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ON THE SEPARATION OF ELASTIC STRAIN ENERGY IN THE GENERAL CASE OF ANISOTROPY: A DIRECT APPROACH

A DIRECT APPROACH to the problem of the separation of elastic strain energy in the case of generally anisotropic materials is described in the present work. It is based on a simple analysis of the strain tensor into a spherical and a deviatoric one. A definition of dilatational and distortional elastic strain densities is introduced, based on the consideration of the geometrical response of a material. Through the generalized Hooke's law, analytic expressions are obtained for the generally anisotropic materials. The present results coincide with the only available in the literature data for anisotropic materials with cubic symmetry. In addition, an application for transversally isotropic materials is presented.

1. Introduction

The separation of elastic strain energy density in the general case of anisotropic materials is a problem long pending for solution. The persisting interest for this solution is judged by its great importance for the description of the limiting surfaces in case of any type of materials. As early as in the first decade of twentieth century Huber [4], Hencky [2] and von Mises [19], [20] stated that a material fails when the distortional strain energy density takes a critical value. This type of failure is called ductile, as it is caused by extensive plastic flow. However, the second main type of failure, that is brittle fracture, is not covered by the Huber-Hencky-Mises (HHM) criterion. It is out of doubt that brittle fracture must be connected with the remaining part of elastic strain energy, i.e. the dilatational one, as it dictates the principle of conservation of energy. Thus, the separation of the elastic strain energy density into its two parts carries high technological importance for a complete description of failure. Application of the HHM-criterion is

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trivial in isotropic materials but not yet available in the general case of anisotropy. Instead, heuristic criteria were introduced, (for example Hill [3], Theocaris [15]), where the general second order surface in the stress space was assumed to be the limiting one. Its nine coefficients are calibrated through experiments and/or symmetry considerations. Rigorous efforts, mainly from the 'Polish School', (Burzyński [1], Rychlewski [10], [11], [12], Olszak et al. [9], Kowalczyk et al. [5], [6]) and from elsewhere (Sutcliffe [14], Tsai et al. [18], Theocaris et al. [16], [17]) resulted to solutions in case of cubic symmetry, only. These solutions cannot be generalized in case of general anisotropy. More disappointing is the conclusion that construction of such a general solution is impossible (e.g. [6], [14]).

However, in a classical book on Continuum Mechanics (Malvern, [8]) it is said that there exists at least one loading system causing pure volume changes and at least one loading system causing pure shape changes in a linear elastic generally anisotropic material. So, there exist two extreme cases of purely dilatational and purely distortional strain energy densities stored in a material. In a more straightforward way, Burzyński [1] stated that *there is no physical reason against the introduction of the decomposition of the elastic energy into these two components* (i.e. dilatational and distortional) in case of anisotropic bodies. Burzyński's statement reflects perfectly the Eucleidian nature of material kinematics, where the only admissible modes of deformation are either linear (dilatation) or angular (distortion). Consequently, the failure to construct a general solution is strange enough, because the above mentioned attempts were developed on faultless theoretical bases. Thus, if any 'error' exists, it must be hidden in the common departing point of the respective analyses. This common departing point is the decomposition of the stress tensor into a sum of terms by means of spectral analysis [5], [10], [14], [16], [17].

Here, a direct approach to this problem is described, which is based on a simple analysis of the strain tensor into a spherical and a distortional part, leaving untouched the stress tensor. Then, the evaluation of the respective two strain energy density parts is performed through the constitutive equations of the general anisotropic material, by means of simple tensor algebra.

2. Theoretical considerations

It is assumed, here, that the decomposition of the stress tensor into a spherical and a deviatoric part is the root of our not fully successful attempts to solve the problem of the separation of elastic strain energy density into two parts connected with dilatation and distortion, respectively.

