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CONTROL AND INFORMATICS

# Eigenvalues assignment in uncontrollable linear systems

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**Abstract.** It is shown that in an uncontrollable linear system $\dot{x} = Ax + Bu$ it is possible to assign arbitrarily the eigenvalues of the closed-loop system with state feedbacks $u = Kx$, $K \in \Re^{n\times n}$ if $\operatorname{rank}[A \;\; B] = n$. The design procedure consists of two steps. In step 1, a nonsingular matrix $M \in \Re^{n\times n}$ is chosen so that the pair $(MA, MB)$ is controllable. In step 2, the feedback matrix $K$ is chosen so that the closed-loop matrix $A_c = A - BK$ has the desired eigenvalues. The procedure is illustrated by a simple example.

**Key words:** controllability; eigenvalues; assignment; linear system; feedback; procedure component.

## 1. INTRODUCTION

The concepts of controllability and observability introduced by Kalman [1,2] have been the basic notions of the modern control theory. It is well-known that if the linear system is controllable then, by the use of state feedback, it is possible to modify the dynamical properties of the closed-loop systems [1–12]. If the linear system is observable, then it is possible to design an observer which reconstructs the state vector of the system [1–12]. Descriptor systems of integer and fractional order have been analyzed in [6,11]. The stabilization of positive descriptor fractional linear systems with two different fractional orders by decentralized controller has been investigated in [11].

In this paper, it will be shown that it is possible to assign arbitrarily the eigenvalues of the closed-loop system with state feedback if $\operatorname{rank}[A \;\; B] = n$. In Section 2 it will be shown that if $\operatorname{rank}[A \;\; B] = n$, then there exists a nonsingular matrix $M \in \Re^{n\times n}$ such that the pair $(MA, MB)$ is controllable. Two procedures for the computation of the matrix $M \in \Re^{n\times n}$ will be proposed and illustrated by simple numerical examples in Section 3. Concluding remarks will be given in Section 4.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n \times m$ real matrices, $I_n$ – the $n \times n$ identity matrix.

## 2. CONTROLLABILITY OF LINEAR SYSTEMS

Consider the linear continuous-time system

$$\dot{x} = Ax + Bu, \tag{1a}$$

$$y = Cx, \tag{1b}$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ are the state, input, and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$.

**Definition 1.** [4, 5, 7, 9, 10, 13] The system (1) (the pair $(A, B)$ is called controllable if there exists an input $u(t) \in \Re^m$, $t \in [0\; t_f]$ which steers the state of the system from the initial state $x(0) \in \Re^n$ to the given final state $x_f = x(t_f)$.

**Theorem 1.** The system (1a) (the pair $(A, B)$ is controllable if and only if one of the following conditions is satisfied:

1. (Kalman condition)

$$\operatorname{rank}\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = n, \tag{2a}$$

2. (Hautus condition)

$$\operatorname{rank}\begin{bmatrix} I_n s - A & B \end{bmatrix} = n, \tag{2b}$$

for $s \in C$ (the field of complex numbers). In the proof of the main result of this section, the following theorem will be used.

**Theorem 2.** (Kronecker–Capelly theorem, [13]) The equation

$$Px = Q, \quad P \in \Re^{n\times p}, \quad Q \in \Re^{n\times q}, \quad n, p, q \geq 1 \tag{3}$$

has a solution $x \in \Re^{p\times q}$ if and only if

$$\operatorname{rank}\begin{bmatrix} P & Q \end{bmatrix} = \operatorname{rank}[P]. \tag{4}$$

**Theorem 3.** If the pair $(A, B)$ is uncontrollable but satisfies the condition

$$\operatorname{rank}\begin{bmatrix} A & B \end{bmatrix} = n, \quad A \in \Re^{n\times n}, \quad B \in \Re^{n\times m}, \tag{5}$$

then there exists a nonsingular matrix $M \in \Re^{n\times n}$ such that the pair

$$\left(\overline{A}, \overline{B}\right), \quad \overline{A} = MA, \quad \overline{B} = MB \tag{6}$$

is controllable.

**Proof.** From (6) we have

$$M\begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} \overline{A} & \overline{B} \end{bmatrix}, \tag{7}$$

where the pair $(\overline{A}, \overline{B})$ is controllable.

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From Theorem 2 applied to equation (7), it follows that there exists a nonsingular matrix $M$ satisfying (7) if the condition (5) holds. □

To compute the desired matrix $M$ the following procedures can be recommended.

**Procedure 1.** By choosing the controllable pair $(\overline{A}, \overline{B})$ and post-multiplying equation (7) by the transposed matrix $[A \;\; B]^T$, we obtain

$$M\left[AA^T \quad BB^T\right] = \overline{A}A^T + \overline{B}B^T. \tag{8}$$

The matrix

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} A^T \\ B^T \end{bmatrix} = AA^T + BB^T \tag{9}$$

is nonsingular since $\operatorname{rank}[A \;\; B] = n$. From (8) we have the desired matrix

$$M = \left(\overline{A}A^T + \overline{B}B^T\right)\left(AA^T + BB^T\right)^{-1}. \tag{10}$$

To find the desired matrix $M$ the following procedure can be also applied.

**Procedure 2.** Choose a nonsingular matrix $M \in \Re^{n\times n}$ and compute the matrix $\left[\overline{A} \;\; \overline{B}\right]$. Check if the pair $[A \;\; B]^T$ is controllable if it is not the case then repeat the procedure for a new matrix $M$.

