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Fixed terminal time fractional optimal control problem for discrete time singular system

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This paper presents the formulation and numerical simulation for linear quadratic optimal control problem (LQOCP) of free terminal state and fixed terminal time fractional order discrete time singular system (FODSS). System dynamics is expressed in terms of Riemann-Liouville fractional derivative (RLFD), and performance index (PI) in terms of state and costate. Because of its complexity, finding analytical and numerical solutions to singular system (SS) is difficult. As a result, we use coordinate transformation to convert FODSS to its corresponding fractional order discrete time nonsingular system (FODNSS). After that, we obtain the necessary conditions by employing a Hamiltonian approach. The relevant conditions are solved using the general solution approach. For the analysis of formulation and solution algorithm, a numerical example is illustrated. Results are obtained for various α values. According to state of the art, this is the first time that a formulation and numerical simulation of free terminal state and fixed terminal time optimal control problem (OCP) of FODSS is presented.

Key words: fractional order differential equation, discrete time singular system, fractional derivative, linear quadratic optimal control problem

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1. Introduction

Physical systems described using fractional derivatives (FDs) are called fractional order systems and represent more accurate behavior [1–3]. Because FD is not a point property, it is a good tool for describing memory and heredity features of diverse systems. This is the fundamental benefit of employing FDs, as such effects are ignored in integer order representation [1]. It should be mentioned here that from the viewpoint of application, FDs emerge in control theory, signal processing, mechanics, electrical engineering, economics, rheology, electrochemistry, bioengineering, biophysics, biology, viscoelasticity, mechatronics, image processing, etc. [1–3]. FDs also appear in OCP.

When at least one FD term appears in either in PI or in system dynamics, or both, a dynamic optimization issue is reduced to a fractional optimal control problem (FOCP) [4]. RLFD and Caputo fractional derivative (CFD) are the most often used FDs. The system dynamics are described using RLFD in this paper.

Only a few works on FOCP have been documented in the literature. In this respect, a general formulation and numerical scheme for FOCPs has been introduced in [4]. In [5, 6], Biswas and Sen proposed formulation of FOCPs at different endpoint conditions. For the solution of state and control, shooting method and Grünwald-Letnikov approximation-based technique have been used. Authors in [7] proposed a solution scheme based on reflection operator for solving FOCPs described by RLFD or CFD. Authors in [8–10] proposed different numerical schemes based on modified Jacobi polynomials, semidefinite programming approach and collocation method along with properties of the Legendre multiwavelets for solving FOCPs. Lotfi et al. proposed different solution schemes, namely penalty variational method [11], Ritz-epsilon method [12], Legendre orthonormal polynomial based method [13], and Ritz-variational method [14] for the solution of constrained multidimensional FOCPs. In these papers, authors first transform the constrained FOCP into an unconstrained FOCP, and then obtain optimal solution. Effati et al. proposed solutions of FOCPs using neural network approach [15], variational iterative method [16], linear programming method [17], modified Adomian decomposition method [18], fixed point approach [19], and hybrid meshless method [20]. In literature, other existing solution schemes for solving FOCPs are based on Bernoulli polynomials in combination with a fractional integral operational matrix [21, 22], Bezier curve [23], hybrid functions [24], nonstandard finite difference [25], Haar wavelets collocation [26], Bernstein polynomials operational matrices of FDs [27], the Legendre wavelets [28], and closed form solution method [29]. Formulation of FOCP with control constraints is discussed in [30]. In [31–33], authors develop formulation and solution of FOCP of discrete-time systems. Some more works of discrete time FOCP are discussed in [34–36].

The above literature discusses the FOCPs of non-singular systems with distinct terminal conditions. SSs have a number of essential properties, including "consistent initial conditions, nonproperness of the transfer matrix, input derivatives in state dynamics, and noncausality" [37]. Because of these special characteristics, SSs are having particular importance, and we can find in various applications, including social, economic, biological and engineering systems [37].

Reported work in the literature on OCP of SSs is not much. Arora and Chauhan [38] present OCP of SSs using block pulse function. Authors in [39–41] discuss LQOCP for SSs. Mohan and Kar [42] propose a solution method for OCP of SSs utilizing shifted Legendre polynomials.

