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THE TWO-ACTIONS THEOREM AND ITS APPLICATION TO COMPOSITE MATERIALS

In the present study a new energy theorem is proposed, "The two actions-theorem", which is valid in linear elastostatic problems. A new formalism concerning the works done by the external actions is introduced. Known energy theorems are proved using the proposed two-actions theorem. A composite materials problem is confronted in terms of the two actions theorem and energy relations are formulated. Finally, it is presented a study on the problems of a composed two material hollow cylinder under internal and external pressure, and of stretching of an infinite plate with an inserted elastic disc of a different material. The proposed energy relations are verified in these applications.

1. Introduction

The known energy theorems have already been analyzed in the literature of Mechanics [1], [2], [3], [4], [5], [6] and have also been used in Variational Methods [6-9]. These theorems are the basis of many calculations in Mechanics. On the other hand, a very interesting problem in composite structures [10], [11], [12], [13], [14], [15], [16] is the variation of the energy in a composite constituent when there is a variation in the actions applied to the composite.

The basic idea of this study is the introduction of a new energy theorem, "The two- actions theorem", based on the external actions' work.

The originality of the present study is that the two-actions theorem is a general energy theorem applied only to linear elasticity problems in the absence of inertial forces. With the proposed theorem, and according to a new formalism, the known energy theorems have been proved in a simple way.

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The two-actions theorem is also applied to a composite materials problem where energy relations based on the new symbolism are formulated. Finally, two applications are made. The first one concerns the problem of a hollow cylinder composed of two materials, with free ends under uniform internal and external pressure. The second one is connected with the problem of stretching of an infinite plate with an inserted elastic disc of a different material. The previously formulated energy relations of the two materials composite problem are verified.

The proposed formalism may play an important role in problem solving and may be more easily grasped by students or practionners. In the case of the composite materials problem, it is proved taking into consideration the superposition principle, how the variation in the actions applied to the composite may influence or not the strain energy of the composite constituents.

The proposed two-actions theorem can also be used as a Variational Method. The formulation of this new energy theorem in the case of variational methods may be a subject of future research.

2. Symbolism of the work of external actions

Let an elastic body (D) subjected to a system of external actions (A).

We define W_{AA} the work that would be done by the system of external actions (A) in acting through the displacements as going from the initial undeformed situation to the final deformed situation of equilibrium. (Fig. 1).

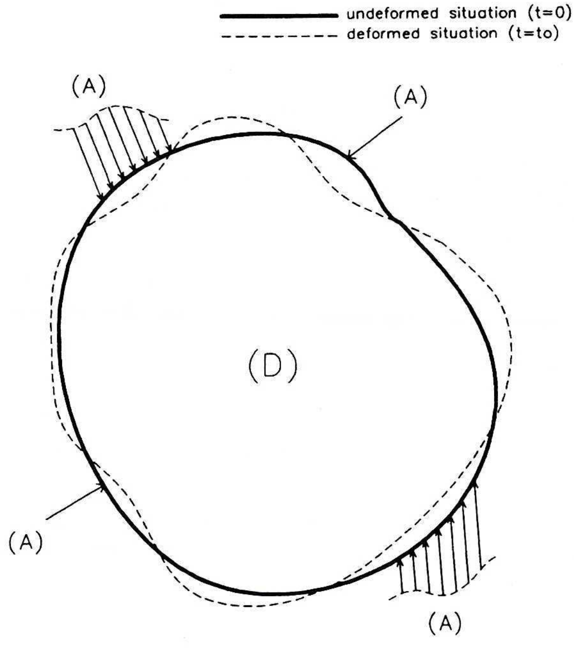

Fig. 1. Elastic body (D) subjected to the system of external actions (A)

In case of lack of kinetic energy, W_{AA} is transformed in strain energy U and it holds

$$W_{AA} = U = \int_V \left(\int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij} \right) dV, \qquad i,j = 1,2,3. \tag{1}$$

Because of the linearity and of the generalized Hooke's Law [2], [3]

$$\int_0^{t_0} \sigma_{ij}\dot{\varepsilon}_{ij} dt = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij} = \int_0^{\varepsilon_{ij}} E_{ijkm}\varepsilon_{km} d\varepsilon_{ij}, \qquad i,j,k,m = 1,2,3, \tag{2}$$

where

$$\sigma_{ij} = \sigma_{ij}(t), \qquad \varepsilon_{ij} = \varepsilon_{ij}(t), \qquad \dot{\varepsilon}_{ij} = \frac{d\varepsilon_{ij}}{dt}, \qquad 0 \le t \le t_0.$$

Taking ito consideration the symmetry property of the elasticity tensor [1-6], relation (1) is written

$$W_{AA} = U = \int_V \left(\frac{1}{2}\sigma_{ij}\varepsilon_{ij} \right) dV = \int_V U_0 dV, \tag{3}$$

where U_0 is the specific strain energy [3] of the elastic body (D).

We define $W_{A,B}$ the work that would be done by the system of external actions (A) (in the state of equilibrium due to (A)) in acting through the displacements' field due to another system of external actions (B), acting on (D), as going from the initial undeformed situation to the final deformed situation of equilibrium $(t = t_0)$ (Fig. 2).

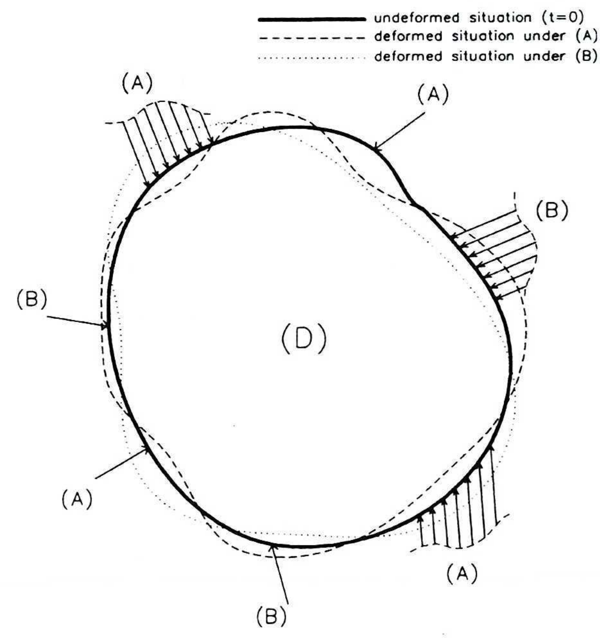

Fig. 2. Elastic body (D) subjected to the systems of external actions (A) or (B)

In case of lack of kinetic energy

$$W_{A,B} = \int_V \left(\int_0^{\varepsilon_{ij}^{(B)}} \sigma_{ij}^{(A)} d\varepsilon_{ij}^{(B)} \right) dV = \int_V \sigma_{ij}^{(A)} \varepsilon_{ij}^{(B)} dV, \qquad i,j = 1,2,3 \tag{4}$$

According to the proposed formalism, if (A) denotes the system of external actions yielding the work and (B) denotes the system of external actions yielding the displacements' field, then "W_{AB}" denotes that both systems of external actions (A) and (B) are progressing together from the initial undeformed situation to the final deformed situation of equilibrium whereas "$W_{A,B}$" denotes that the system of external actions (A) is already in equilibrium while the system of external actions (B) is progressing from the initial undeformed situation to the final deformed situation of equilibrium.

