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FINITE ELEMENT SHAKEDOWN ANALYSIS OF PRE-LOADED PLATE STRUCTURES

In the paper, the authors present the shakedown analysis of the plate structures pre-loaded beyond the elastoplastic range. Two cases of loading are considered, namely: the structure is subjected to the action of two independent sets of loads with constant points of application or one parameter set of loads moves slowly according to an a priori described program. As a result, the safe loading boundary or the shakedown load parameter are calculated, respectively, by means of the finite element method (FEM). Three examples confirmed the effectiveness of the proposed algorithms of analysis.

1. Introduction

In the paper, shakedown analysis of an elastoplastic plate structures loaded according to a specified loading program we consider. In general, the load program covers the kind of load, its distribution on the structure and the way in which the structure is loaded. The proposed procedure constitutes an extension of the method presented in [6], where the shakedown of the elastoplastic frames under moving load was analysed. In that paper, the static method of analysis based on Neal's theorem [12] and the idea of the current elastic domain introduced by König [11] were used. Some other theoretical and experimental results of the shakedown analysis of bar structures were reported in [7], [8] and [9].

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The well-known possibilities of the modern computational methods, like the finite element method, cause that the theoretical results of these papers can now be effectively applied to the analysis of much more complicated structures.

Contrary to the classical shakedown analysis, which can be regarded as a generalization of limit analysis exploiting the Melan's (static) and Koiter's (kinematic) theorems [10], [16], in this paper the real field of residual stresses in a structure is calculated, and the possibilities of shakedown with an emphasis on the protection of the structure against incremental failure are analysed.

It is assumed that the structure can be subjected to the action of a system of forces, whose magnitudes can change within a given range or, if they remain constant, whose points of load applications can slowly change. In particular, two cases of loading are specified.

In the first case, the assumption that the structure is pre-loaded beyond the elastoplastic range is made, and it is next subjected to the action of two independent sets of forces. In this case, the safe loading domain is calculated, where the structure behaves fully elastically for any combination of the load parameter. In the analysis, the geometrical nonlinearities are also taken into account, what means that the safe loading boundary can contain critical points (bifurcation or limit type) appearing during execution of the loading program.

In the second case, one considers the one parameter set of loads moving slowly along the structure, according to the specified load program. Afterwards, when the load program is completed, the structure can still be in the elastic state otherwise some residual stresses can appear. In both situations, the question arises how the load can be increased, or decreased, so that the shakedown could take place for the structure loaded according to the former loading program in a given number of cycles. The loading parameter fulfilling this condition will be called the shakedown load parameter.

The method of analysis is fully numerical, and the basic equations of the problem are formulated using the displacement finite element method in the total Lagrangian description. The nonlinear set of incremental equations is solved by the Newton-Raphson method with the linear constraint equation of the Riks-Wempner type [15].

Plate structures are discretized using the RSE-V/GN element, which is an extension of the elastoplastic RSE-V element [4] on the geometrically nonlinear case with six degrees of freedom per node. Some parts of the paper were presented at the 15th International Conference on Computer Methods in Mechanics [14].

The outline of the paper is as follows. In Section 2, the algorithm of safe loading boundary calculations is described, and in Section 3 the moving load program is defined. The numerical method of computation of the shakedown parameter is presented in Section 4. In Section 5, some modifications of the RSE-V plate element are briefly discussed. In Section 6, three examples are presented in which the validity of the modified plate element RSE-V/GN and effectiveness of the proposed algorithm of the analysis are examined. The paper is closed with some conclusions and references.

2. Method of computation of safe loading boundary

Safe loading boundary can be described for any equilibrium state of the elastoplastic plate structure loaded according to the load program, defined on the load plane R^2. It is assumed that the safe loading boundary is a closed curve, which can be composed of the shakedown boundary and the stability boundary lines. It is well known that, according to the Papkowich's theorem, the stability boundary is convex only for linear problems [13], [15]. In the geometrically nonlinear analysis, a concave boundary can occur. It was an argument for extending the elastoplastic RSE-V element over the geometrically nonlinear case.

The points of the safe loading boundary of the structure with the elastoplastic and geometric nonlinearities taken into account are calculated in two stages. In the stage I, the structure is loaded as long as any plastic deformations have appeared. Afterwards, the safe loading boundary is computed for such deformed structure. It is executed in the stage II by the unloading process in the early defined direction on the plane of load parameters (λ_1, λ_2), as long as at one of the numerical integration points the plastic stress has been attained (an active process after a passive one). The computations are then continued in the opposite direction, and the next point on the safe loading boundary is calculated. Such a "reflection process" of calculations is repeated until the sufficient number of points on the safe loading boundary have been described.

The computations in the constant load direction are continued as long as two conditions are fulfilled, namely: 1) at any Gaussian point of structure the value of yield function is greater than the current yield stress and 2) the distance between two last points calculated on the constant load direction line is smaller than the assumed error. Choosing the first point as the point of the safe loading boundary means that the safe loading domain is estimated from the safe side. It is a different approach from the one assumed in computations of the shakedown boundaries of pre-loaded frames, where the last point was chosen as the point of the safe loading boundary.

