

$$I_R(\nu) = (1-\nu)^{-1} \log\left[\sum_{m=0}^\infty w_m \mathrm{e}^{(1-\nu)I_{REG}}\right], v \neq 1, v > 0, \tag{18}$$

after some mathematical simplifications, where

$$w_m = \sum_{j,i,k,p,s=0}^\infty \binom{-\alpha((\nu+z)+\nu)}{m} (-1)^m \frac{(\nu+j)^i}{i!} \binom{i}{k} \binom{-2\nu}{j} (2\alpha\beta)^\nu \times$$
$$\times \binom{\nu(\alpha-1)+\alpha z+m}{p} \binom{p}{s} (-1)^{s+p+k} \left(\frac{s}{\nu}+1\right)^{-\nu}, \tag{19}$$

and $I_{REG} = (1-\nu)^{-1} \log \int_0^\infty \left[\left(s/\nu+1\right) g(x;\underline{\xi})[G(x;\underline{\xi})]^{\frac{s}{\nu}}\right]^\nu dx$ is Rényi entropy of Exp-G distribution with parameter $s/\nu+1$. Lastly, the Rényi entropy of the HLOPGW-G family of distributions can be obtained directly from Rényi entropy of Exp-G distribution.

4.3 Moments

The r^{th} ordinary moment can be derived using Equation (9) as follows:

$$\mu'_r = E(X^r) = \sum_{p=0}^{\infty} v_p E(Y_p^r), \tag{20}$$

where Y_p follows an Exp-G distribution with power parameter p and v_p is given in equation (10). The s^{th} central moment of X is given by

$$\mu_s = \sum_{r=0}^{s} \binom{s}{r} (-\mu'_1)^{s-r} E(X^r) = \sum_{r=0}^{s} \sum_{p=0}^{\infty} v_p \binom{s}{r} (-\mu'_1)^{s-r} E(Y_p^r).$$

The cumulants of X follow recursively from

$$k_s = \mu'_s - \sum_{r=0}^{s-1} \binom{s-1}{r-1} k_r \mu'_{s-r},$$

where $k_1 = \mu'_1$, $k_2 = \mu'_2 - \mu'^2_1$, $k_3 = \mu'_3 - 3\mu'_2\mu'_1 + \mu'^3_1$, etc. Ordinary moments may also be used to calculate the measures of dispersion, namely, variance, skewness and kurtosis.
The r^{th} incomplete moment of X is given by

$$\phi_r(z) = \int_{-\infty}^{z} x^r f(x)dx = \sum_{p=0}^{\infty} v_p \int_{-\infty}^{z} x^r g_p(x; \underline{\xi})dx. \tag{21}$$

The incomplete moment is very useful and can be used to estimate some important quantities such as Lorenz and Bonferroni curves. These quantities have a wide application in demography, economics, insurance, medicine and reliability. Mathematically, the Lorenz and Bonferroni curves for a given probability p are given by $L(p) = \phi_1(q)/\mu'_1$ and $B(p) = \phi_1(q)/(p\mu'_1)$, respectively, where μ'_1 is given by Equation (20), with $r = 1$ and $q = Q(p)$ is the quantile function of X at p. The incomplete moment (Equation (21)) can be expressed as

$$\phi_r(z) = \sum_{p=0}^{\infty} v_p H_p(z), \tag{22}$$

where $H_p(z) = \int_{-\infty}^{z} x^r g_p(x; \underline{\xi})dx$ is the r^{th} incomplete moment of the Exp-G distribution.
We present the first five moments with the standard deviation (SD or σ), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) for the HLOPGW-U distribution for some parameter values (see Table 3 for details).

Table 3: Moments of the HLOPGW-U distribution for some parameter values

	(0.5,0.7,1.2)	(0.6,0.5,1.1)	(0.2,0.9,1.1)	(0.4,0.8,1.4)	(0.8,0.7,1.5)
$E(X)$	0.2897	0.3378	0.1941	0.1932	0.2698
$E(X^2)$	0.2125	0.2651	0.1401	0.1314	0.1963
$E(X^3)$	0.1715	0.2233	0.1137	0.1015	0.1560
$E(X^4)$	0.1451	0.1948	0.0971	0.0834	0.1301
$E(X^5)$	0.1263	0.1738	0.0853	0.0710	0.1118
SD	0.3585	0.3886	0.3200	0.3067	0.3514
CV	1.2372	1.1501	1.6486	1.5873	1.3021
CS	0.7689	0.5396	1.4263	1.3791	0.8385
CK	1.9504	1.5610	3.4527	3.4072	2.0670