3. The two-actions theorem

Consider an elastic body (D) and two conservative systems [5], [6], of external actions (A) and (B) beginning to act on (D) at $t = 0$.

Let us calculate the work W_{AB} that would be done by the system of external actions (A) in acting through the displacements field due to the system of external actions (B) as both systems are progressing together from the initial undeformed situation to the final deformed situation of equilibrium.

Supposing the validity of the infinitesimal of strains, the linearity of the elastic constitutive law and considering that the work W_{AB} is a function of time, then from (1) it is obtained

$$\dot{W}_{AB} = \int_V \sigma_{ij}^{(A)} \dot{\varepsilon}_{ij}^{(B)} dV, \tag{5}$$

thus the work W_{AB}, is written

$$W_{AB} = \int_0^{t_m} \left(\int_V \sigma_{ij}^{(A)} \dot{\varepsilon}_{ij}^{(B)} dV \right) dt, \tag{6}$$

where t_A the time of evolution of system (A), t_B the time of evolution of system (B) and $t_m = \max(t_A, t_B)$. It is noticed that in relation (6) the external actions (A) and (B) begin to act at $t = 0$, and if $t_m = t_A$ ($t_A > t_B$) then $\dot{\varepsilon}_{ij}^{(B)} = 0$ for $t_B < t \le t_A = t_m$, while if $t_m = t_B$ ($t_A < t_B$) then $\dot{\varepsilon}_{ij}^{(A)} = 0$ (or $\sigma_{ij}^{(A)} = \text{const}$) for $t_A < t < t_B = t_m$.

Because of the conservativeness of (A) and (B), it follows

$$W_{AB} = \int_V \left(\int_0^{t_m} \sigma_{ij}^{(A)} \dot{\varepsilon}_{ij}^{(B)} dt \right) dV. \tag{7}$$

Applying integration by parts to relation (7) and considering that at $t=0$, $\sigma_{ij}^{(A)}=0$ and $\varepsilon_{ij}^{(B)}=0$, we have

$$\int_V\left(\int_0^{\varepsilon_{ij}^{(B)}}\sigma_{ij}^{(A)}d\varepsilon_{ij}^{(B)}\right)dV=\int_V\sigma_{ij}^{(A)}\varepsilon_{ij}^{(B)}dV-\int_V\left(\int_0^{\varepsilon_{km}^{(A)}}\sigma_{km}^{(B)}d\varepsilon_{km}^{(A)}\right)dV \tag{8}$$

or in terms of the proposed formalism

$$W_{AB}+W_{BA}=W_{A,B} \tag{9}$$

From relations (8) and (9), we have the following new energy theorem

THE TWO-ACTIONS THEOREM: *If on an elastic body with a linear constitutive elastic law and an infinitesimal strain field law, two conservative systems of external actions start to act simultaneously without provoking kinetic energy, then the sum of the work done by the first system in acting through the displacements due to the second system as both systems are progressing and the work done by the second system in acting through the displacements due to the first system as again both systems are progressing, is equal to the work that would be done by the first system being already developed in acting through the displacements due to the second system as it is progressing from the initial undeformed situation to its final deformed situation of equilibrium.*

Let $\mathrm{A},\mathrm{B},\Gamma,\Delta$ conservative systems of external actions acting on the elastic body (D).

Because of (9), a permutation property occurs

$$W_{A,B}=W_{B,A} \tag{10}$$

Taking into consideration (4) and (9), a distributive property occurs

$$\begin{gathered}W_{\mathrm{A,B}}+W_{\mathrm{A},\Gamma}=W_{\mathrm{A,(B+}\Gamma)}\\ W_{\mathrm{AB}}+W_{\mathrm{A}\Gamma}=W_{\mathrm{A(B+}\Gamma)}\end{gathered} \tag{11}$$

In the case of displacements superposition from (4), (9) and (11), it is obtained

$$\begin{gathered}W_{\mathrm{A,B}}+W_{\Gamma,\mathrm{B}}=W_{(\mathrm{A}+\Gamma),\mathrm{B}}\\ W_{\mathrm{AB}}+W_{\Gamma\mathrm{B}}=W_{(\mathrm{A}+\Gamma)\mathrm{B}}\end{gathered} \tag{12}$$

Hence, in the general case, it holds

$$\begin{gathered}W_{(\alpha\mathrm{A}+\beta\mathrm{B}),(c\Gamma+d\Delta)}=\alpha cW_{\mathrm{A},\Gamma}+\alpha dW_{\mathrm{A},\Delta}+\beta cW_{\mathrm{B},\Gamma}+\beta dW_{\mathrm{B},\Delta},\quad \alpha,\beta,c,d\in R,\\ W_{(\alpha\mathrm{A}+\beta\mathrm{B})(c\Gamma+d\Delta)}=\alpha cW_{\mathrm{A}\Gamma}+\alpha dW_{\mathrm{A}\Delta}+\beta cW_{\mathrm{B}\Gamma}+\beta dW_{\mathrm{B}\Delta},\quad \alpha,\beta,c,d\in R.\end{gathered} \tag{13}$$

The properties (9) and (13), derived according to the proposed formalism, have also been formulated by other investigators [3], [6], [10].

4. Formulation of the energy theorems in terms of the proposed two-actions theorem

In this paragraph the known energy theorems [2], [3], [6] are proved in terms of the proposed two actions' theorem.

The reciprocity theorem of Betti-Rayleigh is the *permutation property* of the two actions' theorem according to relation (10).

