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# Disturbance-Kalman state for linear offset free MPC

Truong Thanh TUAN, Haslinda ZABIRI, Mohamed Ibrahim Abdul MUTALIB and Dai-Viet N. VO

In model predictive control (MPC), methods of linear offset free MPC are well established such as the disturbance model, the observer method and the state disturbance observer method. However, the observer gain in those methods is difficult to define. Based on the drawbacks observed in those methods, a novel algorithm is proposed to guarantee offset-free MPC under model-plant mismatches and disturbances by combining the two proposed methods which are the proposed Recursive Kalman estimated state method and the proposed Disturbance-Kalman state method. A comparison is made from existing methods to assess the ability of providing offset-free MPC on Wood-Berry distillation column. Results shows that the proposed offset free MPC algorithm has better disturbance rejection performance than the existing algorithms.

**Key words:** Kalman filter; plant-model mismatch; offset free MPC; disturbance model

## 1. Introduction

When dealing with highly interactive multivariable processes such as distillation column, MPC is highly recommended as a controller [1]. The essence of MPC is to optimize a control problem based on a dynamic forecast model to obtain the control move, which is sent as manipulated variable into the plant [2–7]. As any model-based controller, MPC relies on an accurate dynamic model, which is usually a linear state space model [5,8]. The model uses to predict the plant behavior is central to the computation and implementation of MPC [2,4,6]. Therefore, the MPC performance is directly associated with the accuracy of model. However,



T.T. Tuan (corresponding author, e-mail: tuantt@pvu.edu.vn) is with Petrovietnam University, 762 Cach Mang Thang Tam Street, Long Toan Ward, Ba Ria City 78109, Ba Ria Vung Tau Province, Vietnam.

H. Zabiri and M.I.A. Mutalib are with Department of Chemical Engineering, Universiti Teknologi PETRONAS, Bandar Seri Iskandar, 32610 Tronoh, Perak, Malaysia.

Dai-Viet N. Vo is with Center of Excellence for Green Energy and Environmental Nanomaterials (CE@GrEEN), Nguyen Tat Thanh University, 300A Nguyen Tat Thanh, District 4, Ho Chi Minh City 755414, Viet Nam.

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the model is expected to differ from the plant [9–11] because of model-plant mismatch caused by changes in plant characteristics with time and disturbances. This leads to an error between the predicted and measured output and thereby results in offset MPC.

The first method to achieve offset free MPC is obtained by adding a disturbance model to the plant model. An observer is used to estimate the state of the process using the resulting plant-disturbance model. Offset free MPC algorithms augmented with disturbance models under persistent disturbances and set-point changes are reported [12–17]. The offset-free MPC is extended to deal with arbitrary disturbances and references such as sines and ramps [18]. The second method to achieve offset free MPC is the so-called state disturbance method [19]. This method does not require a disturbance model; however, the state of the process should be measurable [20]. When the state of the process cannot be measured, the method is altered by adding an output bias term to get offset free MPC for the linear model in [20] and extended for nonlinear ones in [21–23]. An alternative offset-free MPC is based on a velocity form, in which the input change and state change are used instead of the input value and the state value, respectively [19, 24, 25].

The drawback of the disturbance method is difficult to obtain the disturbance model in real industrial problems. In addition, the disturbance state and output methods use the output error term to compensate the model-plant mismatches which does not fully capture the dynamic behavior of the process comparing to the state error.

The general goal of this paper is to develop a linear offset free model predictive control with the proposed Disturbance-Kalman state method. The MPC with the proposed Disturbance-Kalman for dealing with model-plant mismatches will perform better in terms of disturbance rejection and tracking setpoint than other existing offset-free MPC algorithms.

## 2. Problem statement

In this work, a linear discreate state space model is used by MPC.

A common linear discrete state space model that describes a process is considered, and it is given in model (1) as follows:

$$\begin{aligned} x_{k+1|k} &= Ax_{k|k} + Bu_k \,, \\ y_{k+1|k} &= Cx_{k+1|k} \,, \end{aligned} \tag{1}$$

where subscript $k$ is the sampling time instant ($k$ = integer); subscript $k+1|k$ represents the 1-step ahead prediction at time instant $k$; $k|k$ represents estimated variables at time instant $k$ such as: $x_{k|k}$ is the estimated state at time instant $k$,

$x_{k+1|k}$ is state prediction with 1-step ahead prediction at time instant $k+1$, $x \in R^{n_x}$ denotes the state vector, $u \in R^{n_u}$ denotes the manipulated variable vector, and $y \in R^{n_y}$ denotes the algebraic controlled vector. The discrete state space model corresponds to a one step ahead prediction model. The matrices $A$, $B$, and $C$ are typically termed as a process matrix, an input matrix, and an output matrix, respectively, and are determined by system identification. We assume the model is controllable and observable. Note that matrices $A$, $B$, and $C$ in model (1) are fixed once identification of the model of the MPC is accomplished.

There are two problems leading to offset in predicted output, $y_{k+1|k}$ in model (1):

- If $x_{k|k}$ in model (1) is wrong, it will make $y_{k+1|k}$ wrong even model (1) is correct;
- In the presence of a plant-model mismatch, it will make the output predictive, $y_{k+1|k}$ wrong even the state of the process, $x_{k|k}$ is correct.

Therefore, our research will focus on two points below to attain offset-free MPC by eliminating the offset between the predicted output and the measured output.

- Attainment of the corrected state estimate, $x^m_{k|k}$ of the plant to replace $x_{k|k}$ in model (1) will be presented in section 3.2 with the proposed Recursive Kalman estimated state;
- Addition of the compensated term to model (1), which is equal to the difference between the predicted and measured output, will be presented in section 4.2 with the proposed Disturbance-Kalman state method.

In summary, the method proposed for offset-free MPC will consist of the following steps:

a) obtaining the corrected state of the process (the proposed Recursive Kalman estimated state, $x^m_{k|k}$);

b) combining the corrected state with the proposed state prediction vector (the proposed Disturbance-Kalman state method);

c) integrating the predictive state vector into MPC cost function;

d) comparing studies between the proposed method with other existing methods using the appropriate case studies.