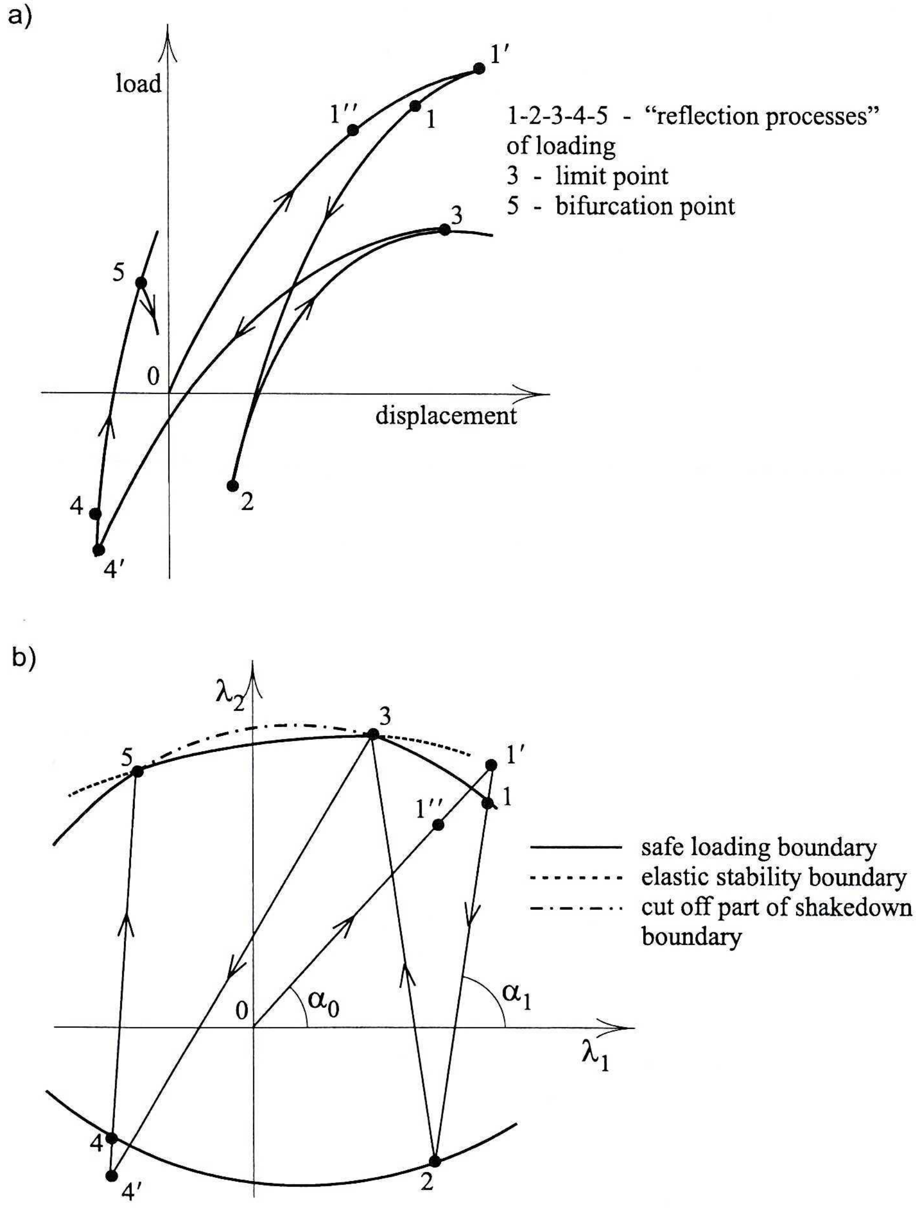


Fig. 1. Equilibrium path for the given load program (a) in relation to the safe loading boundary (b)

The method of calculations is illustrated in Fig. 1. Assuming that for the given load program (in the simplest case defined in terms of angles α_0, α_1, ... on the plane (λ_1, λ_2)) the equilibrium path of the structure has the form shown in Fig. 1a, the safe loading boundary is constructed as it is shown in Fig. 1b. It should be noted that the part 0-1 of the equilibrium path shown in Fig. 1a is nonlinear because of the couple effect of the physical and geometrical nonlinearities. In execution of the loading program, the active processes take place in the parts 1″-1′ and also 1′-1 of the equilibrium path. It means that during the unloading of the external forces at some points of the structure the active processes are still possible. The nonlinearities of the remaining parts of the equilibrium path are only the effect of the geometric nonlinearities (except for the sector 4′-4). The points 1, 2 and 4 of the safe loading boundary shown in Fig. 1b have been calculated in result of appearing of the plastic stresses in the structure. Points 3 and 5, however, are the effect of elastic instabilities of the structure (a snap-through or bifurcation type). These are points of the stability boundary which in such a way form a part of the safe loading boundary.

3. Moving load program

The problem under consideration is the static one, and the term "time" can be identified with any parameter monotonically increasing during the loading of the structure. In the simplest case of one parameter load, the motion of the external forces in the given time can be simulated by changing only values of the load parameter at the nodes of the finite element mesh, where the reference load vector components are non-zero. We introduce the definition of the stage s, $s = 1, ..., S$, as a process of moving the load from the given location on the structure to the neighbouring one. The stage is realized in the number of time steps N. The above definitions are illustrated in Fig. 2, where the way of changing the load from the location i through j to the location k in the time is shown. As it can be seen, the diminution of the load at the location i generates, at the same time, the increase of the load at the location j.