From relations (1) and (9) and for $(A) \equiv (B)$, Clapeyron's theorem [3], [6] is obtained

$$W_{A,A} = 2W_{AA} = 2U \tag{14}$$

Applying relation (9) to the systems of external actions (A) and $(\mathrm{B}) \equiv (\delta\mathrm{A})$, where $(\delta\mathrm{A})$ is a system provoking only virtual displacements consistent with constraints imposed on the body and without creating stresses that change the equilibrium due to $(\mathrm{A})(W_{(\delta\mathrm{A})\mathrm{A}} = 0)$, and taking into account relation (1), the principle of virtual work occurs

$$W_{\mathrm{A},\delta\mathrm{A}} = W_{\mathrm{A}(\delta\mathrm{A})} + W_{(\delta\mathrm{A})\mathrm{A}} = W_{\mathrm{A}(\delta\mathrm{A})} = \delta W_{\mathrm{AA}} = \delta U \tag{15}$$

Applying relation (9) to the systems of external actions $(\mathrm{A}) \equiv (\delta\mathrm{B})$ and (B), where $(\delta\mathrm{B})$ is a system provoking only virtual stresses without creating displacements that change the equilibrium due to $(\mathrm{B})(W_{\mathrm{B}(\delta\mathrm{B})} = 0)$, and taking into account relation (1), the principle of complementary virtual work occurs

$$W_{\delta\mathrm{B},\mathrm{B}} = W_{(\delta\mathrm{B})\mathrm{B}} + W_{\mathrm{B}(\delta\mathrm{B})} = W_{(\delta\mathrm{B})\mathrm{B}} = \delta U^* \tag{16}$$

where U^* is the complementary strain energy.

Applying relation (9) to the systems (A) and $(\delta\mathrm{A})$, where $(\delta\mathrm{A})$ is a virtual system of external actions, a known property of the linear elastic body occurs

$$\delta U = W_{\mathrm{A},\delta A} = W_{\delta A,A} = \delta U^* . \tag{17}$$

Let (A) and $(\mathrm{A} + \delta\mathrm{A})$ systems of external actions. With the use of relations (9) and (13), we have:

$$\Delta U = W_{(\mathrm{A}+\delta\mathrm{A})(\mathrm{A}+\delta\mathrm{A})} - W_{\mathrm{AA}} = \frac{1}{2} W_{(\delta\mathrm{A}),(2\mathrm{A}+\delta\mathrm{A})} \tag{18}$$

In case that $(\delta\mathrm{A})$ becomes infinitesimal small, it is obtained

$$\partial U = \frac{1}{2} W_{(\partial\mathrm{A}),(2\mathrm{A}+\partial\mathrm{A})} = W_{(\partial\mathrm{A}),\mathrm{A}} \tag{19}$$

From relation (19) if $(\partial\mathrm{A}) \equiv \partial Q_i$ (the i component of the load ∂Q), the first theorem of Castigliano occurs

$$\frac{\partial U}{\partial Q_i} = u_i \tag{20}$$

From relations (10) and (18), we get

$$\Delta U = \frac{1}{2} W_{(2\mathrm{A}+\delta\mathrm{A}),(\delta\mathrm{A})}$$

Respectively in the limit when $(\partial \mathrm{A}) \equiv \partial u_i$ (the i component of the displacement $\partial \boldsymbol{u}$), the second theorem of Castigliano occurs

$$\frac{\partial U}{\partial u_i} = Q_i \tag{21}$$

If (A) is a system of external actions, then the potential energy in terms of the proposed symbolism is written

$$\Pi(\mathrm{A}) = W_{\mathrm{AA}} - W_{\mathrm{A,A}}$$

Let $(\delta\mathrm{A})$ be a system of external actions creating a virtual variation $\delta\varepsilon_{ij}$ on the strain field and $\delta\sigma_{ij}$ on the stress field without changing the external forces $(W_{(\mathrm{A}+\delta\mathrm{A}),(\mathrm{A}+\delta\mathrm{A})} = W_{\mathrm{A},(\mathrm{A}+\delta\mathrm{A})})$, then

$$\Pi(\mathrm{A}+\delta\mathrm{A}) = W_{(\mathrm{A}+\delta\mathrm{A})(\mathrm{A}+\delta\mathrm{A})} - W_{\mathrm{A},(\mathrm{A}+\delta\mathrm{A})}$$

Hence

$$\Delta\Pi = \Pi(\mathrm{A}+\Delta\mathrm{A}) - \Pi(\mathrm{A}) = W_{(\delta\mathrm{A})(\delta\mathrm{A})} > 0,$$

as virtual strain energy. Taking into consideration that $(\delta\mathrm{A})$ is an arbitrary virtual system, the theorem of minimum potential energy occurs

$$\Pi(\mathrm{A}) = \min . \tag{22}$$

5. The use of the two-actions theorem in the case of a composite body

Let $(D) = (D_1) \cup (D_2)$ a system of two linear elastic bodies (Fig. 3) and $(\Delta) = (\Delta_1) + (\Delta_2)$ a conservative system [6], [7] of external actions acting on (D) except the bimaterial interface $S_c (S_1 \cap S_2)$ and without the creation of any kinetic energy, where

(Δ_1) is the part of (Δ) acting on the body (D_1),

(Δ_2) is the part of (Δ) acting on the body (D_2).

The body (D_1) is in equilibrium, under the action of the system $(A) = (\Delta_1) + (\Delta_1')$, where $(\Delta_1) \subset (\Delta)$ and (Δ_1') a system, considered as conservative [6], [7] in the present study, acting on the bimaterial interface S_c.

Taking into consideration the superposition principle, let the body (D) be in equilibrium under the system of external actions $\alpha(\Delta) = \alpha(\Delta_1) + \alpha(\Delta_2)$, $\alpha > 1$, without the creation of any kinetic energy. Under the above assumptions, (D_1) is in equilibrium under the action of the system $(\mathrm{B}) = \alpha(\Delta_1) + (\Delta_1'')$, where $\alpha(\Delta_1) \subset \alpha(\Delta)$ and (Δ_1'') a system considered as conservative, acting on the interface $S_c' \left(S_c' \neq S_c\right)$ where S_c' is the new bimaterial interface according to external actions $\alpha(\Delta)$. Hence the body (D_1) is finally in equilibrium under the following system of external actions

$$(\mathrm{B}) = \alpha(\Delta_1) + (\Delta_1'') \Rightarrow (\Gamma) = (\mathrm{B}) - \alpha(\mathrm{A}) = (\Delta_1'') - \alpha(\Delta_1'); \quad (\mathrm{A}) = (\Delta_1) + (\Delta_1').$$