## 3. State observer

This work will propose the method to obtain the corrected state of the process in the presence of a plant-model mismatch without an explicit disturbance model. The proposed method is developed from Kalman filter method.

### 3.1. Kalman filter

The discrete-time Kalman filter equations in [26] are summarized as follows:

State prediction is as follows:

$$x_{k|k-1} = Ax_{k-1|k-1} + Bu_{k-1} \, . \tag{2}$$

Predictive error covariance is as follows:

$$P_{k|k-1} = AP_{k-1|k-1}A' + Q \, . \tag{3}$$

Kalman gain is as follows:

$$K_k = P_{k|k-1}C' \left(CP_{k|k-1}C' + R\right)^{-1} \, . \tag{4}$$

Estimated state is as follows:

$$x_{k|k} = x_{k|k-1} + K_k \left(y_{k|m} - Cx_{k|k-1}\right) . \tag{5}$$

Estimated error covariance is as follows:

$$P_{k|k} = \left(I - K_k C\right) P_{k|k-1} \, , \tag{6}$$

where $y_{k|m}$ is the measured plant output at time instant $k$, $Q$ denotes the process noise covariance matrix, and $R$ denotes the measurement covariance matrix, the transpose of a matrix or vector is denoted by the prime.

There may still be a problem that prevents the Kalman Filter from producing the corrected state estimation as a faulty or an inaccurate mathematical model.

### 3.2. The proposed Recursive Kalman estimated state, $x_k^m$ under model-plant mismatches and disturbances

Kalman filter reduces errors from inaccurate mathematical models based on the correction term $K_k \left(y_{k|m} - Cx_{k|k-1}\right)$ in equation (5). The model-plant mismatch (MPM) is large such as big differential gain between a real plant and a model, Kalman filter may lead to large state estimation error [26], it just helps to reduce a part of the error. In this research, we propose a Recursive Kalman estimated state to obtain the corrected state of the plant.

Equation (5) is used recursively to find the $n$-steps estimated state, $x^{+\ldots n}_{k|k}$ as follows:

$$\begin{aligned} x_{k|k} &= x_{k|k-1} + K_k\left(y_{k|m} - Cx_{k|k-1}\right), \\ x^{+}_{k|k} &= x_{k|k} + K_k\left(y_{k|m} - Cx_{k|k}\right), \\ x^{++}_{k|k} &= x^{+}_{k|k} + K_k\left(y_{k|m} - Cx^{+}_{k|k}\right), \\ x^{+\ldots n}_{k|k} &= x^{+\ldots n-1}_{k|k} + K_k\left(y_{k|m} - Cx^{+\ldots n-1}_{k|k}\right), \end{aligned} \tag{7}$$

$$\begin{aligned} Cx_{k|k} &= Cx_{k|k-1} + CK_k(y_{k|m} - Cx_{k|k-1}) \\ Cx^{+}_{k|k} &= Cx_{k|k} + CK_k(y_{k|m} - Cx_{k|k}) \\ Cx^{++}_{k|k} &= Cx^{+}_{k|k} + CK_k(y_{k|m} - Cx^{+}_{k|k}) \\ Cx^{+\ldots n}_{k|k} &= Cx^{+\ldots n-1}_{k|k} + CK_k(y_{k|m} - Cx^{+\ldots n-1}_{k|k}) \end{aligned}$$

$$\Leftrightarrow \quad \begin{aligned} y_{k|k} &= y_{k|k-1} + CK_k(y_{k|m} - Cx_{k|k-1}) \\ y^{+}_{k|k} &= y_{k|k} + CK_k(y_{k|m} - Cx_{k|k}) \\ y^{++}_{k|k} &= y^{+}_{k|k} + CK_k(y_{k|m} - Cx^{+}_{k|k}) \\ y^{+\ldots n}_{k|k} &= y^{+\ldots n-1}_{k|k} + CK_k\left(y_{k|m} - Cx^{+\ldots n-1}_{k|k}\right), \end{aligned} \tag{8}$$

where $n$ denotes the $n$-steps of recursive Kalman filter, $x^{+\ldots n}_{k|k}$ denotes the $n$-steps estimated state at time instant $k$.

$$CK_k = CP^{-}_{k}C'\left(CP^{-}_{k}C' + R\right)^{-1}. \tag{9}$$

$P$ is symmetric and positive definite [26], implying $CP^{-}_{k}C' \geqslant 0$. $R$ is positive definite [26]. Because of this, the $CK_k$ of equation (9) has all eigenvalues which are positive and less than 1.

**Definition 1** $x^{m}_{k|k}$ *is the corrected state estimate of the process if and only if there exists* $x^{m}_{k|k} = x^{+\ldots n}_{k|k}$ *such that* $y_{k|m} = Cx^{m}_{k|k}$.

If the difference between the predictive output, $y_{k|k-1}$ and the measured plant output, $y_{k|m}$ is large, there will always be a deviation between the estimated output, $y_{k|k}$ and the measured plant output, $y_{k|m}$. Because eigenvalues of $CK_k$ are less than 1, $CK_k$will make $y_{k|k}$ go to $y_{k|m}$. However, $y_{k|k}$ cannot be asymptotically equal to $y_{k|m}$ in one step. Therefore, at each recursive step, a part of the error will be reduced. This means, for a sufficiently large value of n-steps, the error will vanish, and the $n$-steps estimated state, $x^{+\ldots n}_{k|k}$ will asymptotically converge to the corrected state of real plant, $x^{m}_{k|k}$ . The step-by-step description for obtaining the corrected state of the plant, $x^{m}_{k|k}$ is given in Algorithm 1 which is also shown in the flowchart form in Figure 1. The measured output, $y_{k|m}$ is filtered noise, and then the output without noise is sent to Recursive Kalman estimated state to obtain the corrected state, $x^{m}_{k|k}$. If the deviation between the measured plant output, $y_{k|m}$ and the $n$-steps estimated output, $y^{+\ldots n}_{k|k}$ is less than or equal to the value, $\varepsilon$, the $x^{+\ldots n}_{k|k}$ at n recursive steps will be used as the corrected state, $x^{m}_{k|k}$ and sent to MPC cost function.