4.4 Probability Weighted Moments

Probability Weighted Moments (PWMs) are very useful in estimating parameters of distributions which are not in closed form. The (j, i)th PWM, say $\eta_{j,i}$ of $X \sim$ HLOPGW-G $(\alpha, \beta; \underline{\xi})$ distribution is defined by

$$\eta_{j,i} = E(X^j F(X)^i) = \int_{-\infty}^{\infty} x^j f(x) F(x)^i dx.$$

Using Equation (16), we can write

$$\begin{aligned} f(x)F(x)^i = 2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p,k,q,w=0}^{\infty} & \binom{-(\alpha(q+1)+1)}{w} (-1)^w \binom{\beta(k+1)-1}{q} \times \\ & \times \frac{(m+n+1)^p (-1)^{k+n}}{p!} \binom{p}{k} \binom{i-1}{n} \binom{-(i+1)}{m} \alpha\beta \times \\ & \times \left(G(x;\underline{\xi})\right)^{w+\alpha(q+1)-1} g(x;\underline{\xi}), \end{aligned}$$

which simplifies to

$$f(x)F(x)^i = \sum_{w,q=0}^{\infty} h^*_{w,q} g_{w,q}(x;\underline{\xi}),$$

where

$$\begin{aligned} h^*_{w,q} = 2\alpha\beta \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p,k=0}^{\infty} & \binom{-(\alpha(q+1)+1)}{w} (-1)^w \binom{\beta(k+1)-1}{q} \times \\ & \times \frac{(m+n+1)^p (-1)^{k+n}}{p!} \binom{p}{k} \binom{i}{n} \binom{-(i+2)}{m} \times \\ & \times \left(G(x;\underline{\xi})\right)^{w+\alpha(q+1)-1} g(x;\underline{\xi}) \end{aligned}$$

and $g_{w,q}(x;\underline{\xi}) = (w+\alpha(q+1))g(x;\underline{\xi})[G(x;\underline{\xi})]^{w+\alpha(q+1)-1}$ is an Exp-G distribution with power parameter $w+\alpha(q+1)$. The PWM is therefore given by

$$\eta_{j,i} = \sum_{w,q=0}^{\infty} h^{*}_{w,q} \int_{-\infty}^{\infty} x^{j} g_{w,q}(x;\underline{\xi})dx = \sum_{w,q=0}^{\infty} h^{*}_{w,q} E(T^{j}_{w,q}),$$

where $T^{j}_{w,q}$ is j^{th} power of an Exp-G distributed random variable having power parameter $w+\alpha(q+1)$.

4.5 Moment generating functions

The moment generating function (mgf) of HLOPGW-G family of distributions is given by

$$M_{\mathrm{x}}(t) = E(\mathrm{e}^{tX}) = \sum_{p=0}^{\infty} v_p M_p(t),$$

where $M_p(t)$ is the mgf of Exp-G with power parameter p. The mgf of HLOPGW-G family of distributions can be derived directly from that of the Exp-G distribution.

5 Maximum likelihood estimation

Let $X_i \sim HLOPGW-G(\alpha,\beta;\underline{\xi})$, for $i = 1,...n$, with parameter vector $\mathbf{\Delta} = (\alpha,\beta;\underline{\xi})^T$. The log-likelihood for $\ell = \ell(\mathbf{\Delta})$ from a random sample of size n is given by

$$\begin{aligned}
\ell = {} & n\ln(2\alpha\beta) + (\beta-1)\sum_{i=1}^{n}\ln\left[1+\left(\frac{G(x_i;\underline{\xi})}{\overline{G}(x_i;\underline{\xi})}\right)^{\alpha}\right] + \\
& + \sum_{i=1}^{n}\ln\left(1-\left[1+\left(\frac{G(x_i;\underline{\xi})}{\overline{G}(x_i;\underline{\xi})}\right)^{\alpha}\right]^{\beta}\right) + \\
& + (\alpha-1)\sum_{i=1}^{n}\ln\left(G(x_i;\underline{\xi})\right) - (\alpha+1))\sum_{i=1}^{n}\ln\left(\overline{G}(x_i;\underline{\xi})\right) + \sum_{i=1}^{n}\ln(g(x_i;\underline{\xi})) + \\
& - 2\sum_{i=1}^{n}\ln\left(1+\exp\left\{1-\left[1+\left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha}\right]^{\beta}\right\}\right).
\end{aligned}$$