To clarify our hypothesis, we make use of a simple example: Consider a cube from a hypothetical anisotropic material with, say, Young's moduli E_1, E_2 and E_3 and (in order to simplify the generalized Hooke's law) Poisson ratios $\nu_1 = \nu_2 = \nu_3 = 0$. Pure dilatation can be achieved in this cube by applying normal stresses proportional to $1/E_1$, E_2/E_1 and E_3/E_1, respectively. The corresponding diagonal stress tensor, say σ_{ij}^V, is not spherical, although the strain tensor is. The scalar product of these two tensors must be, by physical considerations, pure dilatational strain energy. So, σ_{ij}^V can be considered as the *'... one loading system causing pure volume changes...'* [8]. By a similar way, a second simple loading system, say σ_{ij}^D, can cause pure distortion in the same cube. Simultaneous application of these two loading systems, or any linear combination $\sigma_{ij} = \lambda\sigma_{ij}^V + \mu\sigma_{ij}^D$ of them, causes two discrete, recognizable and measurable changes in the cube, that is, volume and shape changes. This way, an infinite number of loading systems σ_{ij} can be constructed, all of them having in common the property of discrete and a priori quantitatively known development of dilatational and distortional strain energy. So, Burzyński's [1] statement is completely meaningful. However, neither σ_{ij}^V nor σ_{ij}^D are spherical.

The opposite is not true. A given arbitrary stress tensor σ_{ij} cannot *a priori* be decomposed into *two not spherical parts*, σ_{ij}^V and σ_{ij}^D, with the first of them causing pure dilatation. To do this, *strains must be considered at first place*. Otherwise, the physical phenomenon is incomprehensible in terms of stresses and this 'inverse' problem seems to be unsolved. Usual types of stress tensor decomposition result to a sum including one spherical part. This is a common result in the decomposition of any physical quantity, where the first component is, always, a kind of 'mean value' of the quantity. The 'mean value' in case of the stress tensor is, exactly, its spherical part. It is, hence, obvious that application of this spherical stress tensor onto the as above anisotropic cube cannot cause pure dilatation, only. Pure dilatation is achieved by the action of a *'quasi-spherical'* (in the sense of *'acting as spherical'*) stress tensor, like σ_{ij}^V, which is obtained by strains considerations. This quasi-spherical stress tensor in the present example is:

$$\sigma_{ij}^V = \frac{\sigma_{ij}}{E_1} \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix} \tag{1}$$

which is diagonal *but not spherical*.

In other words, stress tensor decomposition is equivalent to decomposing σ_{ij}, which, in general, is the sum of *two not spherical* (but properly obtained through strains) *tensors*, into a sum containing a *spherical term*.

This self-contradicting situation could necessitate a further, 'second order' decomposition of both σ_{ij}^{V} and σ_{ij}^{D} to check whether or not they contain any 'hidden' spherical parts. If so, the physical meaning of decomposition is questionable, as far as more than two parts for σ_{ij} are obtained, some of them showing mixed character. A good example of the mixed character of the decomposition of the stress tensor, without an a priori consideration of strains, is given by Kowalczyk and Ostrowska-Maciejewska [9] where the first term is spherical but an additional 'mixed' term of the form σI.C.s $\neq 0$ is obtained.

To implement the as above approach, let the stress and strain tensors, in a linear elastic material, be σ_{ij} and ε_{ij}, respectively. The strain tensor, in any linear elastic material, can always be separated into a spherical e_{ij} and a distortional $\bar{\varepsilon}_{ij}$ part [7], [8], with:

$$e_{ij} = \frac{1}{3}\varepsilon_{ii}\delta_{ij} = \begin{pmatrix} \boldsymbol{e}_m & 0 & 0 \\ 0 & \boldsymbol{e}_m & 0 \\ 0 & 0 & \boldsymbol{e}_m \end{pmatrix} \tag{2}$$

$$\bar{\varepsilon}_{ij} = \varepsilon_{ij} - e_{ij} = \begin{pmatrix} \varepsilon_{xx} - \boldsymbol{e}_m & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} - \boldsymbol{e}_m & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} - \boldsymbol{e}_m \end{pmatrix} \tag{3}$$

$$\boldsymbol{e}_m = \frac{\varepsilon_{ii}}{3} = \frac{1}{3}(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) = \frac{1}{3}\mathbf{I}_\varepsilon. \tag{4}$$

Here, $\boldsymbol{e}_m$ is the mean dilatation, $\mathbf{I}_\varepsilon$ is the first invariant of the strain tensor and δ_{ij} is the Kronecker delta.