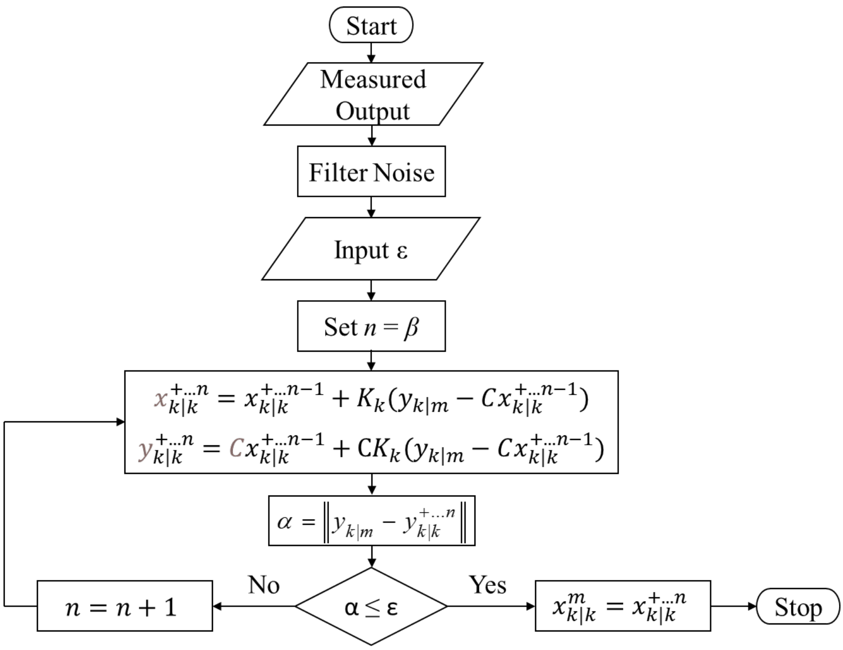


Figure 1: Recursive Kalman estimated state algorithm

**Algorithm 1** Recursive Kalman estimated state

*Step 1: Measure the output.*

*Step 2: Pass the measured output through the noise filter.*

*Step 3: Choose $\varepsilon$ (a small positive number).*

*Step 4: Set $n = \beta$.*

*Step 5: Calculate $x^{+\ldots n}_{k|k}$ using equation (7).*

*Step 6: Calculate $\alpha = \left\| y_{k|m} - y^{+\ldots n}_{k|k} \right\|$.*

*Step 7: Check* $a \leqslant \varepsilon$.

*Step 8: If step 7 is satisfied,* $x^m_{k|k} = x^{+\dots n}_{k|k}$ . *If step 7 is not satisfied, repeat step 5 with* $n = n + 1$.

In Algorithm 1, $\varepsilon$ represents the accuracy of $n$-steps estimated output, $y^{+\dots n}_{k|k}$, $\beta$ is chosen to satisfy $a \approx \varepsilon$. Step 7 and step 8 in Algorithm 1 are to ensure $a \leqslant \varepsilon$.

## 4. The proposed MPC

In this section, the proposed offset-free MPC approach and related algorithms are presented. The proposed method requires both estimation of the corrected state in section 4.1 and the corrected predicted output vector in section 4.2.

In section 4.1, MPC uses the corrected state from recursive Kalman filter as the initial state. However, in the presence of MPM, i.e. large differential in gain or large disturbances or both, the predicted output will be equal to the set point but will not match with the measured output. This problem can be solved to obtain offset free MPC by using the proposed Disturbance-Kalman state method in section 4.2.

### 4.1. MPC with Recursive Kalman estimated state algorithm

MPC is used to determine the approximate input based on the prediction model (discrete state space model) by minimizing control law function (10) as follows:

$$J = \min_{u,y} \sum_{l=1}^{H_p} Q_y \left\| y_{k+l|k} - \bar{y} \right\|^2 + \sum_{j=0}^{H_c} Q_u \left\| u_{k+j|k} - u_{k|m} \right\|^2 + R_y \left\| y_{k+1|k} - \bar{y} \right\|^2 \tag{10}$$

subject to:

discrete state space model (1)

$$\begin{aligned} &u_{\min} \leqslant u_{k+j|k} \leqslant u_{\max}, \quad \forall j \in \{1, \dots, H_c\}, \\ &\left| u_{k+j|k} - u_{k+j-1|k} \right| \leqslant \Delta u, \\ &y_{\min} \leqslant y_{k+l|k} \leqslant y_{\max}, \quad \forall l \in \left\{1, \dots, H_p\right\}, \end{aligned}$$

where $u_{k|m}$ is the recorded controller output at time instant $k$; $\bar{y}$ and $\bar{u}$ are the target output and input, respectively; $Q_u$, $Q_y$ and $R_y$ are weighting parameters; $\Delta u$ is the allowable input change during two consecutive time instants; $H_p$ and $H_c$ are prediction and control horizons, respectively; $(k + l|k)$ represents the $l$-step ahead prediction at time instant $k$; lower bound, $u_{\min}$ and upper bound, $u_{\max}$ form the admissible input set $U$. Additionally, $\| \, . \, \|^2$ is the square of the two-norm of a vector formulated with diagonal matrix and is defined as follows: $\|x\|^2 = x_1^2 + x_2^2 + \ldots + x_n^2$.