The score vector $U = (\frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial \underline{\xi}_k})$ has elements given by:

$$\begin{aligned}
\frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha} + (\beta-1)\sum_{i=1}^{n} \frac{\left(\frac{G(x_i;\underline{\xi})}{\overline{G}(x_i;\underline{\xi})}\right)^{\alpha} \ln\left(\frac{G(x_i;\underline{\xi})}{\overline{G}(x_i;\underline{\xi})}\right)}{\left[1+\left(\frac{G(x_i;\underline{\xi})}{\overline{G}(x_i;\underline{\xi})}\right)^{\alpha}\right]} + \\
&\quad - \sum_{i=1}^{n} \frac{\beta\left(\frac{G(x_i;\underline{\xi})}{\overline{G}(x_i;\underline{\xi})}\right)^{\alpha} \ln\left(\frac{G(x_i;\underline{\xi})}{\overline{G}(x_i;\underline{\xi})}\right)\left[1+\left(\frac{G(x_i;\underline{\xi})}{\overline{G}(x_i;\underline{\xi})}\right)^{\alpha}\right]^{\beta-1}}{\left(1-\left[1+\left(\frac{G(x_i;\underline{\xi})}{\overline{G}(x_i;\underline{\xi})}\right)^{\alpha}\right]^{\beta}\right)} + \\
&\quad + \sum_{i=1}^{n} \ln\left(G(x_i;\underline{\xi})\right) - \sum_{i=1}^{n} \ln\left(\overline{G}(x_i;\underline{\xi})\right) + \\
&\quad + 2\sum_{i=1}^{n}\left[\frac{\beta\left(\frac{G(x_i;\underline{\xi})}{\overline{G}(x_i;\underline{\xi})}\right)^{\alpha} \ln\left(\frac{G(x_i;\underline{\xi})}{\overline{G}(x_i;\underline{\xi})}\right)\left[1+\left(\frac{G(x_i;\underline{\xi})}{\overline{G}(x_i;\underline{\xi})}\right)^{\alpha}\right]^{\beta-1}}{\left(1+\exp\left\{1-\left[1+\left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha}\right]^{\beta}\right\}\right)} \times \right. \\
&\quad \left. \times \exp\left\{1-\left[1+\left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha}\right]^{\beta}\right\}\right],
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \ell}{\partial \beta} &= \frac{n}{\beta} + \sum_{i=1}^{n} \ln\left[1+\left(\frac{G(x_i;\underline{\xi})}{\overline{G}(x_i;\underline{\xi})}\right)^{\alpha}\right] + \\
&\quad - \sum_{i=1}^{n} \frac{\left[1+\left(\frac{G(x_i;\underline{\xi})}{\overline{G}(x_i;\underline{\xi})}\right)^{\alpha}\right]^{\beta} \ln\left[1+\left(\frac{G(x_i;\underline{\xi})}{\overline{G}(x_i;\underline{\xi})}\right)^{\alpha}\right]}{\left(1-\left[1+\left(\frac{G(x_i;\underline{\xi})}{\overline{G}(x_i;\underline{\xi})}\right)^{\alpha}\right]^{\beta}\right)} + \\
&\quad + 2\sum_{i=1}^{n} \frac{\left[1+\left(\frac{G(x_i;\underline{\xi})}{\overline{G}(x_i;\underline{\xi})}\right)^{\alpha}\right]^{\beta} \ln\left[1+\left(\frac{G(x_i;\underline{\xi})}{\overline{G}(x_i;\underline{\xi})}\right)^{\alpha}\right] \exp\left\{1-\left[1+\left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha}\right]^{\beta}\right\}}{\left(1+\exp\left\{1-\left[1+\left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha}\right]^{\beta}\right\}\right)}
\end{aligned}$$