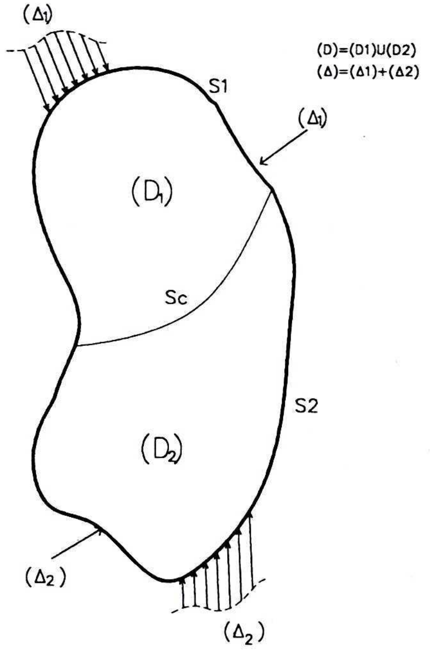

Fig. 3. A system (D) of two elastic bodies subjected to the external actions (Δ)

Applying relations (13), we have

$$W_{\Gamma\Gamma} = W_{(\mathrm{B}-\alpha\mathrm{A})(\mathrm{B}-\alpha\mathrm{A})} = \alpha^2 W_{\mathrm{AA}} - \alpha W_{\mathrm{AB}} - \alpha W_{\mathrm{BA}} + W_{\mathrm{BB}}$$

and taking into consideration the two actions' theorem, it is obtained

$$W_{\Gamma\Gamma} = W_{(\mathrm{B}-\alpha\mathrm{A})(\mathrm{B}-\alpha\mathrm{A})} = \alpha^2 W_{\mathrm{AA}} + W_{\mathrm{BB}} - \alpha W_{\mathrm{A,B}} \tag{23}$$

Relation (23) gives the strain energy of the elastic body (D_1) under the superposition of the following systems of external actions

(i) System: (B),

(ii) System: $-\alpha(\mathrm{A})$, $\alpha > 1$.

In the case that $0 < \alpha - 1 << 1$, the quantity $(a-1)^2$ is approximately zero, thus relation (23) using relations (9) and (13), takes the approximate form

$$
\begin{aligned}
W_{\Gamma\Gamma} &\cong (2a-1)W_{AA} + W_{BB} - aW_{A,B} = \\
&= \left[1+2(a-1)\right]W_{AA} + W_{BB} - \left[1+(a-1)\right]W_{AB} = \\
&= W_{AA} + W_{BB} - W_{AB} - W_{BA} + (a-1)(2W_{AA} - W_{AB} - W_{BA}) = \\
&= W_{A(A-B)} + W_{B(B-A)} + (a-1)\left(W_{A(A-B)} + W_{(A-B)A}\right) = \\
&= -W_{A(B-A)} + W_{B(B-A)} + (a-1)W_{A,(A-B)} = W_{(B-A)(B-A)} + (a-1)W_{A,(A-B)}
\end{aligned}
$$

or

$$
W_{(\mathrm{B-A})(\mathrm{B-A})} \cong W_{(\mathrm{B}-\alpha\mathrm{A})(\mathrm{B}-\alpha\mathrm{A})} - (\alpha-1)W_{\mathrm{A},(\mathrm{A-B})}, \quad 0<\alpha-1<<1, \qquad (\Gamma) \equiv (\mathrm{B}) - \alpha(A). \tag{24}
$$

Relation (24) gives the strain energy of the elastic body (D_1) under the action of the system $(\mathrm{B-A})$. In addition, the variation of the strain energy in (D_1) when the system of external actions (Δ) acting on (D) is replaced by the system $\alpha(\Delta)$, would be

$$\Delta U = W_{\mathrm{BB}} - W_{\mathrm{AA}},$$

where

$$(\mathrm{B}) = (\mathrm{A}) + (\partial\mathrm{A}), \quad (\partial\mathrm{A}) = (\alpha-1)(\Delta_1) + (\Delta_1'' - \Delta_1'); \quad 0<\alpha-1<<1.$$

In the case that $(\partial\mathrm{A})$ is an infinitesimal variation of the system (A), it can be considered that

$$(\mathrm{B}) \cong \alpha(\mathrm{A}), \quad (\partial\mathrm{A}) \cong (\alpha-1)(\mathrm{A}); \quad 0<\alpha-1<<1.$$

Hence

$$\Delta U = W_{\mathrm{BB}} - W_{\mathrm{AA}} \cong \alpha W_{\mathrm{A},(\partial\mathrm{A})}; \quad (\mathrm{B}) = (\mathrm{A}) + (\partial\mathrm{A}), \quad 0<\alpha-1<<1 \tag{25}$$

Relation (23), or in some cases the approximate relation (24), calculates the strain energy $W_{\Gamma\Gamma}$ due to the superposition of the external actions $\Gamma \equiv (B) - \alpha(A)$, $\alpha > 1$. Ensuring that $W_{\Gamma\Gamma}$ is equal or not to zero, an appropriate procedure can be chosen in order to study the behaviour of a particular constituent of the composite body, as it has been analyzed in Section 7.

6. Applications

Two applications are made in the case of a bimaterial composite body in order to verify the validity of the energy relations (23), (24) and (25). The importance of the results combined with an appropriate procedure, is analyzed in Section 7.

6.1. The axisymmetric deformation of a composed two materials hollow cylinder

In this application we are going to express the energy relations (23), (24) and (25), in the case of the axisymmetrical deformation of a composed two material hollow cylinder with uniform internal pressure p and external pressure q and free ends in plane strain conditions (Fig. 4). The system of bodies $(D)=(D_1)\cup(D_2)$ is consisted of the internal hollow cylinder $(D_1)(\lambda_1,\mu_1)$ and the external hollow cylinder $(D_2)(\lambda_2,\mu_2)$, where λ_i $(i=1,2)$ is the Lamé constant and μ_i is the shear modulus. Let, S_c, the bimaterial interface of radius ρ' between the two cylinders.