Function (10) is re-expressed as follows:

$$J = Q_y \left\| vsv(\bar{y}, Hp) - \vec{y}_{k+1}^{H_p} \right\|^2 + Q_u \left\| \vec{u}_k^{H_c} - vsv\left(u_{k|m}, H_c\right) \right\|^2 + R_y \left\| y_{k+1|k} - \bar{y} \right\|^2, \tag{11}$$

where vsv($\bar{y}$, $H_p$) vertically stacks vector $\bar{y}$ with $H_p$ times; vsv($u_{k|m}$, $H_c$) vertically stacks vector $u_{k|m}$ with $H_c$ times.

$$\vec{y}_{k+1}^{H_p} = \begin{bmatrix} y_{k+1|k} \\ y_{k+2|k} \\ \vdots \\ y_{k+H_p|k} \end{bmatrix}; \qquad \vec{u}_k^{H_c} = \begin{bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+H_u-1} \end{bmatrix};$$

$$vsv\left(\bar{y}, H_p\right) = \begin{bmatrix} \bar{y} \\ \vdots \\ \bar{y} \end{bmatrix}; \qquad vsv\left(u_{k|m}, H_c\right) = \begin{bmatrix} u_{k|m} \\ \vdots \\ u_{k|m} \end{bmatrix}$$

The predictive output vector is as follows:

$$\vec{y}_{k+1}^{H_p} = P x_{k|k}^m + H \vec{u}_k^{H_p}, \tag{12}$$

where:

$$P = \begin{bmatrix} C \times A \\ C \times A^2 \\ \vdots \\ C \times A^{H_p} \end{bmatrix}; \quad H = \begin{bmatrix} C \times B & 0 & \cdots & 0 \\ C \times A \times B & C \times B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C \times A^{H_p-1} \times B & C \times A^{H_p-2} \times B & \cdots & C \times B \end{bmatrix}.$$

The predictive output vector in equation (12) is derived from model (1). The corrected state, $x_{k|k}^m$ in equation (12) is obtained by the proposed Recursive Kalman estimated state.

The MPC controller optimizes $J$ function (11) and obtains vector $\vec{u}_k^{H_p}$ as a root. However, only $u_k$ is considered in the vector and sent as manipulated variables to a real plant.

### 4.2. Offset free MPC with the proposed Disturbance-Kalman state method

The main objective of this section is to correct model (1) in the presence of plant-model mismatches. Plant-model mismatches can be solved by modifying model (1) with the proposed Disturbance-Kalman state (DKS) method.

The state prediction in model (1), $x_{k+1|k}$ will be different from the corrected state, $x^m_{k|k}$ of the process in the presence of MPM or/and disturbances. The difference is named as the state disturbance vector which is added to the state prediction, $x_{k+1|k}$ in model (1), then the error correction model after being added the state disturbance vector is as follows:

$$\begin{aligned} x^m_{k+1} &= Ax^m_{k|k} + Bu_k + \left(x^m_{k|k} - x_{k|k-1}\right), \\ y^m_{k+1} &= Cx^m_{k+1}, \end{aligned} \tag{13}$$

where $x_{e|k} = x^m_{k|k} - x_{k|k-1}$ is called as the state disturbance estimate at time instant $k$.

$$\begin{aligned} x^m_{k+1} &= Ax^m_{k|k} + Bu_k + x_{e|k}, \\ y^m_{k+1} &= Cx^m_{k+1}. \end{aligned} \tag{14}$$

$x_{e|k}$ is the term to correct the state prediction of the model under MPM and disturbances.

Using model (14), recursively, it is easy to obtain an $H_p$-step ahead output prediction as follows:

$$\vec{y}^{H_p}_{k+1} = Px^m_{k|k} + H\vec{u}^{H_p}_k + L \times vsv(x_{e|k}, H_p), \tag{15}$$

where $vsv(x_{e|k}, H_p)$ vertically stacks vector $(x_{e|k})$ with $H_p$ times; $x^m_{k|k}$ is obtained by Recursive Kalman estimated state algorithm.

$$L = \begin{bmatrix} C & 0 & \cdots & 0 \\ C \times A & C & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C \times A^{H_p-1} & C \times A^{H_p-2} & \cdots & C \end{bmatrix}.$$

**Algorithm 2** Offset-free MPC with Disturbance-Kalman state

*Step 1: The process state,* $x^m_{k|k}$ *is obtained from Algorithm 1;*

*Step 2: The state disturbance vector is calculated as* $x_{e|k} = x^m_{k|k} - x_{k|k-1}$ *with* $x^m_{k|k}$ *from step 1;*

*Step 3: The optimization problem (11) is solved with predictive output vector calculated by equation (15);*

*Step 4: After the optimal trajectory* $\vec{u}^{H_p}_k$ *is found by the optimization, its first element* $u_k$ *is sent to the process actuators.*

**Proposition 1** *After long enough steps, the predicted output and MPC actions will reach steady state before time instant* $k$*. Namely,* $u_{k-j|k} = u_{ss}$*;* $y^m_{k-j|k} = y_{ss}$*,* $\forall j$*;* $x^m_{k|k} = x_{ss}$*,* $\forall k > 0$*. Using the Algorithm 2 will guarantee the offset-free setpoint tracking if and only if* $y^m_{k+1} = y_{ss}$*.*

**Proof.** Assume that the output and input of a process have already reach the desired value $\{y_{ss},\ u_{ss}\}$ before time instant $k$. Under the proposed Algorithm 2, we will show that the predicted output, $y^m_{k+1} = y_{ss}$ at a steady state.

At steady state, $u_{k-j|k} = u_{ss}$; $y^m_{k-j|k} = y_{ss}$; $x^m_{k|k} = x_{ss}$.

Substituting $x_{ss}$, $u_{ss}$ for $x^m_{k|k}$, $u_k$ in Eq. (13);

$$x^m_{k+1} = Ax_{ss} + Bu_{ss} + \left(x_{ss} - x_{k|k-1}\right).$$

Substituting $Ax^m_{k-1|k-1} + Bu_{k-1}$ for $x_{k|k-1}$

$$x^m_{k+1} = Ax_{ss} + Bu_{ss} + \left(x_{ss} - \left(Ax^m_{k-1|k-1} + Bu_{k-1}\right)\right).$$

Substituting $x_{ss}$, $u_{ss}$ for $x^m_{k-1|k-1}$, $u_{k-1}$

$$x^m_{k+1} = Ax_{ss} + Bu_{ss} + \left(x_{ss} - \left(Ax_{ss} + Bu_{ss}\right)\right).$$

By the commutativity of addition

$$\begin{aligned} x^m_{k+1} &= Ax_{ss} + Bu_{ss} + \left(-\left(Ax_{ss} + Bu_{ss}\right)\right) + x_{ss}\,, \\ x^m_{k+1} &= x_{ss}\,. \end{aligned}$$

Multiplied by $C$

$$\begin{aligned} Cx^m_{k+1} &= Cx_{ss}\,, \\ y^m_{k+1} &= y_{ss}\,. \end{aligned}$$ □

## 5. Case study

Case study is presented in the following sections to demonstrate the efficacy of the proposed offset-free MPC algorithm. The Wood-Berry distillation model is used to represent the plant to be controlled. The Wood-Berry model is a $2 \times 2$ transfer function model of a pilot plant distillation column that separates methanol and water. The reason for this choice is because the amount of mismatch introduced can be easily shown. This case study will be used to demonstrate the advantage of the proposed offset free MPC with two other offset free methods. The other offset free methods will be presented in section 5.1 and section 5.2.