and

$$\frac{\partial\ell}{\partial\underline{\xi}_k} = (\beta-1)\sum_{i=1}^{n}\frac{\alpha\left(\frac{G(x_i;\underline{\xi})}{\overline{G}(x_i;\underline{\xi})}\right)^{\alpha-1}\left[\frac{\partial G(x_i;\underline{\xi})}{\partial\underline{\xi}_k}\overline{G}(x_i;\underline{\xi}) - \frac{\partial\overline{G}(x_i;\underline{\xi})}{\partial\underline{\xi}_k}G(x_i;\underline{\xi})\right]}{\left[1+\left(\frac{G(x_i;\underline{\xi})}{\overline{G}(x_i;\underline{\xi})}\right)^{\alpha}\right]\overline{G}^2(x_i;\underline{\xi})} +$$
$$+\sum_{i=1}^{n}\frac{1}{g(x_i;\underline{\xi})}\frac{\partial g(x_i;\underline{\xi})}{\partial\underline{\xi}_k} +$$
$$-\sum_{i=1}^{n}\left[\frac{\alpha\beta\left[1+\left(\frac{G(x_i;\underline{\xi})}{\overline{G}(x_i;\underline{\xi})}\right)^{\alpha}\right]^{\beta-1}\left(\frac{G(x_i;\underline{\xi})}{\overline{G}(x_i;\underline{\xi})}\right)^{\alpha-1}}{\left(1-\left[1+\left(\frac{G(x_i;\underline{\xi})}{\overline{G}(x_i;\underline{\xi})}\right)^{\alpha}\right]^{\beta}\right)\overline{G}^2(x_i;\underline{\xi})}\right. \times$$
$$\left.\times\left[\frac{\partial G(x_i;\underline{\xi})}{\partial\underline{\xi}_k}\overline{G}(x_i;\underline{\xi}) - \frac{\partial\overline{G}(x_i;\underline{\xi})}{\partial\underline{\xi}_k}G(x_i;\underline{\xi})\right]\right] +$$
$$+(\alpha-1)\sum_{i=1}^{n}\frac{\frac{\partial G(x_i;\underline{\xi})}{\partial\underline{\xi}_k}}{(G(x_i;\underline{\xi}))} - (\alpha+1)\sum_{i=1}^{n}\frac{\frac{\partial\overline{G}(x_i;\underline{\xi})}{\partial\underline{\xi}_k}}{(\overline{G}(x_i;\underline{\xi}))} +$$
$$+2\sum_{i=1}^{n}\left[\frac{\alpha\beta\left[1+\left(\frac{G(x_i;\underline{\xi})}{\overline{G}(x_i;\underline{\xi})}\right)^{\alpha}\right]^{\beta-1}\left(\frac{G(x_i;\underline{\xi})}{\overline{G}(x_i;\underline{\xi})}\right)^{\alpha-1}}{\left(1+\exp\left\{1-\left[1+\left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha}\right]^{\beta}\right\}\right)\overline{G}^2(x_i;\underline{\xi})}\right. \times$$
$$\times\left[\frac{\partial G(x_i;\underline{\xi})}{\partial\underline{\xi}_k}\overline{G}(x_i;\underline{\xi}) - \frac{\partial\overline{G}(x_i;\underline{\xi})}{\partial\underline{\xi}_k}G(x_i;\underline{\xi})\right]\times$$
$$\left.\times\exp\left\{1-\left[1+\left(\frac{G(x;\underline{\xi})}{\overline{G}(x;\underline{\xi})}\right)^{\alpha}\right]^{\beta}\right\}\right],$$

respectively. These functions are not in closed form and can only be solved using iterative methods from applicable softwares. The maximum likelihood estimates of the parameters, denoted by $\hat{\mathbf{\Delta}}$ is obtained by solving the nonlinear equation $(\partial\ell/\partial\alpha, \partial\ell/\partial\beta, \partial\ell/\partial\underline{\xi})^T = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure. To obtain confidence intervals for model parameters $(\alpha,\beta,\underline{\xi})$, and test hypothesis concerning these parameters, the observed information matrix is required and is given by

$$J(\mathbf{\Delta}) = \begin{bmatrix} J_{\alpha\alpha}(\mathbf{\Delta}) & J_{\alpha\beta}(\mathbf{\Delta}) & J_{\alpha\underline{\xi}}(\mathbf{\Delta}) \\ J_{\beta\alpha}(\mathbf{\Delta}) & J_{\beta\beta}(\mathbf{\Delta}) & J_{\beta\underline{\xi}}(\mathbf{\Delta}) \\ J_{\underline{\xi}\alpha}(\mathbf{\Delta}) & J_{\underline{\xi}\beta}(\mathbf{\Delta}) & J_{\underline{\xi}\underline{\xi}}(\mathbf{\Delta}) \end{bmatrix},$$