Only limited work is reported on FOCP of continuous time SSs. In this regard, a "pseudo state space" formulation for FOCP of SSs in terms of RLFD and CFD is introduced in [43, 44]. In literature, authors develop formulation in the sense of RLFD [45–47], CFD [48–50] and distinct numerical schemes [51–53] for FOCP of continuous time SSs at different endpoint conditions. Regarding FODSS, fixed terminal time and fixed terminal state OCP are discussed in [54]. However, free terminal state and fixed terminal time OCP of FODSS have not been discussed so far.

Formulation and numerical simulation for LQOCP of FODSS with free terminal state and fixed terminal time in terms of RLFD are presented in this paper. PI in terms of state and costate is taken into account. We convert the FODSS into its equivalent FODNSS by using transformation [55], and then necessary conditions are obtained. We solve the relevant conditions using the general solution method. An example is used to analyze the formulation and solution strategy.

The remaining part of the paper is framed as follows: In Section 2, LQOCP formulation of FODSS is presented. Numerical algorithm for free terminal state and fixed terminal time OCP of FODSS is presented in Section 3. In Section 4, numerical illustration is carried out for the analysis of formulation and solution algorithm. Eventually, in Section 5, the work's conclusions are presented.

2. LQOCP formulation of FODSS

Consider the FODSS described by Eq. (1) [56]

$$\boldsymbol{E}\Delta^{\alpha}x(k+1)=\boldsymbol{A}x(k)+\boldsymbol{B}u(k),\quad k\in\mathbb{Z}_{+}=\{0,1,\ldots\}, \tag{1}$$

where $\Delta^{\alpha}x(k)$ is given by [56]

$$\Delta^{\alpha}x(k)=\sum_{\iota=0}^{k}(-1)^{\iota}\binom{\alpha}{\iota}x(k-\iota), \tag{2}$$

where $u(k)\in\Re^{m}$, $x(k)\in\Re^{n}$ are the input and state vectors, $\boldsymbol{A}\in\Re^{n\times n}$, $\boldsymbol{E}\in\Re^{n\times n}$, $\boldsymbol{B}\in\Re^{n\times m}$ are the state, singular and input matrices.

Consider a feedback control law

$$u(k) = \boldsymbol{K}x(k) + \nu(k), \tag{3}$$

where $\boldsymbol{K} \in \Re^{m\times n}$ is the gain matrix and $\nu(k) \in \Re^{m}$ is new input vector. Choose the gain matrix $\boldsymbol{K}$ in order to satisfy the relation $\deg(|z\boldsymbol{E} - (\boldsymbol{A} + \boldsymbol{BK})|) = \operatorname{rank}(\boldsymbol{E})$.

By using Eq. (3), we can write Eq. (1) as

$$\boldsymbol{E}\Delta^{\alpha}x(k+1) = (\boldsymbol{A} + \boldsymbol{BK})\,x(k) + \boldsymbol{B}\nu(k). \tag{4}$$

Γ and Λ may be chosen in order to satisfy [37]

$$\Gamma\boldsymbol{E}\Lambda = \operatorname{diag}(I_o, 0), \qquad \Gamma(\boldsymbol{A} + \boldsymbol{BK})\Lambda = \operatorname{diag}(\boldsymbol{Z}, I), \qquad o = \operatorname{rank}(\boldsymbol{E}), \tag{5}$$

where Γ and Λ are non-singular.

We may choose coordinate transformation [37]

$$x(k) = \Lambda\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \qquad x_1(k) \in \Re^{o}, \quad x_2(k) \in \Re^{n-o}. \tag{6}$$

By considering Eqs. (5) and (6), the Eq. (4) is modified as

$$\begin{bmatrix} I_o & 0 \\ 0 & 0 \end{bmatrix}\Delta^{\alpha}\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \boldsymbol{Z} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \end{bmatrix}\nu(k). \tag{7}$$

By considering Eq. (2), the Eq. (7) is modified as

$$\begin{aligned} x_1(k+1) &= \sum_{\iota=0}^{k}\sigma(\iota)x_1(k-\iota) + \boldsymbol{B}_1\nu(k), \\ 0 &= x_2(k) + \boldsymbol{B}_2\nu(k), \end{aligned} \tag{8}$$

where $\sigma(0) = \boldsymbol{Z} + \alpha\, I$, $\sigma(\iota) = (-1)^{\iota}\binom{\alpha}{\iota+1} I$, $\iota = 1, 2, \ldots, k$.