The total elastic strain energy density is given by the scalar products [7]:

$$\begin{cases} T = \frac{1}{2}\sigma_{ij}.\varepsilon_{ij} = \frac{1}{2}\sigma_{ij}.\left(e_{ij} + \bar{\varepsilon}_{ij}\right) = \frac{1}{2}\sigma_{ij}.e_{ij} + \frac{1}{2}\sigma_{ij}.\bar{\varepsilon}_{ij} \text{ or equivalently:} \\ T = \frac{1}{2}\sigma_{ij}.\varepsilon_{ij} = \frac{1}{2}\left(\sigma_m + s_{ij}\right).\varepsilon_{ij} = \frac{1}{2}\sigma_m.\varepsilon_{ij} + \frac{1}{2}s_{ij}.\varepsilon_{ij} \end{cases} \tag{5}$$

where σ_m and s_{ij} are the hydrostatic and deviatoric stress components, respectively. Both expressions hold for any type of linear elastic material, as far as constitutive equations are not yet introduced. Usually, the departing point is somehow different, as indicated by Eq.(6), (see for example Kowalczyk and al. [9])

$$T = \frac{1}{2}\sigma_{ij}.\varepsilon_{ij} = \frac{1}{2}\sigma_{ij}.C_{ijkl}.\sigma_{kl} = \sigma^2\text{I.C.} + \text{s.C.s} + 2\sigma\text{I.C.s} \tag{6}$$

At this point, it must be emphasized that there is no reason to decompose the stress tensor as in Eq.(6), because physical considerations, mentioned in the introductory example, indicate that it drives to physically un-interpretable results. Actually, this is the only difference between the present analysis and previous work. In considering the physical meaning of deformation and its relation with the elastic potential and strain energies it was concluded (as far as our knowledge and intuition imply) that Burzyński's statement is meaningful and there is *no mathematical or physical reason to decompose the stress tensor*. Such a reason, realized by Eq.(6) or similar, emerges in cases where the relationship between material structure (crystallographic classes etc.) and stress fields (driving to multiple stress-triads etc.) is sought. This is not the present case. Here, the aim is to separate, *macroscopically*, strain energy density into two components, acting differently onto the material, in order to obtain a failure surface covering both types (brittle fracture and plastic rupture) in a unified manner. This self-limitation allows for a helpful assumption, namely:

The material has no texture and detailed description of stresses-strains in its body is not necessary. Consequently, both dilatational and distortional strain energy densities are equal to the respective parts of the external work done by the loading system, as far as the material is elastic and dissipative phenomena do not appear.

So, based on the first of Eqs.(5), we can introduce two definitions:

***Definition* 1:** *Dilatational strain energy density, T_V, is the elastic energy corresponding to the general stress tensor σ_{ij} acting on the spherical strain tensor e_{ij}.*

T_V can easily be evaluated by the scalar product of the stress tensor with the spherical strain tensor, i.e.:

$$T_V = \frac{1}{2}\sigma_{ij}.e_{ij} \tag{7}$$

The principle of energy conservation and the present introductory comments, concerning the two geometrically admissible modes of material deformation, dictate that the remaining part of strain energy density must be responsible for distortions, as far as *tertium non est*. Consequently, it is allowed to state:

***Definition* 2:** *Distortional strain energy density, T_D, is the elastic energy corresponding to the general stress tensor σ_{ij} acting on the deviatoric strain tensor $\bar{\varepsilon}_{ij}$.* Equivalently:

Distortional strain energy density, T_D, is the part of the elastic strain energy, remaining after the subtraction of the dilatational strain energy, i.e.:

$$T_D = T - T_V = \frac{1}{2}\sigma_{ij}.\left(e_{ij} + \bar{\varepsilon}_{ij}\right) - \frac{1}{2}\sigma_{ij}.e_{ij} = \frac{1}{2}\sigma_{ij}.\bar{\varepsilon}_{ij} \tag{8}$$

These definitions are proper in the sense that e_{ij} represents pure dilatation (the first mode of admissible deformation) and the strain energy associated with it must be dilatational. Furthermore, the principle of energy conservation necessitates that the remaining energy must be spent for the only remaining second geometrically admissible mode of deformation. In any case, their ultimate validity will be judged by their results.