where $J_{rs} = -\partial^2 \ell(\mathbf{\Delta})/\partial r \partial s$, for $r, s = \alpha, \beta, \underline{\xi}$. Under the usual regularity conditions, $\hat{\mathbf{\Delta}}$ is asymptotically normal distributed, (see Ferguson 1958), that is $\hat{\mathbf{\Delta}} \sim N_3(\mathbf{0}, I^{-1}(\mathbf{\Delta}))$ as $n \to \infty$, where $I(\mathbf{\Delta})$ is the expected information matrix. The asymptotic behavior remains valid if $I(\mathbf{\Delta})$ is replaced by $J(\hat{\mathbf{\Delta}})$, the observed information matrix evaluated at $\hat{\mathbf{\Delta}}$. It must be noted that the asymptotic distribution is not always trivariate normal for all the HLOPGW-G families as this depends on the dimension of the baseline parameter vector, $\underline{\xi}$.

6 Monte-Carlo simulations

In this section, Monte-Carlo simulation study is conducted using the R package to evaluate and examine consistency of the maximum likelihood estimators. We replicate N=1000 times for sample size n=35, 50, 100, 200, 400, 800 and 1000 from the HLOPGW-U distribution. The simulation results are given in Table 4. From these results, we deduce that as the sample size increases, the mean estimates of the parameters tends to be closer to the true parameter values, since RMSEs and average bias converges towards zero in all instances.

7 Applications

The HLOPGW-U model is applied to two real data examples to show the applicability of the proposed distribution when compared to other known non-nested distributions. The best performing model is examined using the goodness-of-fit statistics, namely, -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Cramer von Mises (W^*) and Andersen-Darling (A^*) as given under Chen and Balakrishnan (1995). The model that gives smaller values of these statistics is deemed to have the best fit. The R software is used to estimate model parameters through the nlm function. The results for these Model parameter estimates together with their standard errors (in parenthesis) and the goodness-of-fit-statistics are given in Tables 5 and 6. The fitted densities and observed probability plots (see Chambers et al. 1983) that demonstrate how best these model fits the observed data sets are shown in Figures 7 and 8.
The HLOPGW-U distribution was compared to other competing three parameter non-nested models, namely the Marshall-Olkin Log-logistc (MO-LLoG) distribution by Wenhao (2013), exponentiated-Fréchet (EFr) distribution by Nadarajah and Kotz (2003), Marshall-Olkin extended inverse Weibull (IWMO) by Pakungwati et al. (2018), Marshall-Olkin extended Fréchet (MOEFr) by Barreto-Souza et al. (2013), exponentiated Weibull (EW) by Pal et al. (2006) and Exponentiated Half-Logistic Exponential (EHLE) by (2018). The pdfs of these models are as follows:

Table 4: Simulation Results for HLOPGW-U distribution; Mean, Average Bias and RMSE