The stress and displacement fields for the internal cylinder (D_1) are given [4] by

$$
\begin{aligned}
&\sigma_{rr}(r)=-p\frac{\dfrac{\rho'^2}{r^2}-1}{\dfrac{\rho'^2}{\rho^2}-1}-q'\frac{1-\dfrac{\rho^2}{r^2}}{1-\dfrac{\rho^2}{\rho'^2}},\quad \sigma_{\theta\theta}(r)=p\frac{\dfrac{\rho'^2}{r^2}+1}{\dfrac{\rho'^2}{\rho^2}-1}-q'\frac{1+\dfrac{\rho^2}{r^2}}{1-\dfrac{\rho^2}{\rho'^2}},\\
&\sigma_{r\theta}(r)=0,\\
&u_r(r)=r\frac{p\left[(\lambda_1+\mu_1)\dfrac{\rho'^2}{r^2}+\mu_1\right]-q'\left[\mu_1\dfrac{\rho'^2}{\rho^2}+(\lambda_1+\mu_1)\dfrac{\rho'^2}{r^2}\right]}{2\mu_1(\lambda_1+\mu_1)\left(\dfrac{\rho'^2}{\rho^2}-1\right)},\\
&u_\theta(r)=0,
\end{aligned}
\tag{26}
$$

and for the external cylinder (D_2) by the relations

$$
\begin{aligned}
&\sigma_{rr}(r)=-q'\frac{\dfrac{R^2}{r^2}-1}{\dfrac{R^2}{\rho'^2}-1}-q\frac{1-\dfrac{\rho'^2}{r^2}}{1-\dfrac{\rho'^2}{R^2}},\quad \sigma_{\theta\theta}(r)=q'\frac{\dfrac{R^2}{r^2}+1}{\dfrac{R^2}{\rho'^2}-1}-q\frac{1+\dfrac{\rho'^2}{r^2}}{1-\dfrac{\rho'^2}{R^2}},\\
&\sigma_{r\theta}(r)=0,\\
&u_r(r)=r\frac{q'\left[(\lambda_2+\mu_2)\dfrac{R^2}{r^2}+\mu_2\right]-q\left[\mu_2\dfrac{R^2}{\rho'^2}+(\lambda_2+\mu_2)\dfrac{R^2}{r^2}\right]}{2\mu_2(\lambda_2+\mu_2)\left(\dfrac{R^2}{\rho'^2}-1\right)},\\
&u_\theta(r)=0,
\end{aligned}
\tag{27}
$$

where q' is the uniform pressure acting on S_c.

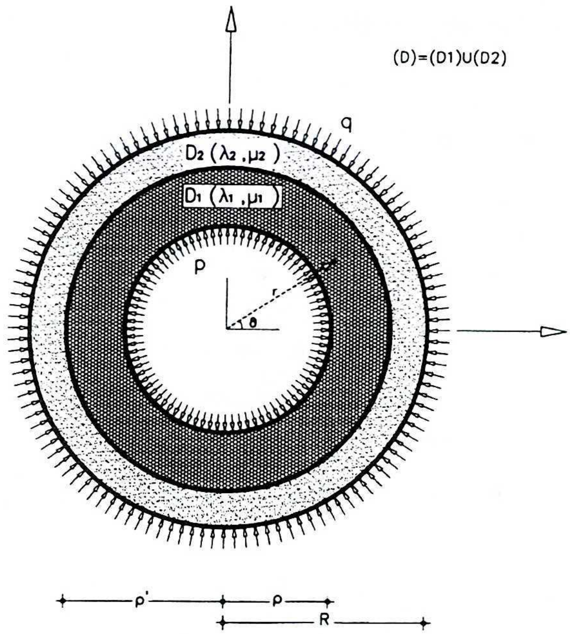

Fig. 4. A two material cylindrical pipe under uniform internal and external pressure

Taking into account the continuity of stresses and displacements between the two cylinders (D_1) and (D_2) and the relations (26) and (27), it is finally obtained

$$q' = \frac{pF_p + qF_q}{F}, \tag{28}$$

where

$$F_p = \rho^2\left(R^2 - \rho'^2\right)\left(\lambda_1 + 2\mu_1\right)\mu_2\left(\lambda_2 + \mu_2\right),$$
$$F_q = R^2\left(\rho'^2 - \rho^2\right)\left(\lambda_2 + 2\mu_2\right)\mu_1\left(\lambda_1 + \mu_1\right),$$
$$F = \left(\rho'^2 - \rho^2\right)\left[\lambda_2 R^2 + \mu_2\left(R^2 + \rho'^2\right)\right]\mu_1\left(\lambda_1 + \mu_1\right) +$$
$$+\left(R^2 - \rho'^2\right)\left[\lambda_1 \rho^2 + \mu_1\left(\rho^2 + \rho'^2\right)\right]\mu_2\left(\lambda_2 + \mu_2\right).$$

The strain energy of the internal cylinder (D_1) per unit length, is given by the formula

$$U = \iint U_0 r d\theta dr = 2\pi \int_{\rho}^{\rho'} U_0 r dr \tag{29}$$

where the elastic potential (specific strain energy) U_0 is given [4] by:

$$U_0 = \frac{\lambda_1}{2}\left(\varepsilon_{rr} + \varepsilon_{\theta\theta}\right)^2 + \mu_1\left(\varepsilon_{rr}^2 + \varepsilon_{\theta\theta}^2\right) =$$
$$= \frac{1}{2\left(\rho'^2 - \rho^2\right)^2}\left[\frac{\left(p\rho^2 - q'\rho'^2\right)^2}{\lambda_1 + \mu_1} + \frac{\left(p - q'\right)^2 \rho^4 \rho'^4}{\mu_1 r^4}\right]. \tag{30}$$

Hence, relation (29) is finally written:

$$U = W_{AA} = \frac{\pi}{2(\rho'^2 - \rho^2)} \left[\frac{(p\rho^2 - q'\rho'^2)^2}{\lambda_1 + \mu_1} + \frac{(p-q')^2 \rho^2 \rho'^2}{\mu_1} \right], \tag{31a}$$

or

$$U = \frac{1}{2}\left[2\pi\rho p u_r(\rho) + 2\pi\rho'(-q')u_r(\rho')\right] = \frac{1}{2}W_{A.A} \tag{31b}$$

where $(A) = (\Delta_1) + (\Delta_1')$, $(\Delta_1) = p$, $(\Delta_1') = q'$.