### 5.1. Disturbance model and observer method

The main concept of this method is to augment the disturbance model with the nominal model, accounting for MPM and disturbances. The model (1) is augmented with a disturbance model as follows:

$$\begin{aligned} \begin{bmatrix} x_{k+1|k} \\ d_{k+1} \end{bmatrix} &= \begin{bmatrix} A & B_d \\ 0 & A_d \end{bmatrix} \begin{bmatrix} x_{k|k} \\ d_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k\,, \\ y_{k+1|k} &= \begin{bmatrix} C & C_d \end{bmatrix} \begin{bmatrix} x_{k+1|k} \\ d_{k+1} \end{bmatrix}, \end{aligned} \tag{16}$$

where $A_d$, $B_d$, and $C_d$ are disturbance model matrices, $d_k$ is disturbance; $d_k \in R^{n_d}$, $A_d \in R^{n_d \times n_d}$, $B_d \in R^{n_x \times n_d}$, $C_d \in R^{n_y \times n_d}$.

The disturbance model matrices ($A_d$, $B_d$, and $C_d$) have appropriate dimension and satisfy the following: rank $\begin{bmatrix} A-\lambda I & B_d \\ 0 & A_d - \lambda_d I \\ C & C_d \end{bmatrix} = n_x + n_d$ which ensures that model (16) is observable.

Consider: $X_{k+1|k} = \begin{bmatrix} x_{k+1|k} \\ d_{k+1} \end{bmatrix}$; $X_{k|k} = \begin{bmatrix} x_{k|k} \\ d_k \end{bmatrix}$; $AA = \begin{bmatrix} A & B_d \\ 0 & A_d \end{bmatrix}$; $BB = \begin{bmatrix} B \\ 0 \end{bmatrix}$; $CC = \begin{bmatrix} C & C_d \end{bmatrix}$.

Therefore, model (16) becomes:

$$\begin{aligned} X_{k+1|k} &= AA \times X_{k|k} + BB \times u_k\,, \\ y_{k+1|k} &= CC \times X_{k+1|k}\,. \end{aligned} \tag{17}$$

Model (17) is considered as the model (1). Hence, the state estimate, $X_{k|k}$ is obtained by equation (5).

The predictive output vector is as follows:

$$\vec{y}_{k+1}^{\,H_p} = PP \times X_{k|k} + HH \times \vec{u}_k^{\,H_p}\,, \tag{18}$$

where:

$$PP = \begin{bmatrix} CC \times AA \\ CC \times AA^2 \\ \vdots \\ CC \times AA^{H_p} \end{bmatrix};$$

$$HH = \begin{bmatrix} CC \times BB & 0 & \cdots & 0 \\ CC \times AA \times BB & CC \times BB & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CC \times AA^{H_p-1} \times BB & CC \times AA^{H_p-2} \times BB & \cdots & CC \times BB \end{bmatrix}.$$

### 5.2. Piotr Tatjewski's method

The work in [20] uses a normal state estimate method which is not the Recursive Kalman estimated state method. Hence, the corrected state of the process cannot be obtained in the presence of plant-model mismatches. Therefore, equation (15) is added a biased output term to compensate the error in the output prediction.

Inserting $x_{k|k}$ into constant state disturbance model in [20] to get state disturbance vector $v_k$.

$$v_k = x_{k|k} - \left(Ax_{k-1|k-1} + Bu_{k-1}\right). \tag{19}$$

By substituting $x_{k|k}$, $v_k$ for $x^m_{k|k}$, $x_{e|k}$ in equation (15) respectively, we get the predictive output trajectory:

$$\vec{y}^{H_p}_{k+1} = P \times x_{k|k} + H \times u_k + L \times vsv(v_k, H_p), \tag{20}$$

where $vsv(v_k,H_p)$ vertically stacks vector $(v_k)$ with $H_p$ times; $x_{k|k}$ is obtained from Kalman filter from equation (5).

Because state estimate $x_{k|k}$ will not be equal to the real process state $x^m_{k|k}$ in the presence of plant-model mismatches, the predicted output trajectory in [20] is extended to:

$$\vec{y}^{H_p}_{k+1} = P \times x_{k|k} + H \times u_k + L \times vsv(v_k, H_p) + vsv((y_{k|m} - Cx_{k|k}), H_p), \tag{21}$$

where $y_{k|m} - Cx_{k|k}$ is the biased output term, $vsv((y_{k|m} - Cx_{k|k}), H_p)$ vertically stacks the biased output term $H_p$ times.

The work in [20] cannot identify the corrected state of the process to obtain the exact state deviation. Therefore, an output deviation is added to compensate the error of the state deviation as shown in equation (21). Our proposed DKS method is able to obtain the corrected state deviation, where the prediction model is only added the state deviation as seen in equation (15).

### 5.3. Summary of offset free MPC methods using in the case study

The performances of the MPC1, MPC2 and MPC3 in Table 1 are measured using the Overall Integral Error (OIE).