Parameter	n	Set I: α=1.5, β=2.3, λ=0.4			Set II: α=2.4, β=2.2, λ=1.0			Set III: α=1.0, β=0.6 , λ=0.2		
		Mean	Bias	RMSE	Mean	Bias	RMSE	Mean	Bias	RMSE
α	35	1.87951	0.37951	0.02292	3.25398	0.85398	0.13859	1.25608	0.25608	0.06165
	50	1.81423	0.31423	0.01466	2.88445	0.48445	0.09954	1.20708	0.20708	0.05434
	100	1.78820	0.28820	0.01322	2.82462	0.42462	0.08474	1.16660	0.16660	0.03770
	200	1.70913	0.20913	0.01107	2.70839	0.30839	0.07360	1.14674	0.14674	0.03230
	400	1.65626	0.15626	0.00993	2.58788	0.18788	0.02719	1.13883	0.13883	0.02596
	800	1.55146	0.05146	0.00657	2.50097	0.10097	0.01936	1.07845	0.07845	0.02160
	1000	1.52254	0.02254	0.00585	2.46558	0.06558	0.01267	1.03982	0.03982	0.01960
β	35	3.13086	0.83086	0.02740	3.25206	1.05206	0.04275	0.96949	0.36949	0.05887
	50	2.80014	0.50014	0.02550	3.03913	0.83913	0.03694	0.95911	0.35911	0.05629
	100	2.67201	0.37201	0.02343	2.88386	0.68386	0.03524	0.92805	0.32805	0.05271
	200	2.60188	0.30188	0.02150	2.62187	0.42187	0.03294	0.89822	0.29822	0.04222
	400	2.57350	0.27350	0.02080	2.43844	0.23844	0.03089	0.81605	0.21605	0.03924
	800	2.45784	0.15784	0.01751	2.36367	0.16367	0.02874	0.64635	0.04635	0.03639
	1000	2.32463	0.02463	0.01602	2.29299	0.09299	0.02519	0.64455	0.04455	0.01638
λ	35	0.65602	0.25602	0.02391	1.65237	0.65237	0.00428	0.42520	0.22520	0.01435
	50	0.57980	0.17980	0.02189	1.38159	0.38159	0.00404	0.40158	0.20158	0.01040
	100	0.56430	0.16430	0.01866	1.33753	0.33753	0.00384	0.37555	0.17555	0.00956
	200	0.52100	0.12100	0.01431	1.12633	0.12633	0.00347	0.35466	0.15466	0.00795
	400	0.50120	0.10120	0.01320	1.08837	0.08837	0.00319	0.34575	0.14575	0.00715
	800	0.47581	0.07581	0.00893	1.07426	0.07426	0.00295	0.31678	0.11678	0.00632
	1000	0.46915	0.06915	0.00716	1.07045	0.07045	0.00254	0.29994	0.09994	0.00326

$$f_{IWMO}(x;\alpha,\theta\lambda) = \frac{\alpha\lambda\theta^{-\lambda}x^{-\lambda-1}\mathrm{e}^{-(\theta x)^{-\lambda}}}{[\alpha-(\alpha-1)\mathrm{e}^{-(\theta x)^{-\lambda}}]^2},$$

for $\alpha,\theta,\lambda > 0$,

$$f_{EFr}(x;\alpha,\lambda,\delta) = \alpha\lambda\delta^{\lambda}\left[1-\mathrm{e}^{-(\delta/x)^{\lambda}}\right]^{\alpha-1}x^{-(1+\lambda)}\mathrm{e}^{-(\lambda+1)(\delta/x)^{\lambda}},$$

for $\alpha,\lambda,\delta > 0$,

$$f_{MOEFr}(x;\alpha,\lambda,\delta) = \frac{\alpha\lambda\delta^{\lambda}x^{-(\lambda+1)}\mathrm{e}^{-(\delta/x)^{\lambda}}}{[1-\bar{\alpha}(1-\mathrm{e}^{-(\delta/x)^{\lambda}})]^2},$$

for $\alpha,\lambda,\delta > 0$,

$$f_{EW}(x;\alpha,\beta,\delta) = \alpha\beta\delta x^{\beta-1}\mathrm{e}^{-\alpha x^{\beta}}(1-\mathrm{e}^{-\alpha x^{\beta}})^{\delta},$$

for $\alpha,\beta,\delta > 0$,

$$f_{MO-LLoG}(x;\alpha,\beta,\gamma) = \frac{\alpha^{\beta}\beta\gamma x^{\beta-1}}{(x^{\beta}+\alpha^{\beta}\gamma)^2},$$

for $\alpha,\beta,\gamma > 0$

and

$$f_{EHLE}(x;\lambda,\alpha,a) = \frac{2a\lambda\alpha\exp(-\alpha\lambda x)(1-\exp-\alpha\lambda x)^{a-1}}{(1+\exp(-\alpha\lambda x))^{a+1}},$$

for $\lambda,\alpha,a > 0$.