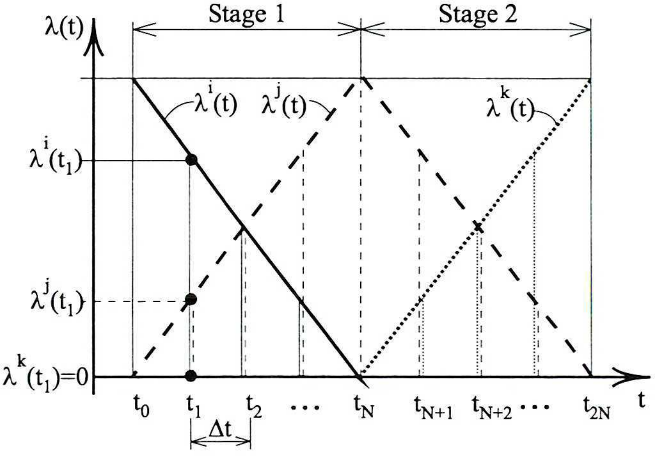


Fig. 2. Discrete moving of the one parameter load from the location i, through j to the location k

The described above motion of the one parameter set of forces is defined by means of the matrix $\underline{M}$ with the number of rows equal to $(S + 1)$, and the number of columns equal to a number of non-zero elements of the reference load vector. This matrix stores all information about the history of loading in the compact form. The consecutive rows of the matrix $\underline{M}$ contain degrees of freedom numbers defining a current location of the forces on the structure.

The load vector component F_r for the locations i, j and k at any time t is given by the formula

$$F_r^l(t) = \lambda^l(t)\bar{F}_r, \quad l = i, j, k \tag{1}$$

where:
$\bar{F}_r$ – reference load vector component r.

In the discrete form, the change of the component F_r during a time step Δt at locations i and j can be written in the form

$$\begin{aligned} F^i_{r,new} &= F^i_{r,old} - \Delta\lambda\bar{F}_r \\ F^j_{r,new} &= F^j_{r,old} + \Delta\lambda\bar{F}_r \end{aligned} \tag{2}$$

where:
$\Delta\lambda = \lambda / N$ – load parameter increment
and $F^i_{r,new} = \lambda\bar{F}_r$ and $F^j_{r,new} = 0$ at the time $t = t_0$, with changing of indices $i \rightarrow j$ and $j \rightarrow k$ for the change of the load location from j to k.

4. Method of computation of the shakedown parameter

In view of the Melan's shakedown theorem, it can be said that the structure will shake down if, for the given self-equilibrated time-independent stress field $\underline{\sigma}_r$, there is an elastic stress field $\underline{\bar{\sigma}}_e$ that would arise in the structure responding purely elastically according to an priori defined load program, in which the condition

$$f(\underline{\sigma}_r(x) + \underline{\bar{\sigma}}_e(\underline{x}, t)) \leq 0, \tag{3}$$

where f is the yield function, is satisfied at all times t and all parts of the structure $\underline{x}$.
Assuming that

$$\underline{\bar{\sigma}}_e(\underline{x}, t) = \lambda\,\underline{\sigma}_e(\underline{x}, t) \tag{4}$$

where:
λ – constant $\lambda > 1$
$\underline{\sigma}_e(\underline{x}, t)$ – elastic reference stress field
Eqn (3) can be rewritten in the form

$$f(\underline{\sigma}_r(\underline{x}) + \lambda \underline{\sigma}_e(\underline{x}, t)) \leq 0, \tag{5}$$

It must be stressed that assumption (4) can be only valid for the case of the so-called weak geometric nonlinearities. In the context of the finite element procedure it means that, if geometric nonlinearities are also taken into account in the analysis, the tangent stiffness matrix of finite elements can be calculated with omission of the stiffness matrix of initial displacements [5].

The shakedown load parameter λ_{SH} is defined as

$$\lambda_{SH} = \min_i (\lambda_i) \tag{6}$$

where $i = 1, 2, ..., I$ and I is the number of integration points of the discretized structure. Calculations are made in two steps.

In the first step, the residual stress field $\underline{\sigma}_r$ is calculated for the given load program. In general, these calculations can be made for any elastoplastic constitutive model of materials. It means that in the case of the material hardening, the current yield stresses σ_p^i must be remembered at all integration points.

In the second step, the elastic structure is loaded according to an a priori defined reference load program (not necessarily the same for which $\underline{\sigma}_r$ was computed), and the elastic reference stress field $\underline{\sigma}_e$ is calculated.

In the paper, the Huber-Mises-Hencky yield function is assumed, and condition (5) takes the simple form of the quadratic equation

$$(\underline{\sigma}_r^i + \lambda_i \underline{\sigma}_e^i)^T \cdot \underline{P} \cdot (\underline{\sigma}_r^i + \lambda_i \underline{\sigma}_e^i) - \frac{2}{3}(\sigma_p^i)^2 = 0, \qquad i = 1, 2, ..., I \tag{7}$$

where the matrix convention is used and $\underline{P}$ is the so-called projection matrix.

5. Plate finite element RSE-V/GN

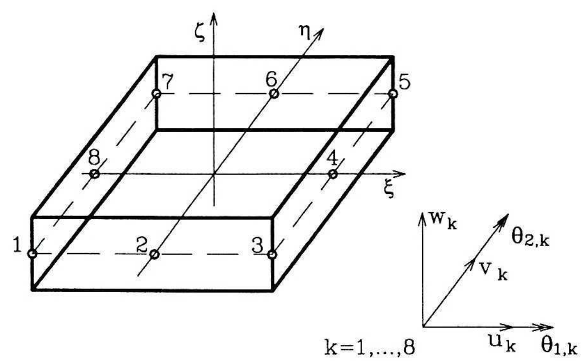
For purposes of the paper, the rectangular plate finite element RSE-V/GN (Resultant Surface Element – Volume approach /Geometrically Nonlinear) has been elaborated. The elastoplastic physical properties of RSE-V element are broadly discussed in [4]. In the following, the RS-V/GN element is presented with paying attention to its nonlinear geometrical properties.