It is evident from Eqs. (5) (7) and (8) that for $\varepsilon_{ij} = e_{ij}$ (change of volume only) $T_D = 0$ and for $\varepsilon_{ij} = \bar{\varepsilon}_{ij}$ (change of shape only) $T_V = 0$.

In explicit form Eq.(7) can be written as:

$$T_V = \frac{1}{2}\left(\sigma_{xx}\boldsymbol{e}_m + \sigma_{yy}\boldsymbol{e}_m + \sigma_{zz}\boldsymbol{e}_m\right) = \frac{1}{2}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})\boldsymbol{e}_m = \frac{1}{2}\sigma_{ii}\boldsymbol{e}_m = \frac{1}{6}\mathbf{I}_\sigma\mathbf{I}_\varepsilon \tag{9}$$

where $\mathbf{I}_\sigma$ and $\mathbf{I}_\varepsilon$ are the first stress and strain invariants, respectively. It implies that T_V is invariant for any rotation of the coordinate system. In addition, its expression is exactly the same as in the case of isotropic materials – an expected result. Because both T and T_V are invariants, T_D ($T_D = T - T_V$) is an invariant quantity, as well. Furthermore, Eq.(8) can be explicitly written as:

$$\begin{aligned} T_D = \frac{1}{2}[&\sigma_{xx}(\varepsilon_{xx} - e_m) + \sigma_{yy}(\varepsilon_{yy} - e_m) + \sigma_{zz}(\varepsilon_{zz} - e_m) \\ &+2(\sigma_{xy}\varepsilon_{xy} + \sigma_{xz}\varepsilon_{xz} + \sigma_{yz}\varepsilon_{yz})] \end{aligned} \tag{10}$$

which, also, reduces to the well known expression for T_D, in case of isotropic materials.

In the general case of a linear elastic anisotropic material Hooke's law is [7]:

$$\varepsilon_{ij} = C_{ijkl}\sigma_{kl} \text{ or } \sigma_{ij} = S_{ijkl}\varepsilon_{kl}, \tag{11}$$

where C_{ijkl} is the compliance tensor and S_{ijkl} the respective stiffness tensor of the material. Hence, T, T_V and T_D, in terms of the compliance tensor (written in the form of a (6×6) tensor) and stresses take the general forms:

$$T = \frac{1}{2}\left(C_{11}\sigma_{xx}^2 + C_{22}\sigma_{yy}^2 + C_{33}\sigma_{zz}^2\right) + 2[C_{44}\sigma_{yz}^2 + C_{55}\sigma_{xz}^2 + C_{66}\sigma_{xy}^2$$
$$+ C_{12}\sigma_{xx}\sigma_{yy} + C_{13}\sigma_{xx}\sigma_{zz} + C_{14}\sigma_{xx}\sigma_{yz} + C_{15}\sigma_{xx}\sigma_{xz} + C_{16}\sigma_{xx}\sigma_{xy}$$
$$+ C_{23}\sigma_{yy}\sigma_{zz} + C_{24}\sigma_{yy}\sigma_{yz} + C_{25}\sigma_{yy}\sigma_{xz} + C_{26}\sigma_{yy}\sigma_{xy} \qquad (12)$$
$$+ C_{34}\sigma_{zz}\sigma_{yz} + C_{35}\sigma_{zz}\sigma_{xz} + C_{36}\sigma_{zz}\sigma_{xy}$$
$$+ C_{45}\sigma_{xz}\sigma_{yz} + C_{46}\sigma_{xy}\sigma_{yz} + C_{56}\sigma_{xy}\sigma_{xz}]$$