7.1 Half-way house data (failure times in days: 49 cases)

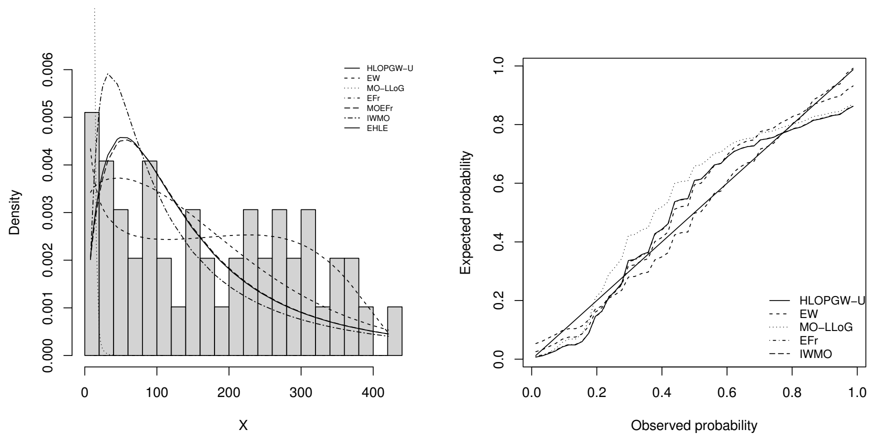
The first data set presents failure times in days (49 cases) for half-way house parolees in the District of Columbia and it was first studied by Stollmack and Harris (1974). The data is given by: 13, 16, 20, 22, 22, 25, 32, 45, 49, 59, 64, 70, 88, 8, 89, 93, 95, 10, 112, 116, 122, 147, 150, 151, 177, 179, 190, 204, 207, 221, 233, 240, 245, 247, 264, 267, 272, 283, 291, 301, 307, 320, 337, 343, 352, 362, 367, 396, 421.

It is noted that the HLOPGW-U distribution gives a better fit to the half-way house data set compared to other non-nested models considered in this paper. In addition, from the fitted density plots (see Figure 7), we confirm that the proposed model fits the heavy-tailed data well.

7.2 Leukemia data

The second data set relates to lifetimes in days of forty three (43) blood patients who had leukemia and was first collected from one hospital in Saudi Arabia and reported by Abouammoh et al. (1994). The data is as follows: 115, 181, 255, 418, 441, 461,

Figure 7: Fitted pdfs and observed probability plots for half-way house data



516, 739, 743, 789, 807, 865, 924, 983, 1025, 1062, 1063, 1165, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1519, 1578, 1578, 1599, 1603, 1605, 1696, 1735, 1799, 1815, 1852, 1899, 1925, 1965.

Figure 8: Fitted pdfs and probability plots for leukemia data

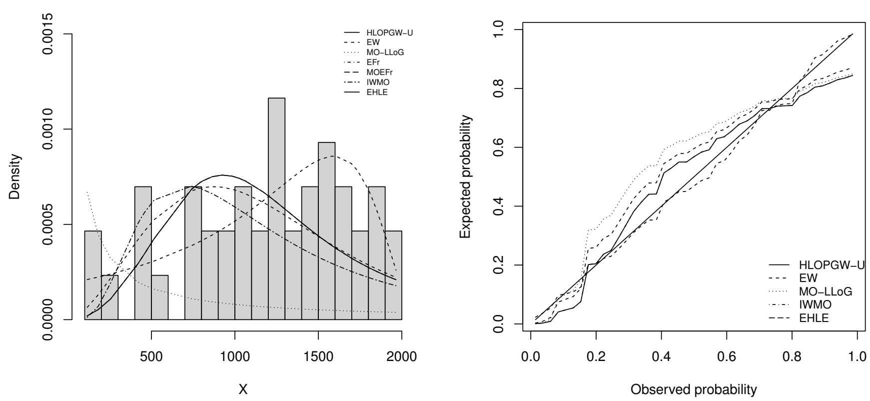


Table 5: The Models parameter estimates and goodness-of-fit statistics for half-way house data