We consider quadratic PI as

$$J = \sum_{k=0}^{N-1}\left[x^T(k)\boldsymbol{\Phi}x(k) + u^T(k)\boldsymbol{\Theta}u(k)\right], \tag{9}$$

where $\boldsymbol{\Phi} \in \Re^{n\times n} > 0$ and $\boldsymbol{\Theta} \in \Re^{m\times m} > 0$.

By considering Eqs. (6) and (8) we write

$$\begin{aligned} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} &= \begin{bmatrix} I & 0 \\ \boldsymbol{K} & I \end{bmatrix}\begin{bmatrix} x(k) \\ \nu(k) \end{bmatrix} = \begin{bmatrix} \Lambda & 0 \\ \boldsymbol{K}\Lambda & I \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \\ \nu(k) \end{bmatrix} \\ &= \begin{bmatrix} \Lambda & 0 \\ \boldsymbol{K}\Lambda & I \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & -\boldsymbol{B}_2 \\ 0 & I \end{bmatrix}\begin{bmatrix} x_1(k) \\ \nu(k) \end{bmatrix}. \end{aligned} \tag{10}$$

By using Eq. (10), we can modify Eq. (9) as

$$
\begin{aligned}
J &= \sum_{k=0}^{N-1} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} \Phi & 0 \\ 0 & \Theta \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \\
&= \sum_{k=0}^{N-1} \left\{ \begin{bmatrix} x_1(k) \\ \nu(k) \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & -\boldsymbol{B}_2 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} \Lambda & 0 \\ \boldsymbol{K}\Lambda & I \end{bmatrix}^T \begin{bmatrix} \Phi & 0 \\ 0 & \Theta \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ \boldsymbol{K}\Lambda & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -\boldsymbol{B}_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1(k) \\ \nu(k) \end{bmatrix} \right\} \\
&= \sum_{k=0}^{N-1} \begin{bmatrix} x_1(k) \\ \nu(k) \end{bmatrix}^T \begin{bmatrix} \overline{\boldsymbol{\Phi}} & \Sigma \\ \Sigma^T & \overline{\boldsymbol{\Theta}} \end{bmatrix} \begin{bmatrix} x_1(k) \\ \nu(k) \end{bmatrix} \\
&= \sum_{k=0}^{N-1} \left[x_1^T(k)\overline{\boldsymbol{\Phi}}x_1(k) + x_1^T(k)\Sigma\nu(k) + \nu^T(k)\Sigma^T x_1(k) + \nu^T(k)\overline{\boldsymbol{\Theta}}\nu(k) \right].
\end{aligned}
$$

Here we can note that the matrix $\begin{bmatrix} \Phi & 0 \\ 0 & \Theta \end{bmatrix}$ is symmetric positive definite (SPD) and matrix $\begin{bmatrix} \Lambda & 0 \\ \boldsymbol{K}\Lambda & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -\boldsymbol{B}_2 \\ 0 & I \end{bmatrix}$ is of full column rank. Therefore, the matrix

$$
\begin{bmatrix} \overline{\boldsymbol{\Phi}} & \Sigma \\ \Sigma^T & \overline{\boldsymbol{\Theta}} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & -\boldsymbol{B}_2 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} \Lambda & 0 \\ \boldsymbol{K}\Lambda & I \end{bmatrix}^T \begin{bmatrix} \Phi & 0 \\ 0 & \Theta \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ \boldsymbol{K}\Lambda & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -\boldsymbol{B}_2 \\ 0 & I \end{bmatrix}
$$

is SPD and, therefore, are matrices $\overline{\boldsymbol{\Phi}}$ and $\overline{\boldsymbol{\Theta}}$ [37, 51].

Finally, PI becomes

$$
J = \sum_{k=0}^{N-1} \left[x_1^T(k)\widetilde{\Phi}x(k) + \vartheta^T(k)\overline{\boldsymbol{\Theta}}\vartheta(k) \right], \tag{11}
$$

where $\widetilde{\Phi} = \overline{\boldsymbol{\Phi}} - \Sigma\overline{\boldsymbol{\Theta}}^{-1}\Sigma^T$, $\vartheta(k) = \nu(k) + \overline{\boldsymbol{\Theta}}^{-1}\Sigma^T x_1(k)$ and

$$
\begin{bmatrix} \widetilde{\Phi} & 0 \\ 0 & \overline{\boldsymbol{\Theta}} \end{bmatrix} = \begin{bmatrix} I & -\Sigma\overline{\boldsymbol{\Theta}}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \overline{\boldsymbol{\Phi}} & \Sigma \\ \Sigma^T & \overline{\boldsymbol{\Theta}} \end{bmatrix} \begin{bmatrix} I & -\Sigma\overline{\boldsymbol{\Theta}}^{-1} \\ 0 & I \end{bmatrix}^T.
$$