**Example 1.** Consider the uncontrollable system (1) with the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{11}$$

From (11) it follows that it is impossible to stabilize the system by state feedback $u = Kx$, $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$. The system with (11) satisfies condition (5) since

$$\operatorname{rank}\begin{bmatrix} A & B \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} = 2 = n. \tag{12}$$

According to Procedure 1, we choose the controllable pair in the form

$$\overline{A} = \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}, \qquad \overline{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{13}$$

Using (10) and (13) we obtain

$$\begin{aligned} M &= \left(\overline{A}A^T + \overline{B}B^T\right)\left(AA^T + BB^T\right)^{-1} \\ &= \left[\begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}\right] \\ &\quad \times \left[\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}\right]^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}. \end{aligned} \tag{14}$$

According to Procedure 2, we choose the matrix $M$ for example in the form

$$M = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}. \tag{15}$$

In this case

$$\begin{aligned} \overline{A} &= MA = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}, \\ \overline{B} &= MB = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned} \tag{16}$$

The pair (16) is controllable since

$$\operatorname{rank}\begin{bmatrix} B & AB \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = 2 = n. \tag{17}$$

Note that if we choose

$$M = A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \tag{18}$$

then

$$\overline{A} = MA = A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \overline{B} = MB = \begin{bmatrix} 0 \\ -1 \end{bmatrix}. \tag{19}$$

and the pair $(\overline{A}, \overline{B})$ is uncontrollable.

The above considerations for the pair $(A, B)$ can be extended to the pair $(A, C)$ of the system (1).

**Definition 2.** [4, 5, 7, 9, 10, 13] The linear system (1) is called observable if knowing its input $u(t) \in \Re^m$ and its output $y(t) \in \Re^p$ for $t \in \begin{bmatrix} 0 & t_f \end{bmatrix}$ it is possible find its unique initial condition $x(0) \in \Re^n$.

**Theorem 4.** If the pair $(A, C)$ is unobservable but satisfies the condition

$$\operatorname{rank}\begin{bmatrix} A \\ C \end{bmatrix} = n, \qquad A \in \Re^{n\times n}, \quad C \in \Re^{p\times n}, \tag{20}$$

then there exists a nonsingular matrix $\overline{M} \in \Re^{n\times n}$ such that the pair

$$\left(\overline{A}, \overline{C}\right), \quad \overline{A} = A\overline{M}, \quad \overline{C} = C\overline{M} \tag{21}$$

is observable.

**The proof** is similar (dual) to the proof of Theorem 3.

## 3. STABILIZATION OF THE UNCONTROLLABLE LINEAR SYSTEMS BY STATE FEEDBACKS

Consider the linear system (1) with an uncontrollable pair $(A, B)$. We are looking for the state feedback matrix $K$ such that the closed-loop matrix

$$\hat{A}_c = \overline{A} - \overline{B}K \tag{22}$$

has the desired eigenvalues (Fig. 1).

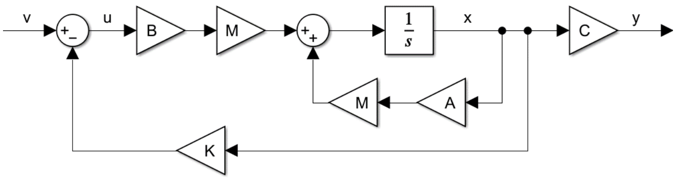


**Fig. 1.** Linear system with state feedback

First, we choose the matrix $M$ such that the pair

$$\overline{A} = MA, \qquad \overline{B} = MB \tag{23}$$

is controllable and next using one of the well-known approaches [3, 5, 6, 9, 10, 12] of the eigenvalues assignment we choose the matrix $K$ such that the matrix $\hat{A}_c$ has the desired eigenvalues.

To solve the problem the following procedure can be applied.

**Procedure 3.**

**Step 1.** Using the approach of Section 2 find the matrix $M$ such that the pair (23) is controllable.

**Step 2.** Using one of the well-known approaches of the eigenvalues assignment find the matrix $K$ such that the closed-loop matrix (22) has the desired eigenvalues.

**Example 2.** (Continuance of Example 1) For the uncontrollable system with the matrices (11) find the state feedback matrix $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ such that the close–loop matrix (22) has the eigenvalues $s_1 = -2$, $s_2 = -3$.

Using the procedure we obtain the following:

**Step 1.** For the matrix $M$ of the form (15), we have obtained the controllable pair

$$\overline{A} = \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{24}$$

**Step 2.** In this case, the close-loop matrix has the form

$$\begin{aligned} \hat{A}_c = \overline{A} - \overline{B}K &= \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \\ &= \begin{bmatrix} 2-k_1 & -1-k_2 \\ 3-k_1 & -1-k_2 \end{bmatrix} \end{aligned} \tag{25}$$

and

$$\begin{aligned} \det\left[I_2 s - \hat{A}_c\right] &= \begin{bmatrix} s-2+k_1 & 1+k_2 \\ -3+k_1 & s+1+k_2 \end{bmatrix} \\ &= s^2 + (k_1+k_2-1)s + k_2 + 1 \\ &= (s-s_1)(s-s_2) = s^2 - (s_1+s_2)s + s_1 s_2 \,. \end{aligned} \tag{26}$$

From (26) for the desired eigenvalues $s_1 = -2$, $s_2 = -3$ we have $k_1 = 1$, $k_2 = 5$ and the desired state feedback matrix has the form $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 1 & 5 \end{bmatrix}$.

## 4. CONCLUDING REMARKS

It has been shown that it is possible to assign arbitrarily the eigenvalues of the closed-loop system with state feedback if condition (5) is satisfied. It is also shown that if condition (5) is satisfied, then there exists a nonsingular matrix $M \in \Re^{n\times n}$ such that the pair $(MA, MB)$ is controllable (Theorem 3). Two procedures for the computation of the matrix $M \in \Re^{n\times n}$ have been proposed and illustrated by a simple numerical example. This approach can be extended to linear discrete-time linear systems and fractional orders linear systems.

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