Model	Estimates			Statistics							
	α	β	λ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	P-value
HLOPGW-U	0.6323	1.4380	0.1825	590.94	596.94	597.48	602.62	0.0338	0.2684	0.0703	0.97
	(0.1124)	(0.2898)	(0.0553)								
OPGW-U	1.0465	40.4810	1.01×10^4	594.48	600.48	601.01	606.15	0.0755	0.5129	0.0932	0.79
	(0.1296)	(18.2100)	(0.0081)								
Weibull	-	0.0011	1.3056	600.53	604.53	604.79	608.32	0.1455	0.9488	0.1169	0.52
	-	(0.0009)	(0.1576)								
	α	β	δ								
EW	1.2358	0.2499	37.5554	619.64	625.64	626.17	631.31	0.3787	2.2668	0.1706	0.12
	(0.7570)	(0.0760)	(19.7592)								
	α	β	γ								
MO-LLoG	18.7236	1.6231	24.6109	614.51	620.51	621.04	626.18	0.3040	1.8809	0.1360	0.33
	(4.5697)	(0.1940)	(2.1419)								
	α	δ	λ								
EFr	5.8717	6.8262	0.0599	673.89	679.89	680.42	685.56	0.6275	3.5043	0.3072	$1.93\ \times10^{-3}$
	(7.7003)	(1.0339)	(0.1813)								
MOEFr	116.2064	6.8742	1.6021	614.31	620.31	620.84	625.99	16.6901	96.7639	0.9928	$< 2.2\times10^{-16}$
	(9.8739)	(5.6548)	(0.2030)								
	α	λ	θ								
IWMO	1.16×10^2	1.6021	0.1455	614.30	620.30	620.83	625.98	0.3129	1.9192	0.1375	0.31
	(2.71×10^{-5})	(0.1818)	(5.52×10^{-2})								
	λ	α	a								
EHLE	3.0255	2.68×10^{-3}	1.0677	600.30	606.30	606.83	611.97	0.1396	0.9148	0.1098	0.60
	(3.32×10^{-7})	(3.82×10^{-4})	(0.1907)								

Table 6: The Model parameter estimates and goodness of fit statistics for leukemia data

Model	Estimates			Statistics							
	α	β	λ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	P-value
HLOPGW-U	1.0049	0.8746	0.2267	648.17	654.17	654.79	659.46	0.0199	0.1471	0.0650	0.99
	(0.2293)	(0.2289)	(0.0493)								
OPGW-U	1.8505	20.8620	1.003×10^{4}	651.10	657.10	657.72	662.39	0.0341	0.2557	0.0776	0.95
	(0.2370)	(7.8754)	(0.0041)								
Weibull	-	1.65×10^{-7}	2.1767	660.28	664.28	664.59	667.80	0.1482	0.9563	0.1456	0.32
	-	(1.23×10^{-9})	(0.9613)								
	α	β	δ								
EW	0.0667	0.5572	15.5090	685.74	691.74	692.36	697.03	0.4393	2.5642	0.1978	0.07
	(0.0287)	(0.1595)	(11.3452)								
	α	β	γ								
MO-LLoG	202.9888	3.1368	228.1210	673.59	679.59	680.21	684.88	0.3152	1.8977	0.1557	0.25
	(47.3798)	(0.4129)	(13.4405)								
	α	δ	λ								
EFr	1.8668	167.4499	0.2905	758.63	764.63	765.25	769.92	0.7989	4.4088	0.3942	3.14×10^{-6}
	(0.5507)	(45.1699)	(0.1253)								
MOEFr	1.46×10^{4}	53.8340	3.1362	673.59	679.59	680.21	684.88	13.5791	84.0707	0.9992	$< 2.200\times10^{-16}$
	(0.0277)	(22.5960)	(0.0411)								
	α	λ	θ								
IWMO	1.52×10^{4}	3.1362	0.0019	673.59	679.59	680.21	684.88	0.3158	1.9008	0.1557	0.25
	(2.77×10^{-8})	(0.0412)	(7.93×10^{-3})								
	λ	α	a								
EHLE	1.1326	1.39×10^{-3}	2.6078	667.43	673.43	674.05	678.72	0.2459	1.5160	0.1494	0.29
	(4.11×10^{-7})	(1.72×10^{-4})	(0.0583)								

Furthermore, from the results shown in Table 6 that the HLOPGW-U distribution fit the leukemia data set better than other non-nested models. The HLOPGW-U distribution gives smaller values of the the goodness-of-fit statistics that is A^*, W^* and SS. Furthermore, according to the fitted density plots (see Figure 8), we note that the HLOPGW-U model fit the relief times data better than the selected non-nested.

8 Conclusions

A new family of generalized distribution namely the Half Logistic odd Power Generalized Wiebull-G (HLOPGW-G) family of distributions. The new distribution is a linear combination of the exponentiated-G distribution. The maximum likelihood estimation technique is used to estimate the model parameters. We applied the HLOPGW-U to two data sets that are heavily-skewed. Our new proposed model performs better than the selected equal-parameter non-nested models. Lastly, we hope that the new family of generalized distributions will find wider applicability in various disciplines such as finance, economics, engineering, reliablity and survival analysis just to mention a few.

Acknowledgements

The authors would like to thank the anonymous reviewers and Editors for the recommendations and useful comments that greatly improved this article to its current state.

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