Let (B) a system acting on (D_1), consisting of an internal uniform pressure $\alpha p\,(\alpha > 1)$ and a uniform external pressure q'' due to (D_2). Hence, the superposition $(\Gamma) \equiv (B) - \alpha(A)$ is consisted only from the uniform external pressure $q'' - \alpha q'$ acting on S_c. Applying the relation (31a) for the systems (A), (B) and (Γ), we get

$$W_{\Gamma\Gamma} = \frac{\pi(q'' - \alpha q')^2 \rho'^2}{2(\rho'^2 - \rho^2)} \left(\frac{\rho'^2}{\lambda_1 + \mu_1} + \frac{\rho^2}{\mu_1} \right),$$

$$W_{AA} = \frac{\pi}{2(\rho'^2 - \rho^2)} \left[\frac{(p\rho^2 - q'\rho'^2)^2}{\lambda_1 + \mu_1} + \frac{(p-q')^2 \rho^2 \rho'^2}{\mu_1} \right], \tag{32}$$

$$W_{BB} = \frac{\pi}{2(\rho'^2 - \rho^2)} \left[\frac{(\alpha p\rho^2 - q''\rho'^2)^2}{\lambda_1 + \mu_1} + \frac{(\alpha p - q'')^2 \rho^2 \rho'^2}{\mu_1} \right],$$

and because of the definition of $W_{A.B}$ (§ 2)

$$\begin{aligned} W_{A.B} &= 2\pi\rho p u_{r_B}(\rho) - 2\pi\rho' q' u_{r_B}(\rho') = \\ &= \frac{\pi}{2(\rho'^2 - \rho^2)} \left\{ \frac{2}{\lambda_1 + \mu_1} (p\rho^2 - q'\rho'^2)(\alpha p\rho^2 - q''\rho'^2) + \right. \\ &\left. + \frac{2}{\mu_1} \rho^2 \rho'^2 (p - q')(\alpha p - q'') \right\}. \end{aligned} \tag{33}$$

Using relations (32) and (33), relation (23) is proved.

Let consider now the system $(A - B)$ acting on (D_1) and consisting of an internal uniform pressure $(1-\alpha)p$ and an external uniform pressure $q' - q''$ with $0 < \alpha - 1 << 1$. Applying the relation (31a) for the system $(A - B)$ and the superposition system $(\Gamma) \equiv (B) - \alpha(A)$, we obtain

$$W_{(A-B)(A-B)} \cong \frac{\pi}{2(\rho'^2 - \rho^2)} \left[\frac{A}{\lambda_1 + \mu_1} + \frac{B}{\mu_1} \right], \tag{34}$$

where

$$A=(q'-q'')\rho'^2\left[(q'-q'')\rho'^2-2(1-\alpha)p\rho^2\right],$$
$$B=(q'-q'')\left[q'-q''-2(1-\alpha)p\right]\rho^2\rho'^2,$$

and

$$W_{(\mathrm{B}-\alpha\mathrm{A})(\mathrm{B}-\alpha\mathrm{A})}=W_{\Gamma\Gamma}=\frac{\pi}{2(\rho'^2-\rho^2)}\left[\frac{(q''-\alpha q')^2\rho'^4}{\lambda_1+\mu_1}+\frac{(q''-\alpha q')^2\rho^2\rho'^2}{\mu_1}\right] \tag{35}$$

Because of the definition of $W_{\mathrm{A.B}}$ (§ 2)

$$W_{\mathrm{A.(A-B)}}=2\pi\rho p u_{r_{(\mathrm{A-B})}}(\rho)-2\pi\rho' q' u_{r_{(\mathrm{A-B})}}(\rho') \tag{36}$$

Using relations (34), (35), (36) and (26) the relation (24) is proved.

Finally, let us consider a system $(\mathrm{B})\cong\alpha(\mathrm{A})$, $0<\alpha-1<<1$, acting on (D_1) and consisting of an internal uniform pressure αp and an external uniform pressure $q''\cong\alpha q'$. In this case, the superposition system (Γ) is zero, and the system $(\mathrm{B})-(\mathrm{A})=(\partial\mathrm{A})$ is consisting of an internal pressure $(\alpha-1)p$ and an external pressure $(\alpha-1)q'$. Applying relations (32), we get

$$W_{\mathrm{BB}}-W_{\mathrm{AA}}\cong\frac{\pi(\alpha-1)}{\rho'^2-\rho^2}\left[\frac{1}{\lambda_1+\mu_1}(p\rho^2-q'\rho'^2)^2+\frac{1}{\mu_1}(p-q')^2\rho^2\rho'^2\right] \tag{37}$$

Taking into consideration the definition of $W_{\mathrm{A.B}}$ (§2) and relations (26), the relation (25) is finally proved.

6.2. Stretching of an infinite plate with an inserted bonded elastic disc

Let an infinite medium D_2 (μ_2,κ_2) with a circular hole of radius R_2 into which another circular elastic disc with different elastic properties $D_1(\mu_1,\kappa_1)$ and with an originally larger radius $R_1=R_2+\varepsilon$ $(\varepsilon>0)$ is inserted (Fig. 5), where $\mu_i\,(i=1,2)$ is the shear modulus, $\kappa_i=3-4\nu_i$ for plane strain or $\kappa_i=(3-4\nu_i)/(1+\nu_i)$ for generalized plane stress conditions and ν_i is the Poisson's ratio. The infinite matrix D_2 is also subjected to biaxial tension at infinity.

The stress and displacement fields for the inclusion, are given [17] by

$$\sigma_{1_{rr}}=\frac{p+q}{2}\Lambda+\frac{p-q}{2}M\cos 2\theta-P,$$
$$\sigma_{1_{\theta\theta}}=\frac{p+q}{2}\Lambda-\frac{p-q}{2}M\cos 2\theta-P,$$
$$\sigma_{1_{r\theta}}=-\frac{p-q}{2}M\sin 2\theta,$$

$$
\begin{gathered}
2\mu_1 u_{1_r} = \frac{p+q}{4}\Lambda(\kappa_1 - 1) r + \frac{p-q}{2} M r \cos 2\theta - \frac{P}{2}(\kappa_1 - 1) r, \\
2\mu_1 u_{1_\theta} = -\frac{p-q}{2} M r \sin 2\theta,
\end{gathered}
\tag{38}
$$

where

$$
\Lambda = \frac{\mu_1(\kappa_2 + 1)}{2\mu_1 + \mu_2(\kappa_1 - 1)}, \qquad M = \frac{\mu_1(\kappa_2 + 1)}{\mu_2 + \kappa_2 \mu_1}, \qquad P = \frac{4\mu_2 \Lambda \varepsilon}{R_2(\kappa_2 + 1)}.
$$

Let, (A) the stress and displacement fields at the boundary of the inclusion $(r = R_1 \cong R_2)$ due to the biaxial tension at infinity (Fig. 5), given by

$$
\begin{gathered}
N_{(\theta)}^{(A)} = \sigma_{1_{rr}}^{(A)} = \frac{p+q}{2}\Lambda + \frac{p-q}{2} M \cos 2\theta - P, \\
T_{(\theta)}^{(A)} = \sigma_{1_{r\theta}}^{(A)} = -\frac{p-q}{2} M \sin 2\theta, \\
2\mu_1 u_{1_r}^{(A)}(R_2, \theta) = \frac{p+q}{4}\Lambda R_2(\kappa_1 - 1) + \frac{p-q}{2} M R_2 \cos 2\theta - \frac{P}{2}(\kappa_1 - 1) R_2, \\
2\mu_1 u_{1_\theta}^{(A)}(R_2, \theta) = -\frac{p-q}{2} M R_2 \sin 2\theta.
\end{gathered}
\tag{39}
$$