5.1. Geometry of RSE-V/GN element

The Reissner-Mindlin plate theory is applied to a finite element formulation with transverse shear included. The element has 8 nodes localized on the middle surface, and 6 degrees of freedom per node. The sixth degree of freedom (the so-called drilling degree of freedom) has been additionally introduced, in the same way as it is proposed in the ADINA system [1], taking into account the possibilities of analysis of space structures discretized with plate elements. The element stiffness matrix is constructed by superimposing the artificial stiffness corresponding to the drilling degrees of freedom. In the program, this stiffness is set equal to

$$K_{\varphi 3} = \frac{K_{\varphi 1} + K_{\varphi 2}}{2} \times 2.5E - 4$$

in order to remove the zero stiffness, but yet not to affect the analysis results significantly, where $K_{\varphi 1}$, $K_{\varphi 2}$, $K_{\varphi 3}$ are components of the stiffness matrix which correspond with angular degrees of freedom ($\theta_{1,k}$, $\theta_{2,k}$ and $\theta_{3,k}$).



$\xi = x/a \quad \eta = y/b \quad \zeta = 2z/t$

$\xi, \eta, \zeta \in [-1,1]$ - local coordinates

$(\underline{q}_k^e)^T = [u_k \; v_k \; w_k \; \theta_{1,k} \; \theta_{2,k}]$

$(\underline{q}^e)^T = [\underline{q}_1^e \ldots \underline{q}_8^e]$ - vector of nodal displacement

Fig. 3. RSE-V plate finite element

In such a way, the stiffness matrix of the RSE-V/GN element is the simple extension of the stiffness matrix of the RSE-V element shown in Fig. 3.

5.2. Physical equations

Material nonlinearities are taken into account assuming the plastic flow theory associated with the Huber-Mises-Hencky yield criterion. The linear relation between stress and strain rates is

$$\underline{\dot{\sigma}} = \underline{E}_{ep}\,\underline{\dot{\varepsilon}} \tag{8}$$

where:
$\underline{E}_{ep}$ – consistent stiffness matrix
$\underline{\dot{\sigma}}^T = [\dot{\sigma}_x\ \dot{\sigma}_y\ \dot{\tau}_{xy}\ \dot{\tau}_{xz}\ \dot{\tau}_{yz}]$ – vector of stress rates
$\underline{\dot{\varepsilon}}^T = [\dot{\varepsilon}_x\ \dot{\varepsilon}_y\ \dot{\gamma}_{xy}\ \dot{\gamma}_{xz}\ \dot{\gamma}_{yz}]$ – vector of strain rates
and $(.)^T$ denotes matrix transposition.

The integration of the elastoplastic constitutive relations is performed in the local formulation. This means that the constitutive equations are solved at all discretisation points along the thickness, and then stresses are integrated to result in the generalized stresses on the middle surface of the plate.

In the paper, the integration of Eqn (8) along the loading path has been made using the return-mapping algorithm. The problem has come at last to the one nonlinear equation with the plastic flow parameter $\Delta\lambda$ as the unknown and solved with the help of the Newton-Raphson method.

5.3. Geometrical equations

The relation between the generalized strain vector $\underline{e}$ on the middle surface and the strain vector $\underline{\varepsilon}$ has the form

$$\underline{\varepsilon}(\xi, \eta, \zeta) = \underline{H}\left(\zeta \cdot \frac{1}{2}\right) \cdot \underline{\varepsilon}(\xi, \eta, 0) \tag{9}$$

where $\underline{H}$ is so-called 'hypothesis' matrix

$$\underline{H}\left(z = \zeta \cdot \frac{1}{2}\right) = \begin{bmatrix} 1 & 0 & 0 & z & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & z & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{10}$$

with

$$\underline{e}^T = [e_x\ e_y\ \gamma_{xy}\ \kappa_x\ \kappa_y\ \chi_{xy}\ \gamma_{xz}\ \gamma_{yz}] \tag{11}$$

and

$$
\begin{array}{ll}
e_x = u,_x + \frac{1}{2}w,_x^2 & e_y = v,_y + \frac{1}{2}w,_y^2 \\
\gamma_{xy} = u,_y + v,_x + w,_x \cdot w,_y & \kappa_x = \theta_{2,x} \\
\kappa_y = -\theta_{1,y} & \chi_{xy} = \theta_{2,y} - \theta_{1,x} \\
\gamma_{xy} = \theta_2 + w,_z & \gamma_{yz} = -\theta_1 + w,_y
\end{array}
\tag{12}
$$

where $(.)_{,x} \equiv \partial(.)/\partial x$.