In $\begin{bmatrix} \widetilde{\Phi} & 0 \\ 0 & \overline{\boldsymbol{\Theta}} \end{bmatrix} = \begin{bmatrix} I & -\Sigma\overline{\boldsymbol{\Theta}}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \overline{\boldsymbol{\Phi}} & \Sigma \\ \Sigma^T & \overline{\boldsymbol{\Theta}} \end{bmatrix} \begin{bmatrix} I & -\Sigma\overline{\boldsymbol{\Theta}}^{-1} \\ 0 & I \end{bmatrix}^T$, the matrix $\begin{bmatrix} I & -\Sigma\overline{\boldsymbol{\Theta}}^{-1} \\ 0 & I \end{bmatrix}$ is nonsingular, and the matrix $\begin{bmatrix} \overline{\boldsymbol{\Phi}} & \Sigma \\ \Sigma^T & \overline{\boldsymbol{\Theta}} \end{bmatrix}$ is SPD. Therefore, the matrix $\begin{bmatrix} \widetilde{\Phi} & 0 \\ 0 & \overline{\boldsymbol{\Theta}} \end{bmatrix}$ is SPD and so is matrix $\widetilde{\Phi}$ [37, 51].

Substitute $\nu(k) = \vartheta(k) - \overline{\boldsymbol{\Theta}}^{-1}\Sigma^T x_1(k)$ in Eq. (8), we get

$$x_1(k+1) = \sum_{\iota=0}^{k} \rho(\iota) x_1(k-\iota) + \boldsymbol{B}_1 \vartheta(k), \tag{12}$$

where

$$\rho(0) = \boldsymbol{Z} + \alpha I - \boldsymbol{B}_1 \overline{\boldsymbol{\Theta}}^{-1}\Sigma^T, \qquad \rho(\iota) = (-1)^{\iota} \binom{\alpha}{\iota+1} I, \quad \iota = 1, 2, \ldots, k.$$

Therefore, the FODSS given by Eq. (1) is transformed into its equivalent FODNSS given by Eq. (12). We can now use OCP strategy to generate a new control vector $\vartheta(k)$ that minimizes the new PI defined by Eq. (11).

By using Lagrange's multiplier $\lambda(k)$, we write augmented PI (J_a) as

$$\begin{aligned} J_a = \sum_{k=0}^{N-1} \Bigg(& x_1^T(k) \widetilde{\Phi} x_1(k) + \vartheta^T(k) \overline{\boldsymbol{\Theta}} \vartheta(k) \\ & + \left[\sum_{\iota=0}^{k} \rho(\iota) x_1(k-\iota) + \boldsymbol{B}_1 \vartheta(k) - x_1(k+1) \right]^T \lambda(k+1) \Bigg). \end{aligned}$$

Hamiltonian function can be defined as

$$\begin{aligned} \mathrm{H}(k) = {} & x_1^T(k) \widetilde{\Phi} x_1(k) + \vartheta^T(k) \overline{\boldsymbol{\Theta}} \vartheta(k) \\ & + \left[\sum_{\iota=0}^{k} \rho(\iota) x_1(k-\iota) + \boldsymbol{B}_1 \vartheta(k) \right]^T \lambda(k+1). \end{aligned}$$

By using Hamiltonian, we write J_a as

$$J_a = x_1^T(0)\lambda(0) - x_1^T(N)\lambda(N) + \sum_{k=0}^{N-1} \left[\mathrm{H}(k) - x_1^T(k)\lambda(k) \right].$$

We can write the first variation of J_a as

$$\begin{aligned} \delta J_a = -\lambda(N)\delta x_1^T(N) + \sum_{k=0}^{N-1} \Bigg[& \left(\frac{\partial \mathrm{H}(k)}{\partial x_1^T(k)} - \lambda(k) \right) \delta x_1^T(k) + \frac{\partial \mathrm{H}(k)}{\partial \vartheta^T(k)} \delta \vartheta^T(k) \\ & + \left(\frac{\partial \mathrm{H}(k-1)}{\partial \lambda^T(k)} - x_1(k) \right) \delta \lambda^T(k) \Bigg]. \end{aligned}$$