The OIE is defined as follow:

$$OIE = \sqrt{\frac{rIAE^2 + rISE^2 + rITAE^2}{3}}. \tag{22}$$

It should be noted that, the maximum of *OIE* is 1 because the maximum of *rIAE*, *rISE* and *rITAE* is 1.

$$\text{where} \qquad rIAE = \frac{IAE}{IAE0}; \quad rISE = \frac{ISE}{ISE0}; \quad rITAE = \frac{ITAE}{ITAE0}. \tag{23}$$

Table 1: Recursive Kalman estimated state

| | MPC1 | MPC2 | MPC3 |
|---|---|---|---|
| Offset free MPC with | the proposed Disturbance-Kalman state method | Piotr Tatjewski's method | disturbance model and observer method |
| The predictive output vector | equation (15) | equation (21) | equation (18) |
| State estimation method | Recursive Kalman estimated state algorithm in section 3.2 | Kalman filter | Kalman filter |

*IAE0*, *ISE0* and *ITAE0* are the maximum values of *IAE*, *ISE* and *ITAE* of MPC1, MPC2, MPC3, respectively. The definitions of the Integral absolute error (IAE), Integral squared error (ISE) and Integral time absolute error (ITAE) are available in [27].

### 5.4. Wood-Berry distillation column

Distillation column is a common process for separation of liquid mixtures. Figure 2 shows a simple specific distillation column scheme with Wood-Berry setting [28]. The feed stream enters at the middle tray. The target is to separate methanol from the methanol-water mixture. The methanol composition in top product, $x_D$ and bottom product, $x_B$ is controlled by manipulating the reflux flowrate, $R$ and steam flowrate, $S$. The model used in MPC will be chosen to be different from the process, i.e. it will have plant-model mismatches.

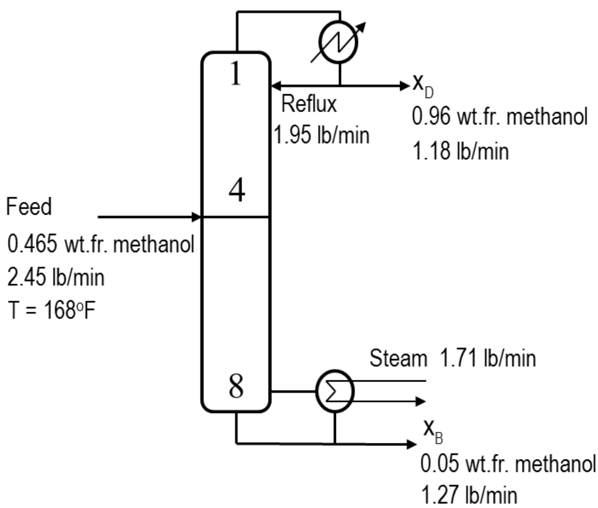


Figure 2: Wood-Berry distillation column

The Wood-Berry linear model was introduced in [28] and used to represent the plant and disturbance as given by

$$\begin{bmatrix} x_D(s) \\ x_B(s) \end{bmatrix} = \begin{bmatrix} \dfrac{12.8e^{-1s}}{16.7s+1} & \dfrac{-18.9e^{-3s}}{21.0s+1} \\ \dfrac{6.6e^{-7s}}{10.9s+1} & \dfrac{-19.4e^{-3s}}{14.4s+1} \end{bmatrix} \begin{bmatrix} R(s) \\ S(s) \end{bmatrix} + \begin{bmatrix} \dfrac{3.8e^{-8s}}{14.9s+1} \\ \dfrac{4.9e^{-30s}}{13.2s+1} \end{bmatrix} D(s), \tag{24}$$

where $R$ is the reflux flowrate, $S$ is the steam flowrate, $D$ is the disturbance, $x_D$ is the top product purity, $x_B$ is the bottom product purity.

This linear model is valid around a typical steady state condition: $x_D = 0.96$ and $x_B = 0.05$.

The theoretical possibilities of mismatch parameters are the gain ($K$), the time constant ($\tau$) and time delay ($\theta$). $\Delta K$, $\Delta\tau$, and $\Delta\theta$ are gain, time constant and time delay mismatches respectively, as shown in equation (25):

$$G_p = \frac{(K+\Delta K)e^{-(\theta+\Delta\theta)s}}{(\tau+\Delta\tau)\,s+1}\,. \tag{25}$$

The model used in MPC in the presence of plant-model mismatches is given by (26). The gain, time delay and time constant mismatches introduced in (26) are given by (27):

$$\begin{bmatrix} x_D(s) \\ x_B(s) \end{bmatrix} = \begin{bmatrix} \dfrac{6.4}{25.05s+1} & \dfrac{-12.6}{42.0s+1} \\ \dfrac{13.2}{7.267s+1} & \dfrac{-29.1}{7.2s+1} \end{bmatrix} \begin{bmatrix} R(s) \\ S(s) \end{bmatrix}, \tag{26}$$

$$\begin{gathered} \Delta K(\%) = \begin{bmatrix} -50\% & -33.33\% \\ +100\% & +50\% \end{bmatrix}; \\ \Delta\theta = \begin{bmatrix} -1 & -3 \\ -7 & -3 \end{bmatrix}; \qquad \Delta\tau(\%) = \begin{bmatrix} +50\% & +100\% \\ -33.33\% & -50\% \end{bmatrix}. \end{gathered} \tag{27}$$

The transfer function model (26) is converted to discrete state space form with sampling time, $Ts = 1$ s using Matlab as given by (28).

$$\begin{aligned} x_{k+1|k} &= \begin{bmatrix} 0.9609 & 0 & 0 & 0 \\ 0 & 0.8714 & 0 & 0 \\ 0 & 0 & 0.9765 & 0 \\ 0 & 0 & 0 & 0.8703 \end{bmatrix} x_{k|k} + \begin{bmatrix} 0.4902 & 0 \\ 0.9342 & 0 \\ 0 & 0.4941 \\ 0 & 1.8673 \end{bmatrix} \begin{bmatrix} R \\ S \end{bmatrix}_k, \\ \begin{bmatrix} x_D \\ x_B \end{bmatrix}_{k+1|k} &= \begin{bmatrix} 0.5110 & 0 & -0.6 & 0 \\ 0 & 1.8165 & 0 & -2.0208 \end{bmatrix} x_{k+1|k}\,. \end{aligned} \tag{28}$$