Assembling the terms that are linear in the displacement increments in $\Delta \underline{e}_L$ vector and the terms that are nonlinear in the displacement increments in $\Delta \underline{e}_{NL}$ vector, one can write the following formula

$$\Delta \underline{e} = \Delta \underline{e}_L + \Delta \underline{e}_{NL} \tag{13}$$

where:

$$
\begin{array}{l}
(\Delta \underline{e}_L)^T = \\
[\Delta u,_x + w,_x \cdot \Delta w,_x \quad \Delta v,_y + w,_y \cdot \Delta w,_y \quad \Delta u,_y + \Delta v,_x + w,_x \quad \Delta w,_y + w,_y \cdot \Delta w,_x \\
\Delta \theta_{2,x} - \Delta \theta_{1,y} \quad \Delta \theta_{2,y} - \Delta \theta_{1,x} \quad \Delta \theta_2 + \Delta w,_x \quad -\Delta \theta_1 + \Delta w,_y] \\
(\Delta \underline{e}_{NL})^T = \left[\frac{1}{2}\Delta w,_x^2 \quad \frac{1}{2}\Delta w,_y^2 \quad \Delta w,_x \cdot \Delta w,_y \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \right]
\end{array}
\tag{14}
$$

5.4. Approximations

The finite element approximations of the unknown increments of the generalized displacement functions $\Delta \underline{d}^e$ can be expressed in the matrix form

$$\Delta \underline{d}^e = \underline{N}^e \Delta \underline{q}^e \tag{15}$$

where:

$(\Delta \underline{d}^e)^T = [\Delta u^e \quad \Delta v^e \quad \Delta w^e \quad \Delta \theta_1^e \quad \Delta \theta_2^e]$ – vector of generalized displacement function increments

$\underline{N}^e$ – shape function matrix

$\Delta \underline{q}^e$ – vector of nodal displacement increments, Fig. 3.

The matrix of the shape function is composed of the serendipity shape functions [2]

$$N_k^e = (1 + \xi\xi_k)(1 + \eta\eta_k)(\xi\xi_k + \eta\eta_k - 1)/4 \quad \text{for} \quad k = 1,3,5,7$$

$$N_m^e = \left(\xi_m^2 (1 + \xi\xi_m)(1 - \eta^2) + \eta_m^2 (1 + \eta\eta_m)(1 - \xi^2)\right)/2 \quad \text{for} \quad m = 2,4,6,8 \tag{16}$$

5.5. Incremental equilibrium equations of the element

Incremental equilibrium equations of the element can be derived from the linearized equation of the incremental principle of virtual work. The final form of these equations can be written in the matrix form

$$(\underline{K}_L^e + \underline{K}_{NL}^e)\,\Delta\underline{q}^e = \underline{f}_{ext}^e - \underline{f}_{int}^e. \tag{17}$$

In the above equation, $\underline{f}_{ext}^e$ is the external load vector and the following matrices and vectors are defined:
incremental stiffness matrix

$$\underline{K}_L^e = \int_{\Omega_o} (\underline{B}_L^e)^T \underline{D}_{ep}^e \underline{B}_L^e \, d\Omega_o \tag{18}$$

where:

$\underline{D}_{ep}^e = \int_t \underline{H}^T(z)\underline{E}_{ep}^e(z)\underline{H}(z)dz$ – constitutive resultant matrix

generalized stress vector

$$\underline{S}^e = \int_t \underline{H}^T(z)\underline{\sigma}(z)dz \tag{19}$$

initial stress matrix

$$\underline{K}_{NL}^e = \int_{\Omega_o} \left(S_x^e \underline{B}_{NL,x}^e + S_y^e \underline{B}_{NL,y}^e + S_{xy}^e (\underline{B}_{NL,xy}^e + \underline{B}_{NL,yx}^e)\right) d\Omega_o \tag{20}$$

where:
S_x^e, S_y^e, S_{xy}^e – the generalized stress vector $\underline{S}^e$ components
$\underline{B}_L^e$, $\underline{B}_{NL}^e$ – linear and nonlinear matrices of derivatives of the shape functions connected with $\Delta\underline{e}_L$ and $\Delta\underline{e}_{NL}$, respectively

internal force vector

$$\underline{f}_{int}^e = \int_{\Omega_o} (\underline{B}_L^e)^T \underline{S}^e \, d\Omega_o. \tag{21}$$

The matrices and vectors have been calculated numerically using Gaussian quadrature: 3 points – for integration in z direction and 4 points – for integration on the middle surface.

After standard assembling procedure, the following incremental equilibrium equation of the structure is formulated

$$\underline{K}_T \, \Delta\underline{Q} = \Delta\underline{F}_{ext} + \underline{R} \tag{22}$$

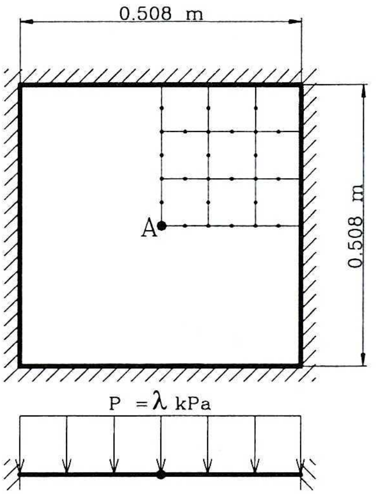
where:
$\underline{K}_T = \underline{K}_L + \underline{K}_{NL}$ – global tangent stiffness matrix
$\Delta\underline{Q}$ – vector of nodal displacement increments.
In the case of two parameter loads, the following vectors must be also defined:
$\Delta\underline{F}_{ext} = \Delta\lambda_1 \underline{\bar{F}}_{1ext} + \Delta\lambda_2 \underline{\bar{F}}_{2ext}$ – vector of nodal load increments
$\underline{R} = \lambda_1 \underline{\bar{F}}_{1ext} + \lambda_2 \underline{\bar{F}}_{2ext} - \underline{F}_{int}$ – vector of residual forces
λ_i, $\Delta\lambda_i$ – total and incremental load parameters, respectively, $i = 1, 2$
$\underline{\bar{F}}_{1ext}$, $\underline{\bar{F}}_{2ext}$ – vectors of reference loads

The computations have been made using the extended set of equations, where the incremental FE equations (22) were completed by the linear Riks-Wempner equation [3]. It enables us to specify different control methods of a deformation process of the structure (load control, displacement control, arc-length control).