For optimum $\delta J_a = 0$ [4]. Which yields

$$x_1(k+1) = \frac{\partial \mathrm{H}(k)}{\partial \lambda^T(k+1)} = \sum_{\iota=0}^{k} \rho(\iota) x_1(k-\iota) + \boldsymbol{B}_1 \vartheta(k), \tag{13}$$

$$\lambda(k) = \sum_{k=0}^{N-1} \frac{\partial \mathrm{H}(k)}{\partial x_1^T(k)} = \left[\widetilde{\Phi} + \widetilde{\Phi}^T\right] x_1(k) + \sum_{\iota=0}^{N-k-1} \rho^T(\iota) \lambda(k+\iota+1), \tag{14}$$

$$\frac{\partial \mathrm{H}(k)}{\partial \vartheta^T(k)} = 0 \quad \Rightarrow \quad \vartheta(k) = -\left[\overline{\boldsymbol{\Theta}} + \overline{\boldsymbol{\Theta}}^T\right]^{-1} \boldsymbol{B}_1^T \lambda(k+1). \tag{15}$$

Finally, δJ_a becomes $-\lambda(N)\delta x_1^T(N) = 0$.

3. Numerical algorithm

A solution strategy [31–33] is described here in order to solve optimal conditions (13), (14) and (15) with the given initial condition $x(k=0) = x_0$.

Apply the z-transform to Eqs. (13) and (15) by considering the initial condition, and then apply the inverse z-transform, we can obtain the solution of state equation (13) as

$$x_1(k) = \hbar(k) x_1(0) - \sum_{\iota=0}^{k-1} \hbar(k-\iota-1) \boldsymbol{B}_1 \left[\overline{\boldsymbol{\Theta}} + \overline{\boldsymbol{\Theta}}^T\right]^{-1} \boldsymbol{B}_1^T \lambda(\iota+1), \tag{16}$$

where $\hbar(0) = I_n$, $\hbar(k) = \sum_{\iota=0}^{k-1} \rho(\iota)\hbar(k-\iota-1)$.

We can write Eq. (16) in matrix form as

$$\begin{bmatrix} x_1(1) \\ \vdots \\ x_1(N) \end{bmatrix} = \begin{bmatrix} \hbar(1) \\ \vdots \\ \hbar(N) \end{bmatrix} x_1(0) + \begin{bmatrix} \hbar(0) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \hbar(N-1) & \cdots & \hbar(0) \end{bmatrix} \left[-\boldsymbol{B}_1 \left[\overline{\boldsymbol{\Theta}} + \overline{\boldsymbol{\Theta}}^T\right]^{-1} \boldsymbol{B}_1^T\right] \begin{bmatrix} \lambda(1) \\ \vdots \\ \lambda(N) \end{bmatrix}. \tag{17}$$

The matrix form of Eq. 14) is

$$\begin{bmatrix}\lambda(1)\\ \vdots \\ \lambda(N)\end{bmatrix} = \begin{bmatrix}\hbar^T(N-1)\\ \vdots \\ \hbar^T(0)\end{bmatrix}\lambda(N) + \begin{bmatrix}0 & \hbar^T(0) & \cdots & \hbar^T(N-2)\\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \hbar^T(0)\\ 0 & 0 & \cdots & 0\end{bmatrix}\left[\widetilde{\Phi}+\widetilde{\Phi}^T\right]\begin{bmatrix}x_1(0)\\ \vdots \\ x_1(N-1)\end{bmatrix}. \tag{18}$$

In the present case terminal state $x(N)$ is free, therefore, the variation $\delta x^T(N) \neq 0$. Therefore, the transversality condition is $\lambda(N) = 0$.