The choice of disturbance model:

$$A_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \qquad C_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \qquad B_d = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{29}$$

According to model (17), the model (28) augmented with the disturbance model (29) becomes:

$$X_{k+1|k} = \begin{bmatrix} 0.9609 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.8714 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.9765 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8703 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} X_{k|k} + \begin{bmatrix} 0.4902 & 0 \\ 0.9342 & 0 \\ 0 & 0.4941 \\ 0 & 1.8673 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R \\ S \end{bmatrix}_k,$$
$$\begin{bmatrix} x_D \\ x_B \end{bmatrix}_{k+1|k} = \begin{bmatrix} 0.5110 & 0 & -0.6 & 0 & 1 & 0 \\ 0 & 1.8165 & 0 & -2.0208 & 0 & 1 \end{bmatrix} X_{k+1|k}, \tag{30}$$

MPC1 and MPC2 use model (28); MPC3 uses model (30).

The MPC cost functions with tuning parameters is presented as in equation (31):

$$J = \left\| vsv(\bar{y}, 10) - \vec{y}^{\,10}_{k+1} \right\|^2 + 20 \left\| \vec{u}^{\,6}_k - vsv\left(u_{k|m}, 6\right) \right\|^2 + 5 \left\| y_{k+1|k} - \bar{y} \right\|^2 \tag{31}$$

All MPCs use equation (31) as a cost function. The disturbance $D$ is introduced to the Wood-Berry process by equation (24), which will contribute the disturbances $d_{x_D}$ and $d_{x_B}$ to $x_D$ and $x_B$, respectively as depicted in Figure 3.

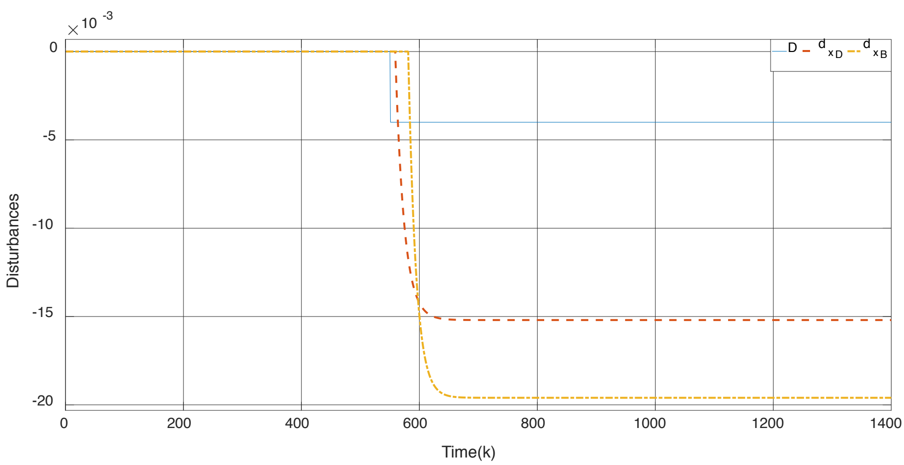


Figure 3: Disturbances plot with respect to time

MPC1 uses $x^m_{k|k}$ to substitute $x_{k|k}$ in (28), where $x^m_{k|k}$ is obtained by Algorithm 1 with $\beta = 1000$ and $\varepsilon = 10^{-12}$. And, the predictive output vector in MPC1 is calculated by Equation (15).

The parameters of the Kalman filter used to test offset-free MPCs are written in Matlab form as follows:

$P_{0|0} = \text{diag}(\text{repmat}([1\ 1],\ 1,\ \text{size}(A, 1)/2))$;

$Q = 1e - 6 \cdot \text{eye}(\text{size}(A, 1))$;

$R = 1e - 1$.

## 6. Results and discussion

In the methodology it is indicated that MPC1, MPC2 and MPC3 refer to MPC1 using the proposed offset free algorithm, MPC2 using the Piotr Tatjewski's method and MPC3 with the augmented disturbance model, respectively. The results of MPC1, MPC2 and MPC3 in Figures 4a and 4b show the CVs (methanol composition in top product, $x_D$ and bottom product, $x_B$) and in Figures 4c and 4d show the MVs (the reflux flowrate, $R$ and steam flowrate, $S$). The test includes set-point tracking and disturbance rejection problems.

Figure 4 shows the response of the controllers in tracking set point changes with $t_k$ from 1 ($k$) to 500 ($k$). It shows that MPC1 tracks set point faster than MPC2 and MPC3 and in all cases the MVs move reasonably smoothly. Note that, MPC3 uses an actual disturbance model which in practice might not be easy to obtain. MPC2, which is well tuned also results in a very sluggish response and takes a long time to remove the offset. It is observed from Figure 4, Table 2 and Table 3 and also from the algorithm that MPC1 performs better in removing offset and requires no disturbance model.

Figure 4 shows the result of the three MPCs for the disturbance rejection problem with $t_k$ from 501 ($k$) to 1400 ($k$). It is observed from the figure that MPC1 has smaller overshoot and settles quickly removing the offset. Table 2 and Table 3 show the overall integral errors (OIEs) of the three MPCs for tracking set point changes and disturbance rejection, respectively. The results in Table 2 show the comparative performance of MPC1 and MPC3 for set-point tracking which has much smaller overall integral error (OIE) than that of MPC2. Also note that in MPC3, the authors assume the disturbance model is available or a disturbance model with good accuracy can be obtained. However, in practice obtaining disturbance model with good accuracy is a difficult task. This is not required by the proposed algorithm whose performance is demonstrated by MPC1.