6. Examples

6.1. Verification example of the RSE-V/GN element

The quadratic plate, clamped on boundaries, under uniform lateral load has been calculated in order to verify the RSE-V/GN plate element. Geometry, loading and discretisation of the plate are shown in Fig. 4. Taking into account the symmetry, only the quarter of the structure has been considered and discretized.



E=69 GPa – Young modulus
ν=0.3 – Poisson's ratio
σ_p=248 MPa – yield stress
h=0.00254 m – thickness of the plate

Fig. 4. Geometry, loading and discretisation of the plate

Figure 5 shows the equilibrium paths obtained by means of different approaches. In the case when only the physical nonlinearities are assumed, the results of computations using RSE-V and RSE-V/GN elements are, of course, the same, Fig. 5a. In this example, however, using RSE-V/GN elements, the great influences of geometrical nonlinearities on deformation of the plate have been detected, Fig. 5b. The influence of the elastoplastic nonlinearities are clearly seen only for $w_A > 0.016$m. The validity of the results of computations was confirmed by additional calculations using the ADINA system (8 node shell element). This example confirms the rightness of proposed RSE-V/GN element and demonstrates substantial difference between geometrically linear and nonlinear analysis.

In Fig. 6, the plastic zone of the geometrically and physically nonlinear plate discretized with RSE-V/GN element and loaded up to P = 557.4 kPa (see Fig. 5b) is shown (approximate drafts). The similar plastic regions were obtained using ADINA system.

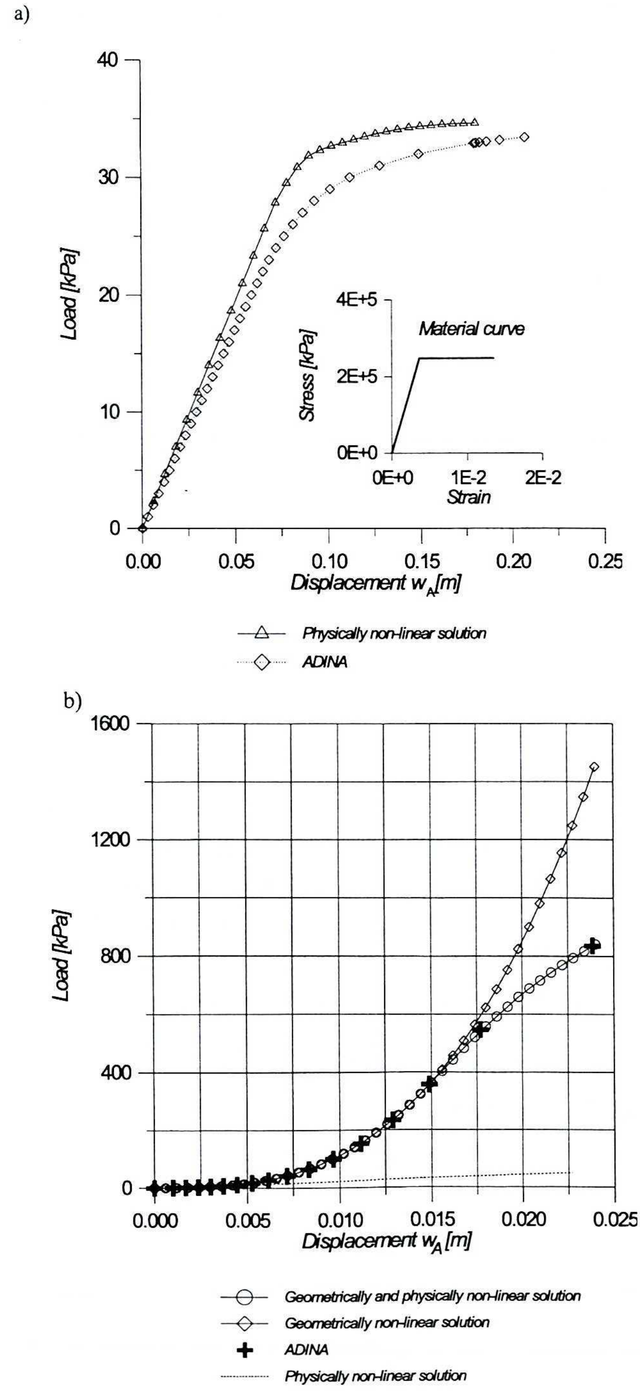


Fig. 5. Equilibrium paths for the plate a) RSE-V elements discretisation in comparison with ADINA b) RSE-V/GN elements discretisation in comparison with ADINA

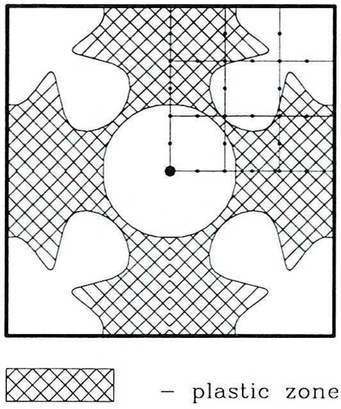
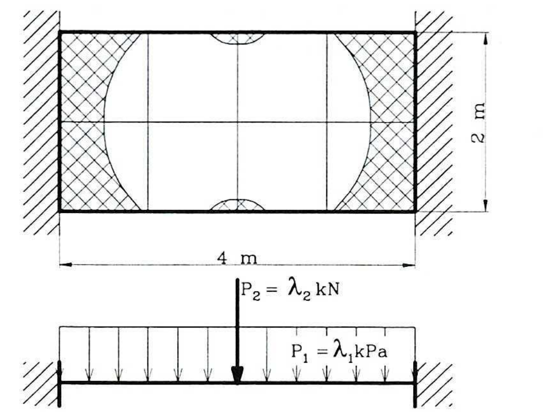