By using $\lambda(N) = 0$ in Eq. (18), we get

$$\begin{bmatrix}\lambda(1)\\ \vdots \\ \lambda(N)\end{bmatrix} = \begin{bmatrix}0 & \hbar^T(0) & \cdots & \hbar^T(N-2)\\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \hbar^T(0)\\ 0 & 0 & \cdots & 0\end{bmatrix}\left[\widetilde{\Phi}+\widetilde{\Phi}^T\right]\begin{bmatrix}x_1(0)\\ \vdots \\ x_1(N-1)\end{bmatrix}. \tag{19}$$

By considering $\tau_1 = \left[\overline{\mathbf{\Theta}}+\overline{\mathbf{\Theta}}^T\right]$ and $\tau_2 = \left[\widetilde{\Phi}+\widetilde{\Phi}^T\right]$, Eq. (19) can be modified as

$$\begin{bmatrix}\lambda(1)\\ \lambda(2)\\ \vdots \\ \lambda(N-1)\\ \lambda(N)\end{bmatrix} = \begin{bmatrix}\hbar^T(0)\tau_2 & \hbar^T(1)\tau_2 & \cdots & \hbar^T(N-2)\tau_2 & 0\\ 0 & \hbar^T(0)\tau_2 & \cdots & \hbar^T(N-3)\tau_2 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & \hbar^T(0)\tau_2 & 0\\ 0 & 0 & \cdots & 0 & 0\end{bmatrix}\begin{bmatrix}x_1(1)\\ x_1(2)\\ \vdots \\ x_1(N-1)\\ x_1(N)\end{bmatrix}. \tag{20}$$

Equation (17) can be modified by using Eq. (20) as

$$\begin{bmatrix}\psi_{11} & \psi_{12} & \cdots & \psi_{1,N-1} & \psi_{1,N}\\ \psi_{21} & \psi_{22} & \cdots & \psi_{2,N-1} & \psi_{2,N}\\ \vdots & \vdots & \ddots & \cdots & \vdots\\ \psi_{N-1,1} & \psi_{N-1,2} & \cdots & \psi_{N-1,N-1} & \psi_{N-1,N}\\ \psi_{N,1} & \psi_{N,2} & \cdots & \psi_{N,N-1} & \psi_{N,N}\end{bmatrix}\begin{bmatrix}x_1(1)\\ x_1(2)\\ \vdots \\ x_1(N-1)\\ x_1(N)\end{bmatrix} = \begin{bmatrix}\hbar(1)\\ \hbar(2)\\ \vdots \\ \hbar(N-1)\\ \hbar(N)\end{bmatrix}x_1(0), \tag{21}$$

where

$$\begin{bmatrix} \psi_{11} & \psi_{12} & \cdots & \psi_{1,N-1} & \psi_{1,N} \\ \psi_{21} & \psi_{22} & \cdots & \psi_{2,N-1} & \psi_{2,N} \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ \psi_{N-1,1} & \psi_{N-1,2} & \cdots & \psi_{N-1,N-1} & \psi_{N-1,N} \\ \psi_{N,1} & \psi_{N,2} & \cdots & \psi_{N,N-1} & \psi_{N,N} \end{bmatrix} = \begin{bmatrix} I_n & 0 & \cdots & 0 & 0 \\ 0 & I_n & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_n & 0 \\ 0 & 0 & \cdots & 0 & I_n \end{bmatrix}$$

$$+ \begin{bmatrix} \hbar(0) & 0 & \cdots & 0 & 0 \\ \hbar(1) & \hbar(0) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hbar(N-2) & \hbar(N-3) & \cdots & \hbar(0) & 0 \\ \hbar(N-1) & \hbar(N-2) & \cdots & \hbar(1) & \hbar(0) \end{bmatrix} \boldsymbol{B}_1 \tau_1^{-1} \boldsymbol{B}_1^T$$

$$+ \begin{bmatrix} \hbar^T(0)\tau_2 & \hbar^T(1)\tau_2 & \cdots & \hbar^T(N-2)\tau_2 & 0 \\ 0 & \hbar^T(0)\tau_2 & \cdots & \hbar^T(N-3)\tau_2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \hbar^T(0)\tau_2 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Optimal state vector $x_1(k)$ can be obtained from the Eq. (21) as

$$\begin{bmatrix} x_1(1) \\ x_1(2) \\ \vdots \\ x_1(N-1) \\ x_1(N) \end{bmatrix} = \begin{bmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1,N-1} & \mu_{1,N} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2,N-1} & \mu_{2,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{N-1,1} & \mu_{N-1,2} & \cdots & \mu_{N-1,N-1} & \mu_{N-1,N} \\ \mu_{N,1} & \mu_{N,2} & \cdots & \mu_{N,N-1} & \mu_{N,N} \end{bmatrix} \begin{bmatrix} \hbar(1) \\ \hbar(2) \\ \vdots \\ \hbar(N-1) \\ \hbar(N) \end{bmatrix} x_1(0), \qquad (22)$$