$$T_V = \frac{1}{6}[(C_{11} + C_{12} + C_{13})\sigma_{xx}^2 + (C_{12} + C_{22} + C_{23})\sigma_{yy}^2 + (C_{13} + C_{23} + C_{33})\sigma_{zz}^2$$
$$+ (C_{11} + 2C_{12} + C_{13} + C_{22} + C_{23})\sigma_{xx}\sigma_{yy}$$
$$+ (C_{12} + C_{13} + C_{22} + 2C_{23} + C_{33})\sigma_{yy}\sigma_{zz}$$
$$+ (C_{11} + C_{12} + 2C_{13} + C_{23} + C_{33})\sigma_{zz}\sigma_{xx}$$
$$+ (C_{14} + C_{24} + C_{34})\sigma_{xx}\sigma_{yz} + (C_{15} + C_{25} + C_{35})\sigma_{xx}\sigma_{xz} \qquad (13)$$
$$+ (C_{16} + C_{26} + C_{36})\sigma_{xx}\sigma_{xy} + (C_{14} + C_{24} + C_{34})\sigma_{yy}\sigma_{yz}$$
$$+ (C_{15} + C_{25} + C_{35})\sigma_{yy}\sigma_{xz} + (C_{16} + C_{26} + C_{36})\sigma_{yy}\sigma_{xy}$$
$$+ (C_{14} + C_{24} + C_{34})\sigma_{zz}\sigma_{yz} + (C_{15} + C_{25} + C_{35})\sigma_{zz}\sigma_{xz}$$
$$+ (C_{16} + C_{26} + C_{36})\sigma_{zz}\sigma_{xy}]$$

$$T_D = \frac{1}{6}\Big[(2C_{11} - C_{12} - C_{13})\,\sigma_{xx}^2 + (2C_{22} - C_{12} - C_{23})\,\sigma_{yy}^2$$
$$+ (2C_{33} - C_{13} - C_{23})\,\sigma_{zz}^2 + 3\left(C_{44}\sigma_{yz}^2 + C_{55}\sigma_{xz}^2 + C_{66}\sigma_{xy}^2\right)$$
$$+ 6\left(C_{56}\sigma_{xy}\sigma_{xz} + C_{46}\sigma_{yz}\sigma_{xy} + C_{45}\sigma_{yz}\sigma_{xz}\right)$$
$$+ (4C_{12} - C_{11} - C_{13} - C_{22} - C_{23})\,\sigma_{xx}\sigma_{yy} + (5C_{16} - C_{26} - C_{36})\,\sigma_{xx}\sigma_{xy}$$
$$+ (5C_{15} - C_{25} - C_{35})\,\sigma_{xx}\sigma_{xz} + (5C_{14} - C_{24} - C_{34})\,\sigma_{xx}\sigma_{yz}$$
$$+ (4C_{13} - C_{11} - C_{12} - C_{33} - C_{23})\,\sigma_{xx}\sigma_{zz}$$
$$+ (4C_{23} - C_{22} - C_{12} - C_{33} - C_{13})\,\sigma_{yy}\sigma_{zz} + (5C_{26} - C_{16} - C_{36})\,\sigma_{yy}\sigma_{xy}$$
$$+ (5C_{25} - C_{15} - C_{35})\,\sigma_{yy}\sigma_{xz} + (5C_{24} - C_{14} - C_{34})\,\sigma_{yy}\sigma_{yz}$$
$$+ (5C_{36} - C_{16} - C_{26})\,\sigma_{zz}\sigma_{xy} + (5C_{35} - C_{15} - C_{25})\,\sigma_{zz}\sigma_{xz}$$
$$+ (5C_{34} - C_{14} - C_{24})\,\sigma_{zz}\sigma_{yz}] \qquad (14)$$

Eqs.(12) to (14) can, easily, be expressed in terms of strain and stiffness tensors. In case of pure dilatation and pure distortion they are:

i) Pure dilatation:

$$\varepsilon_{ij} = \frac{1}{3}e_m\delta_{ij} \;\Rightarrow\; T_D = 0 \;\Rightarrow$$

$$T = T_V = \frac{1}{2}(S_{11} + 2S_{12} + 2S_{13} + S_{22} + 2S_{23} + S_{33})\, e_m^2 \tag{15}$$

ii) Pure distortion:

$$\varepsilon_{ij} = \bar{\varepsilon}_{ij}, \quad \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{e_m}{3} = 0 \;\Rightarrow\; T_V = 0 \;\Rightarrow$$