Fig. 6. Approximate plastic domains for the plate with RSE-V/GN elements discretisation

6.2. Safe loading boundary for the plate under two parameter loading

In the second example, the plate, clumped on two shorter edges, shown in Fig. 7, has been analysed. The plate has been discretized using only 8 RSE-V/GN elements without the material hardening. The elastic domain and the safe loading domain of the plate initially loaded in the direction $\alpha_o = 45°$ up to the point A ($\lambda_1 = \lambda_2 = 183.86$) are shown in Fig. 8.



E=2100 GPa – Young modulus
ν=0.3 – Poisson's ratio
σ_p=248 MPa – yield stress
h=0.02 m – thickness of the plate

Fig. 7. Geometry and load of the plate

The process of loading and unloading from this point was repeated assuming increment of the angle equal ± 5°. The safe loading boundary shown in Fig. 8 is only composed of shakedown curves, however, locally limit points were also observed, close to the points on the shakedown boundary (not shown in Fig. 8).

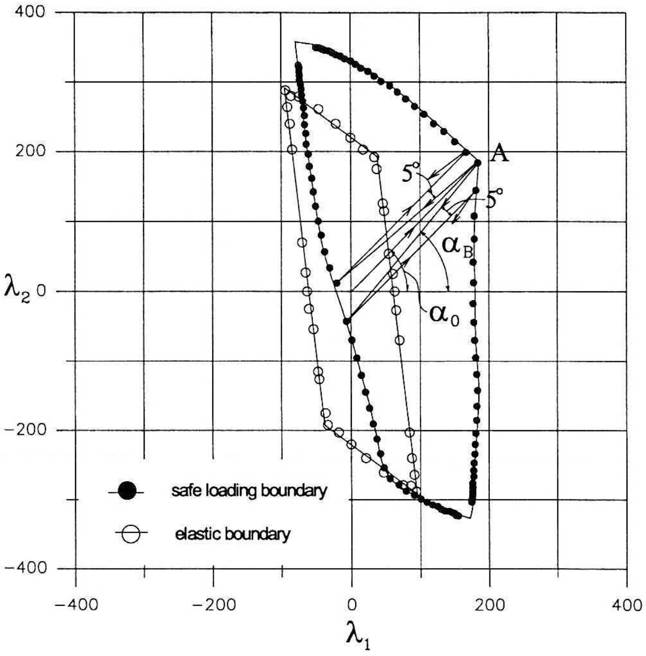


Fig. 8. Safe loading domain for the plate

6.3. Shakedown load parameter for the elastoplastic cylindrical panel under one parameter moving load

In this example, the elastoplastic cylindrical panel loaded by one moving force is analysed. The geometry of the structure and material parameters are shown in Fig. 9. The structure has been discretized using 16 RSE-V elements with the isotropic material hardening and the scalar-valued hardening parameter based on the strain-hardening hypothesis. The load program is defined as some numbers of cycles: A $\rightarrow$ B $\rightarrow$ A, Fig. 9. The distribution of $\underline{\sigma}_e$ has been calculated for the same load program as used for calculation of $\underline{\sigma}_r$.

The results of the calculations are shown in Fig. 10 and in Table 1. In Fig. 10, the shakedown load parameter λ_{SH} is computed for different values of λ and different numbers of cycles (1, 2 or 3). It is observed that after crossing the value $\lambda = 3.45$, the value of $\Delta\lambda_{SH}$ decreases and after then λ_{SH} reaches the maximum at $\lambda_{SH} \approx 3.46$. For the larger values of λ, the values of λ_{SH} decrease. However, in comparison to each other, the values of λ_{SH} increase after a greater number of load cycles. The details of calculations are put together in Tab. 1. This example shows the possibilities of calculation of λ_{SH} for the real structure loaded beyond the elastoplastic range and then repeatedly loaded according to a priori defined load program and the given number of cycles.