Table 3 clearly shows that MPC1 with OIE for $x_D$ and $x_B$ equal to 0.3550 and 0.6068, respectively, has much better performance than both MPC2 and MPC3. The second best, i.e., MPC2, has OIE for $x_D$ and $x_B$ equal to 0.7652 and 0.9253,

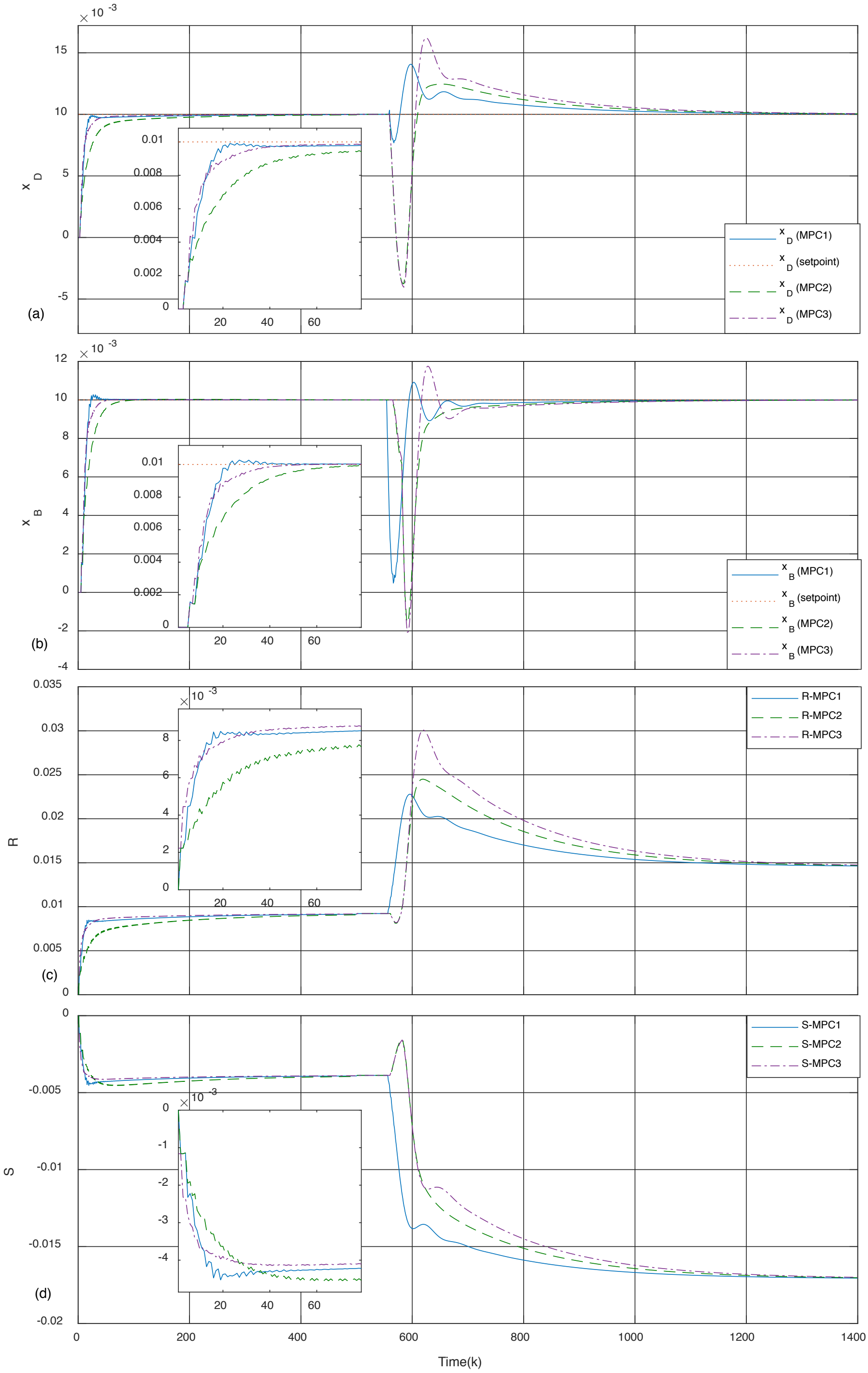


Figure 4: Comparison between MPC1, MPC2 and MPC3

respectively. Also note that in Table 3, for disturbance rejection, the performance of MPC1 which uses the proposed algorithm is much better in all the three performance measurements compared to MPC2 and MPC3.

Table 2: Recursive Kalman estimated state algorithm

| | rIAE | | rISE | | rITAE | | OIE | |
|---|---|---|---|---|---|---|---|---|
| | $x_D$ | $x_B$ | $x_D$ | $x_B$ | $x_D$ | $x_B$ | $x_D$ | $x_B$ |
| MPC1 | 0.4996 | 0.6348 | 0.6299 | 0.8028 | 0.4556 | 0.4424 | 0.5335 | 0.6437 |
| MPC2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| MPC3 | 0.4656 | 0.6486 | 0.5906 | 0.7607 | 0.4010 | 0.4746 | 0.4921 | 0.6389 |

Table 3: Offset free MPC Methods

| | rIAE | | rISE | | rITAE | | OIE | |
|---|---|---|---|---|---|---|---|---|
| | $x_D$ | $x_B$ | $x_D$ | $x_B$ | $x_D$ | $x_B$ | $x_D$ | $x_B$ |
| MPC1 | 0.3979 | 0.6991 | 0.1167 | 0.6400 | 0.4539 | 0.4544 | 0.3550 | 0.6068 |
| MPC2 | 0.7762 | 0.9652 | 0.7799 | 0.9486 | 0.7389 | 0.8585 | 0.7652 | 0.9253 |
| MPC3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

## 7. Conclusion

In this study, the comparison between three offset-free MPCs is presented. The important step in the proposed offset free MPC technique (MPC1) focuses on attaining the corrected state, $x^m_{k|k}$ of a plant and adding compensated term to model (1). The corrected state is obtained by the proposed Kalman estimated state method. In the proposed Disturbance-Kalman state method, the state disturbance vector is added to model (1) as the compensated term.

The effectiveness of the offset free MPC algorithm is demonstrated through Wood-Berry distillation column case study to show the effectiveness of the proposed algorithm in the presence of MPM and disturbances. The result shows that the proposed offset-free MPC algorithm with OIE for $x_D$ and $x_B$ equal to 0.3550 and 0.6068, respectively is much better in disturbance rejection performance than other two existing offset-free MPC algorithms.

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