$$\begin{aligned} T = T_D = 2\{&S_{66}\varepsilon_{xy}^2 + S_{55}\varepsilon_{xz}^2 + S_{44}\varepsilon_{yz}^2 + 2S_{56}\varepsilon_{xy}\varepsilon_{xz} + 2S_{46}\varepsilon_{xy}\varepsilon_{yz} + 2S_{45}\varepsilon_{xz}\varepsilon_{yz}\} \\ &+\frac{1}{2}\{(S_{11} - 2S_{12} + S_{22})\varepsilon_{yy}^2 + 4(S_{24} - S_{14})\varepsilon_{yy}\varepsilon_{yz} + 4(S_{25} - S_{15})\varepsilon_{yy}\varepsilon_{xz} \\ &+4(S_{26} - S_{16})\varepsilon_{yy}\varepsilon_{xy} + 2(S_{11} + S_{23} - S_{12} - S_{13})\varepsilon_{yy}\varepsilon_{zz} \\ &+(S_{11} - 2S_{13} + S_{33})\varepsilon_{zz}^2 + 4(S_{34} - S_{14})\varepsilon_{zz}\varepsilon_{yz} + 4(S_{35} - S_{15})\varepsilon_{zz}\varepsilon_{xz} \\ &+4(S_{36} - S_{16})\varepsilon_{zz}\varepsilon_{xy}\} \end{aligned} \tag{16}$$

It is an easy task to prove that Eqs.(12) to (14) reduce, exactly, to the well-known expressions, in case of linear elastic isotropic materials, when the non-zero elements of the compliance tensor are $C_{33} = C_{11}$, $C_{13} = C_{12}$, $C_{44} = 2(C_{11} - C_{12})$.

3. Application

In this section, two applications will be presented.

i) Cubic symmetry: The present solution coincides with the only available solution [e.g. 16] for the case of a material with cubic symmetry, where the non-zero elements of the compliance tensor are:

$$C_{11} = C_{22} = C_{33}, \; C_{12} = C_{13} = C_{23}, \; C_{44} = C_{55} = C_{66} \tag{17}$$

In this case, Eqs.(12) to (14) reduce to:

$$\begin{aligned} T = \frac{1}{2}\Big[&C_{11}\left(\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2\right) + C_{44}\left(\sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yz}^2\right) \\ &+ 2C_{12}\left(\sigma_{xx}\sigma_{yy} + \sigma_{xx}\sigma_{zz} + \sigma_{yy}\sigma_{zz}\right)\Big] \end{aligned} \tag{18}$$

$$T_V = \frac{1}{6}(C_{11} + 2C_{12})\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)^2 \tag{19}$$

$$\begin{aligned} T_D = &2(C_{11} - C_{12})\left(\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 - \sigma_{xx}\sigma_{yy} - \sigma_{xx}\sigma_{zz} - \sigma_{yy}\sigma_{zz}\right) \\ &+ \frac{1}{2}C_{44}\left(\sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yz}^2\right) \end{aligned} \tag{20}$$

ii) Transversal isotropy: In this case, the non-zero elements of the compliance tensor are:

$$C_{11} = C_{22},\ C_{33},\ C_{12},\ C_{13} = C_{23},\ C_{44} = C_{55},\ C_{66} = 2(C_{11} - C_{12}) \tag{21}$$

From Eqs.(12) to (14) we obtain:

$$\begin{aligned} T = \frac{1}{2}\left[C_{11}\left(\sigma_{xx}^2 + \sigma_{yy}^2\right) + C_{33}\sigma_{zz}^2 + (C_{11} - C_{12})\,\sigma_{xy}^2 + C_{44}\left(\sigma_{xz}^2 + \sigma_{yz}^2\right)\right] \\ + C_{12}\sigma_{xx}\sigma_{yy} + C_{13}\left(\sigma_{xx}\sigma_{zz} + \sigma_{yy}\sigma_{zz}\right) \end{aligned} \tag{22}$$

$$\begin{aligned} T_V = \frac{1}{6}[(C_{11} + C_{12} + C_{13})\left(\sigma_{xx}^2 + \sigma_{yy}^2\right) + (2C_{13} + C_{33})\,\sigma_{zz}^2 \\ + 2(C_{11} + C_{12} + C_{13})\,\sigma_{xx}\sigma_{yy} + (C_{11} + C_{12} + 3C_{13} + C_{33})\left(\sigma_{xx}\sigma_{zz} + \sigma_{yy}\sigma_{zz}\right)] \end{aligned} \tag{23}$$