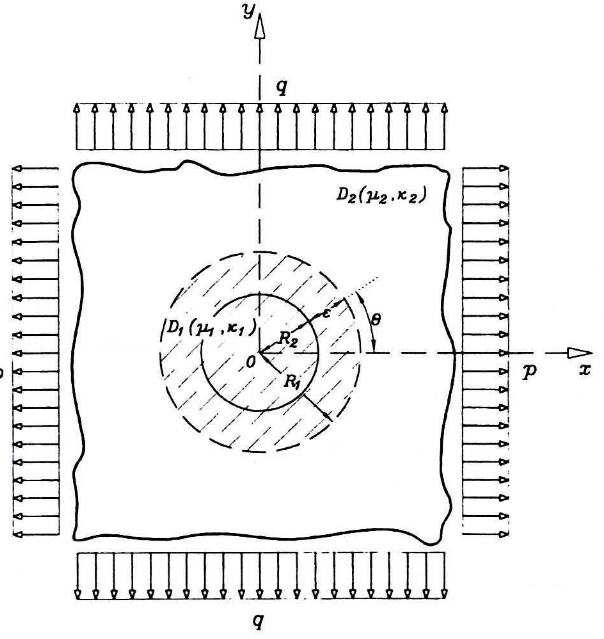

Fig. 5. Stretching of an infinite plate with an inserted elastic disc

Let, (B) the stress and displacement fields at the boundary of the inclusion $(r = R_1 \cong R_2)$ due to the actions $ap(a>1)$ and q at infinity, given by

$$N_{(\theta)}^{(B)} = \sigma_{1_{rr}}^{(B)} = \frac{ap+q}{2}\Lambda + \frac{ap-q}{2}M\cos 2\theta - P,$$
$$T_{(\theta)}^{(B)} = \sigma_{1_{r\theta}}^{(B)} = -\frac{\alpha p - q}{2}M\sin 2\theta,$$
$$2\mu_1 u_{1_r}^{(B)}(R_2,\theta) = \frac{\alpha p+q}{4}\Lambda R_2(\kappa_1 - 1) + \frac{\alpha p - q}{2}MR_2\cos 2\theta - \frac{P}{2}(\kappa_1 - 1)R_2, \tag{40}$$
$$2\mu_1 u_{1_\theta}^{(B)}(R_2,\theta) = -\frac{\alpha p - q}{2}MR_2\sin 2\theta.$$

If the loading $(\Gamma) = (\mathrm{B}) - a(A)(a>1)$ is considered as the superposition of the actions (B) and $-a(A)$ at the boundary of the inclusion, we have

$$N_{(\theta)}^{(\Gamma)} = N_{(\theta)}^{(B)} - \alpha N_{(\theta)}^{(A)} = P(\alpha-1) - \frac{q}{2}(\Lambda - M\cos 2\theta)(\alpha - 1),$$
$$T_{(\theta)}^{(\Gamma)} = T_{(\theta)}^{(B)} - \alpha T_{(\theta)}^{(A)} = -\frac{q}{2}M(\alpha-1)\sin 2\theta,$$
$$2\mu_1 u_{1_r}^{(\Gamma)}(R_2,\theta) = \frac{PR_2}{2}(\kappa_1 - 1)(\alpha - 1) - \frac{qR_2}{4}\left[\Lambda(\kappa_1 - 1) - 2M\cos 2\theta\right](\alpha - 1),$$
$$2\mu_1 u_{1_\theta}^{(\Gamma)}(R_2,\theta) = -\frac{qR_2}{2}(\alpha - 1)M\sin 2\theta.$$
$$\tag{41}$$

Taking into consideration the relation for the strain energy

$$U = \frac{1}{2}\int_0^{2\pi}\left[N(\theta)u_{1_r}(R_2,\theta) + T(\theta)u_{1_\theta}(R_2,\theta)\right]R_2 d\theta\,, \tag{42}$$

we obtain for the different actions (A), (B) and (Γ)

$$\begin{aligned} U^{(A)} = W_{AA} = & \frac{(p+q)^2 R_2^2\pi}{16\mu_1}\Lambda^2(\kappa_1 - 1) + \\ & + \frac{(p-q)^2 R_2^2\pi}{8\mu_1}M^2 + \frac{PR_2^2\pi}{4\mu_1}(\kappa_1 - 1)(P - (p+q)\Lambda), \end{aligned} \tag{43}$$

$$\begin{aligned} U^{(B)} = W_{BB} = & \frac{(\alpha p+q)^2 R_2^2\pi}{16\mu_1}\Lambda^2(\kappa_1 - 1) + \\ & + \frac{(\alpha p-q)^2 R_2^2\pi}{8\mu_1}M^2 + \frac{PR_2^2\pi}{4\mu_1}(\kappa_1 - 1)(P - (\alpha p+q)\Lambda), \end{aligned} \tag{44}$$

$$
\begin{aligned}
U^{(\Gamma)} = W_{\Gamma\Gamma} = & \frac{(\alpha-1)^2 PR_2^2\pi}{4\mu_1}(\kappa_1-1)(P-q\Lambda)+ \\
& +\frac{(\alpha-1)^2 q^2 R_2^2\pi}{16\mu_1}\left[(\kappa_1-1)\Lambda^2+2M^2\right],
\end{aligned}
\tag{45}
$$

and because of the definition of $W_{A,B}$ (relation (4))

$$
\begin{aligned}
W_{A,B} = & \frac{R_2^2\pi}{8\mu_1}(p+q)(\alpha p+q)\Lambda^2(\kappa_1-1)+\frac{R_2^2\pi}{4\mu_1}(p-q)(\alpha p-q)M^2+ \\
& +\frac{PR_2^2\pi}{4\mu_1}(\kappa_1-1)\left(2P-\left[(\alpha+1)p+2q\right]\Lambda\right).
\end{aligned}
\tag{46}
$$

Using relations (43), (44), (45) and (46), relation (23) is proved for the inclusion when the infinite matrix is subjected to biaxial tension at infinity.