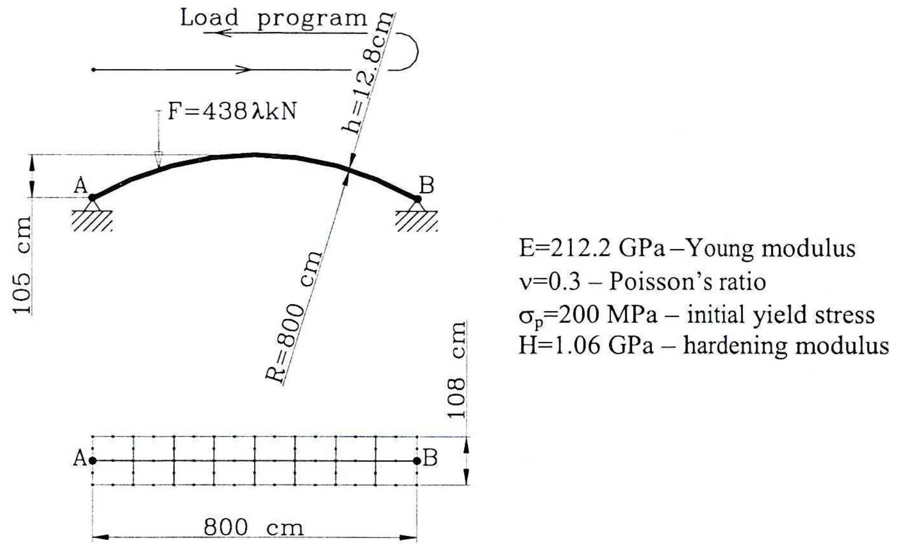


Fig. 9. Elastoplastic cylindrical panel under moving load

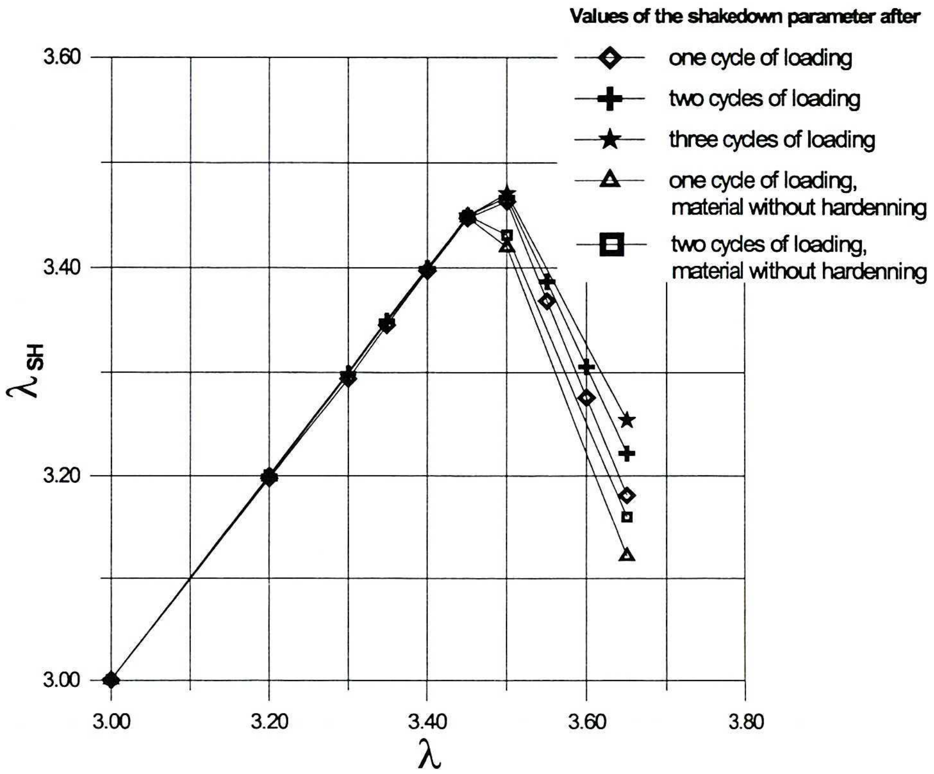


Fig. 10. Shakedown parameter vs. load parameter for the cylindrical panel

Table 1.

Results of calculation for the cylindrical panel

λ	λ_{SH} – 1 cycle	λ_{SH} – 2 cycles	λ_{SH} – 3 cycles	λ_{SH} – 1 cycle, (material without hardening)	λ_{SH} – 2 cycles, (material without hardening)
2.5292	2.5292				
2.53				2.5302	
2.8	2.8007			2.8034	
3	3.0002	3.0002		3.0007	3.0007
3.2	3.1976	3.1999		3.1994	3.2009
3.3	3.2938	3.2991			
3.35	3.3453	3.3493			
3.4	3.3974	3.3996			
3.45	3.4474	3.4497	3.4497	3.4485	3.4502
3.5	3.4635	3.4676	3.4711	3.42	3.4315
3.55	3.3684	3.3871			
3.6	3.2758	3.3056			
3.65	3.1811	3.222	3.2539	3.1217	3.16

7. Conclusions

In the paper, we have analysed the possibilities of the shakedown of plate structures into the real field of the residual stresses. Two cases of load have been considered, namely the cyclic loading and the moving load. In the first case the safe loading boundaries and in the second case the shakedown load parameters have been computed, respectively. Because the method of the analysis is purely numerical, the proposed approach can not be generalised, but examples have shown that it is effective and can be easily extended into the analysis of more complicated plate structures and different sets of loads. In particular, using the method of contour lines, the three parameters load can be considered (e.g. assuming λ_3 = const.). The extension of the RSE-V plate element into the case of geometrical nonlinearities facilitates the analysis of influence of these nonlinearities on the shape of the safe loading boundary. However, it must be stressed that, because the processes considered are non-conservative, the results of the analysis are valid only for a priori defined load program. The assumption of the Huber-Mises-Henckey yield criterion also simplified the method of solution very much.

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Skończenie elementowa analiza przystosowania wstępnie przeciążonych konstrukcji płytowych

S t r e s z c z e n i e

Praca dotyczy analizy przystosowania wstępnie przeciążonych sprężysto-plastyczny konstrukcji płytowych. Rozpatrywane są dwa przypadki: pierwszy przypadek dotyczy obciążenia dwuparametrowego o ustalonym miejscem przyłożenia. W drugim przypadku konstrukcja poddawana jest działaniu jednoparametrowego obciążenia ruchomego. Następnie przy użyciu algorytmów opartych na metodzie elementów skończonych, wyznaczana jest granica bezpiecznego obciążenia (przypadek pierwszy) lub wartość parametru przystosowania (przypadek drugi). Na koniec przedstawione są przykłady analizy numerycznej potwierdzające efektywność proponowanych algorytmów.