$$\begin{aligned} T_D = \frac{1}{6}[(2C_{11} - C_{12} - C_{13})\left(\sigma_{xx}^2 + \sigma_{yy}^2\right) + 2(C_{33} - C_{13})\,\sigma_{zz}^2 \\ + 2(-C_{11} + 2C_{12} - C_{13})\,\sigma_{xx}\sigma_{yy} \\ + (-C_{11} - C_{12} + 3C_{13} - C_{33})\left(\sigma_{xx}\sigma_{zz} + \sigma_{yy}\sigma_{zz}\right) \\ + 6(C_{11} - C_{12})\,\sigma_{xy}^2 + 3C_{44}\left(\sigma_{xz}^2 + \sigma_{yz}^2\right)] \end{aligned} \tag{24}$$

Direct verification of the present results is not possible, due to the lack of similar data in the literature. However, they reduce to Eqs.(18) to (20) for cubic symmetry and to the classical expressions for isotropic materials, by proper zeroing of elastic constants. Perhaps, the most convincing argument for the validity of Eqs.(22) to (24) is that the coefficients of stress components are the same for stresses in the (xy)-plane of isotropy but different for stress components including suffix-z, an intuitively expected behavior.

Finally, it is interesting to mention that the total elastic strain energy density, T, given by Eq.(12), coincides numerically with its respective value obtained from stress tensor decomposition (e.g. [13]), in spite of the fact that, in the latter case, components with mixed character appear. In other words, strain-tensor decomposition results in only two components with clear physical meaning, although stress-tensor decomposition results in additional terms with obscure physical content.

4. Conclusions

In the present work it was proved that, in the general case of anisotropic materials, the decomposition of the total elastic strain energy density into

its two components, i.e. dilatational and distortional, can be performed by starting from the decomposition of the strain tensor, instead of the stress one. Available in the literature analytic expressions coincide with the present results. This approach has the advantage that it is, a priori, assured that the volume and shape changes of the elementary volume of the material are under control. This strain tensor decomposition is, always, permissible, regardless the type of the linear elastic material. Then, a simple multiplication of the spherical and the distortional strain tensors with the respective complete stress tensor results, through the compliance tensor, into two products. They, by definition, represent dilatational and distortional strain energy densities, respectively. This way, difficulties in identifying additional energy terms, obtained through stress-tensor decomposition, are by-passed.

It must be emphasized that any type of spectral analysis of the stress tensor results into a sum including a spherical stress component, which invokes shear strains in case of anisotropic materials. That is why the two types of decomposition described by Eqs.(5) and (6) decouple. The role of the spherical stress component in anisotropic materials plays the so-called quasi-spherical tensor. A simplified form of this tensor (given by Eq.(1)), indicates that it is diagonal but not spherical.

The present results coincide with similar results for special cases of anisotropy, found in the literature. Finally, the present approach is quite adequate to be applied for the formation of failure surfaces in the general case of anisotropy.

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O separacji energii odkształceń sprężystych w ogólnym przypadku anizotropii. Podejście bezpośrednie

Streszczenie

W pracy przedstawiono PODEJŚCIE BEZPOŚREDNIE do problemu separacji energii odkształceń sprężystych dla ogólnego przypadku materiałów anizotropowych. Podejście jest oparte na prostym rozkładzie tensora odkształceń na tensory sferyczny i skośny. Na podstawie rozważań nad geometrią odpowiedzi materiału wprowadzono definicję gęstości odkształcenia sprężystego, objętościowego i postaciowego. Za pośrednictwem uogólnionego prawa Hooke'a uzyskano wyrażenia analityczne dla materiałów o ogólnej anizotropii. Przedstawione wyniki są zgodne z jedynymi dostępnymi w literaturze danymi dla materiałów o symetrii regularnej. Pokazano ponadto zastosowanie dla przypadku odkształcenia sprężystego materiałów transwersalnie izotropowych.