Let $(\mathrm{A}-\mathrm{B})$ the stress and displacement fields at the boundary of the inclusion $(r=R_1\cong R_2)$ due to the unidirectional tension at infinity with $0<a-1<<1$. Applying relations (38) with $(1-a)p$ and $q=0$, we obtain

$$
\begin{aligned}
N_{(\theta)}^{(A-B)} &= \frac{(1-\alpha)}{2}(\Lambda+M\cos 2\theta)-P, \\
T_{(\theta)}^{(A-B)} &= -\frac{(1-\alpha)P}{2}M\sin 2\theta, \\
2\mu_1 u_{1_r}^{(A-B)}(R_2,\theta) &= \frac{(1-\alpha)pR_2}{4}\left[\Lambda(\kappa_1-1)+2M\cos 2\theta\right]-\frac{PR_2}{2}(\kappa_1-1), \\
2\mu_1 u_{1_\theta}^{(A-B)}(R_2,\theta) &= -\frac{(1-\alpha)P}{2}MR_2\sin 2\theta.
\end{aligned}
\tag{47}
$$

In a similar way, and taking into account that $(\alpha-1)^2\cong 0$, we may obtain the following relations

$$
W_{(A-B)(A-B)} \cong \frac{PR_2^2\pi}{4\mu_1}(\kappa_1-1)\left[P-(1-\alpha)p\Lambda\right], \tag{48}
$$

$$
\begin{aligned}
W_{A,(A-B)} = & \frac{R_2^2\pi(1-\alpha)p}{8\mu_1}\left[(p+q)\Lambda^2(\kappa_1-1)+2(p-q)M^2-2P\Lambda(\kappa_1-1)\right]+ \\
& +\frac{PR_2^2\pi}{4\mu_1}(\kappa_1-1)\left[2P-(p+q)\Lambda\right].
\end{aligned}
\tag{49}
$$

It is also obtained from relation (45) with $(\alpha-1)^2\cong 0$, that

$$
W_{\Gamma\Gamma}\cong 0. \tag{50}
$$

Thus

$$W_{\Gamma\Gamma} - (\alpha - 1) W_{A,(A-B)} \cong \frac{P R_2^2 \pi (1-\alpha)}{4\mu_1} (\kappa_1 - 1)\left[2P - (p+q)\Lambda\right]. \tag{51}$$

Taking into consideration relations (48) and (51), relation (24) is valid if

$$P = 0 \Leftrightarrow \varepsilon = 0. \tag{52}$$

From relations (43), (44) and in the case that $(a-1)^2 \cong 0$, we get

$$W_{BB} - W_{AA} \cong \frac{R_2^2 \pi (\alpha - 1) p}{8\mu_1}\left[\Lambda^2 (\kappa_1 - 1)(p+q) + 2M^2 (p-q) - 2P\Lambda(\kappa_1 - 1)\right]. \tag{53}$$

In addition, from relation (49) and for $(B)-(A)=(\partial A)$, we have

$$\begin{aligned} \alpha W_{A,\partial A} = {} & \frac{R_2^2 \pi (\alpha - 1) p}{8\mu_1}\left[\Lambda^2 (\kappa_1 - 1)(p+q) + 2M^2 (p-q) - 2P\Lambda(\kappa_1 - 1)\right] - \\ & - \frac{\alpha P R_2^2 \pi}{4\mu_1} (\kappa_1 - 1)\left(2P - (p+q)\Lambda\right). \end{aligned} \tag{54}$$

From relations (53) and (54), relation (25) is valid if

$$P = 0 \Leftrightarrow \varepsilon = 0. \tag{55}$$

7. Discussion and conclusions

The proposed two-actions theorem and the introduced formalism (equation (9)), because of their generality and simplicity, may play an important role in understanding and an important practical role in the application of the energy theorems. The presented methodology simplifies the energy analysis, as it has been proved in Section 4, because all the known energy theorems are derived easily from the application of the two-actions theorem. On the other hand, the simplicity of the proposed formalism and the two-actions theorem, result in new energy relations in the case of composite bodies.

Based on the new formalism and the superposition principle, the new energy relations are introduced in Sections 5 and 6 in the case of a two material composite body. From the two applications it is concluded that the strain energy $W_{\Gamma\Gamma}$ due to the superposition of the external actions $(B)-\alpha(A)$, $\alpha > 1$, is nil in the composite constituent (D_1) only in the case that:

The action (B) on (D_1) caused by the external action $\alpha(\Delta)$ on the composite body (D), is an exact multiplier of the action (A) on (D_1) due to (Δ), in the case that at the interface between the two bodies (D_1) and (D_2), there is not any imposed deformation.

Hence, there is no difference between the following two procedures:

- At first the action on the composite body (D) is increased, and secondly the (D_1) body is separated from the composite.
- At first the (D_1) body is separated from the composite (D) and secondly the action on (D_1) is increased.

Namely, the procedure "increase and separate" is equivalent to the procedure "separate and increase" for a constituent of the composite.
On the contrary $W_{\Gamma\Gamma}$ is not nil in the cases that:

(i) The action (B) on (D_1) caused by the external action $\alpha(\Delta)$ on (D), is not an exact multiplier of the action (A) on (D_1) due to (Δ).

(ii) There is some imposed deformation at the interface between the two bodies.

In this case, there is a difference between the previously mentioned procedures. Namely, "increasing the action on the composite (D) and after separating the (D_1) body" is different from "separating the (D_1) body from the composite (D) and after increasing the action on (D_1)".

The proposed energy relations help the investigator to select the appropriate procedure in order to study the behaviour of a constituent of the composite when the external actions on the composite vary. Evidently, the proposed energy relations may also be applied to the case of a composite consisting of more than two different bodies or for analyzing a part of a composite.

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Twierdzenie o podwójnych oddziaływaniach i jego zastosowanie do materiałów kompozytowych

Streszczenie

W pracy przedstawiono nowe twierdzenie, "Twierdzenie o podwójnych oddziaływaniach", które ma zastosowanie w liniowych zagadnieniach elastostatycznych. Wprowadzono nowy formalizm dla wyrażenia pracy wykonanej przez oddziaływania zewnętrzne. Twierdzenie o podwójnych oddziaływaniach może posłużyć do weryfikacji znanych twierdzeń związanych z energią. Twierdzenie wykorzystano w zagadnieniu dotyczącym materiałów kompozytowych formułując zależności energetyczne. Ponadto, zaprezentowano dwa przykłady aplikacji twierdzenia: w zastosowaniu do cylindra złożonego z dwu cylindrów z różnych materiałów poddanego ciśnieniu wewnętrznemu i zewnętrznemu, oraz w zagadnieniu rozciągania nieskończonej płyty z wbudowanym elastycznym dyskiem z innego materiału. W aplikacjach tych zweryfikowano zaproponowane związki energetyczne.