where

$$\begin{bmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1,N-1} & \mu_{1,N} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2,N-1} & \mu_{2,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{N-1,1} & \mu_{N-1,2} & \cdots & \mu_{N-1,N-1} & \mu_{N-1,N} \\ \mu_{N,1} & \mu_{N,2} & \cdots & \mu_{N,N-1} & \mu_{N,N} \end{bmatrix} = \begin{bmatrix} \psi_{11} & \psi_{12} & \cdots & \psi_{1,N-1} & \psi_{1,N} \\ \psi_{21} & \psi_{22} & \cdots & \psi_{2,N-1} & \psi_{2,N} \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ \psi_{N-1,1} & \psi_{N-1,2} & \cdots & \psi_{N-1,N-1} & \psi_{N-1,N} \\ \psi_{N,1} & \psi_{N,2} & \cdots & \psi_{N,N-1} & \psi_{N,N} \end{bmatrix}^{-1}.$$

Co-state vector $\lambda(k)$ can be obtained by substituting Eq. (22) in Eq. (20). Once $\lambda(k)$ is known, we can obtain $\vartheta(k)$ by using Eq. (15). After getting $\vartheta(k)$, we can obtain $\nu(k)$ by using the relation $\nu(k) = \vartheta(k) - \overline{\boldsymbol{\Theta}}^{-1}\Sigma^T x_1(k)$. Thereafter, $u(k)$ and $x_2(k)$ can be obtained by using the Eqs. (3) and (8).

4. Numerical illustration

Consider a FODSS described by

$$\boldsymbol{E}\Delta^{\alpha}x(k+1) = \boldsymbol{A}x(k) + \boldsymbol{B}u(k),$$

where

$$\boldsymbol{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

which optimizes the PI

$$J = \sum_{k=0}^{N-1} \left[x^T(k)\Phi x(k) + u^T(k)\Theta u(k) \right],$$

where

$$\Phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \Theta = [2]$$

with the given conditions as

$$x_1(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad N = 10.$$

Matrices $\boldsymbol{K}$, Γ and Λ may be chosen as $\boldsymbol{K} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, $\Gamma = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ [57].

Let $x_1(k) = \begin{bmatrix} x_{11}(k) \\ x_{12}(k) \end{bmatrix}$, $\begin{bmatrix} x_{11}(0) \\ x_{12}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Figures 1–5 shows the results obtained using foregoing considerations.

The problem is solved for various α values. The optimal states, optimal control, and minimum value of PI for the free terminal state problem are shown in Figures 1-5. The amplitudes of optimal states, optimal control, and minimum value of PI in these responses increase as α is increased. From this, we observed that PI reduces as α is decreased and demands for small control effort. As a result, we argue that considering FOCP can provide numerous advantages over equivalent integer order OCP.

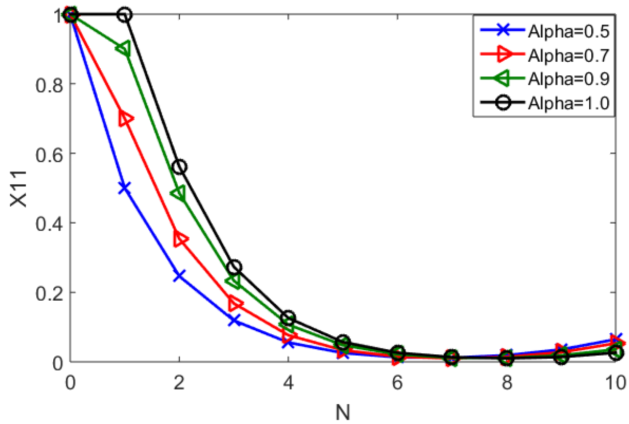

Figure 1: Optimal state x_{11} for $\alpha = 0.5,\ 0.7,\ 0.9,\ 1.0$

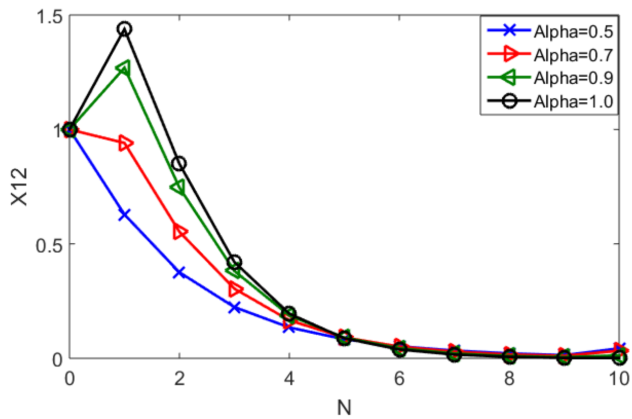

Figure 2: Optimal state x_{12} for $\alpha = 0.5,\ 0.7,\ 0.9,\ 1.0$

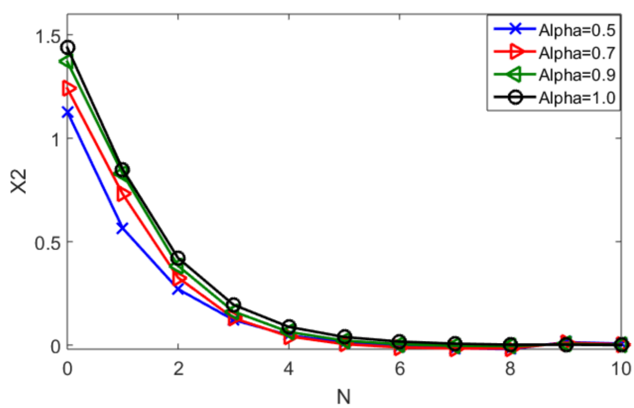

Figure 3: Optimal state x_2 for $\alpha = 0.5,\ 0.7,\ 0.9,\ 1.0$

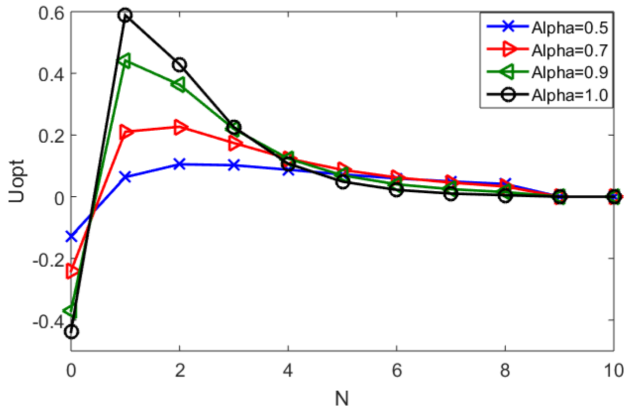

Figure 4: Optimal control u_{opt} for $\alpha = 0.5,\ 0.7,\ 0.9,\ 1.0$

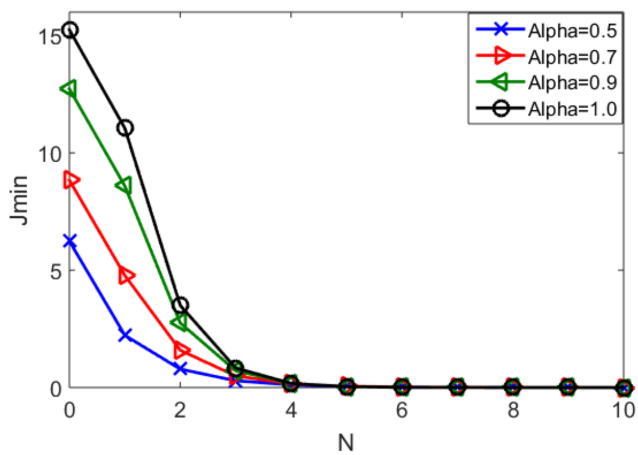

Figure 5: Minimum value of PI $J_{\min}$ for $\alpha = 0.5,\ 0.7,\ 0.9,\ 1.0$

5. Conclusions

LQOCP formulation and numerical algorithm for FODSS has been discussed in this work. PI is considered in quadratic form. FDEs are described in the sense of RLFD. By using transformation, we have converted FODSS into its equivalent FODNSS and then applied optimal control theory for obtaining necessary conditions. The necessary conditions are solved using a general solution approach. For various values of α, optimal states, optimal control, and the minimum value of PI are determined. As a result of the findings, we noticed that as α rises, the amplitudes of states and control rise as well. We also observe that when α decreases, then minimum value of PI is decreased. From this, we argue that considering FOCP can provide numerous advantages over equivalent integer order OCP. According to author's knowledge, this is the first time a formulation and numerical simulation of free terminal state and fixed terminal time OCP of FODSS is presented.

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