THE ARCHIVE OF MECHANICAL ENGINEERING

Vol. XLVIII 2001 Number 2

Key words: *uniaxial stress, multiaxial stress, fatigue limits, fatigue criteria*

JANUSZ KOLENDA *)

ON FATIGUE SAFETY CRITERIA OF MACHINERY PARTS IN DETERMINISTIC AND PROBABILISTIC APPROACH

The paper deals with the safety criteria of design for an infinite fatigue life of machinery parts. Uniaxial and multiaxial zero mean stress states are considered. In the latter case, constant-amplitude in-phase stress components, as well as random-amplitude synchronous stress components, are taken into account. Dimensionless and relative safety margins for these stress states are defined. The presented criteria refer to ductile materials showing true fatigue limits. Transformation rules in the plane are given for fatigue limits referenced to coordinate system different than the components of the plane stress.

1. Background

In the design for an infinite fatigue life of engineering members under constant-amplitude uniaxial loading conditions, the fatigue safety factor, f, and the fatigue safety margin, M, are given by

$$f = \frac{F}{\sigma} \tag{1}$$

$$M = F - \sigma \tag{2}$$

where F is the fatigue limit and σ is the stress amplitude. Neither materials that do not show true fatigue limit nor the influence of mean stress on the fatigue safety will be considered in this paper. Having calculated the fatigue safety factor, one can define the dimensionless fatigue safety margin, m, by

$$m = f - 1 \tag{3}$$

Instead of these safety margins, the relative fatigue safety margin

*) 1. *Technical University of Gdańsk; ul. G. Narutowicza 11/12, 80-952 Gdańsk, Poland; E-mail: sekoca@pg.gda.pl*
2. *Polish Naval Academy; ul. Śmidowicza 71, 81-919 Gdynia, Poland*

$$\mu = \frac{M}{F} = \frac{m}{f} \tag{4}$$

can be introduced, that is

$$\mu = 1 - \frac{1}{f} \tag{5}$$

The criterion of an infinite fatigue life can be expressed as

$$\mu \geq 0 \tag{6}$$

The limiting state of infinite fatigue life is reached at $\mu = 0$. A fatigue damage occurs at $\mu < 0$.

When σ is a random variable, an infinite fatigue life may be expected if

$$\mathrm{E}\{\mu\} \geq 0 \tag{7}$$

or, in a more conservative approach,

$$E\{\mu\} - js_{\mu} \geq 0, \quad j > 0 \tag{8}$$

where: $E\{\ \}$ — expected value,
j — arbitrary number (see the Example),
s_{μ} — standard deviation of μ.

The probability of an infinite fatigue life, P, is

$$P\{\sigma \leq F\} = D_{\sigma}(F) \tag{9}$$

where D_{σ} is the distribution function of σ.

Relationships similar to Eqs (5) through (8) can be written in the case of multiaxial stress [1]. For example, at in-phase bending and torsion, the fatigue safety factor is [2]

$$f = \left(f_b^{-2} + f_t^{-2}\right)^{-1/2} \tag{10}$$

so that

$$\mu = 1 - \left(f_b^{-2} + f_t^{-2}\right)^{1/2} \tag{11}$$

Here

$$f_b = \frac{F_b}{\sigma_b}, \qquad f_t = \frac{F_t}{\sigma_t}$$

are the partial fatigue safety factors, where F_b, F_t and σ_b, σ_t are the fatigue limits and stress amplitudes at fully reversed bending and torsion, respectively. Adequate size factors, notch sensitivity indices, etc., can be included in a usual way [2].

The safe region and the failure region in the σ_b, σ_t pane are separated by an elliptic curve given by the interaction equation

$$f^{-2} - 1 = 0 \tag{12}$$

Eq. (10) and the following approximate relationship for ductile materials [2]

$$\frac{F_b}{F_t}=\frac{Z_{go}}{Z_{so}}=\sqrt{3} \tag{13}$$

lead to

$$f=F_b\left(\sigma_b^2+3\sigma_t^2\right)^{-1/2} \tag{14}$$

Eq. (14) confirms the well-known fact that in-phase stress can be treated by the distortion-energy theory [3], [4]. Therefore, this theory will be also used below.

2. General state of in-phase stress

According to the distortion-energy theory [5], two stress states are equivalent in terms of effort of the material if the strain energies of distortion in both these states are equal.

Let us assume that the stress components in combined in-phase bending and torsion and in the general state of in-phase stress have the same frequency, and that the maximum strain energies of distortion per unit volume in both these states are equal, i.e.,

$$\frac{1+\nu}{3E}\left(\sigma_b^2+3\sigma_t^2\right)=\frac{1+\nu}{3E}\left[\sigma_x^2+\sigma_z^2+\sigma_z^2-\sigma_x\sigma_y-\sigma_y\sigma_z-\sigma_z\sigma_x+\right.$$
$$\left.+3\left(\sigma_{xy}^2+\sigma_{yz}^2+\sigma_{zx}^2\right)\right] \tag{15}$$

where: E — Young modulus,
ν — Poisson's ratio,
σ_i — amplitude of i-th Cartesian component in the general state of in-phase stress ($i = x, y, z, xy, yz, zx$).

Combining Eqs (13) through (15) yields

$$f=F_bF_t\left[F_t^2\left(\sigma_x^2+\sigma_y^2+\sigma_z^2-\sigma_x\sigma_y-\sigma_y\sigma_z-\sigma_z\sigma_x\right)+\right.$$
$$\left.+F_b^2\left(\sigma_{xy}^2+\sigma_{yz}^2+\sigma_{zx}^2\right)\right]^{-1/2} \tag{16}$$

In order to account for other load modes (tension-compression, shear) and material anisotropy, the amplitudes $\sigma_x,\sigma_y,\sigma_z$ in Eq. (16) can be multiplied by modifying factors F_b/F_x, F_b/F_y, F_b/F_z and $\sigma_{xy},\sigma_{yz},\sigma_{zx}$ by F_t/F_{xy}, F_t/F_{yz}, F_t/F_{zx}, respectively, where F_i is the fatigue limit at the load producing stress of an amplitude σ_i. Then Eq. (16) becomes [1]

$$f=\left[\sum_i f_i^{-2}-\left(f_xf_y\right)^{-1}-\left(f_yf_z\right)^{-1}-\left(f_zf_x\right)^{-1}\right]^{-1/2} \tag{17}$$

where $f_i = F_i/\sigma_i$ is the i-th partial fatigue safety factor. In Eq. (17), f represents the fatigue safety factor in the general state of in-phase stress which after substitution in Eqs (5) and 12) transforms μ into the relative fatigue safety margin in design for an infinite fatigue life under multiaxial in-phase loading conditions and Eq. (12) into the interaction equation of in-phase Cartesian stress components. If there are bending stresses, the relevant signs „–" in Eq. (17) may have to be replaced by „+". Eq. (17) includes material constants that have simple physical interpretation, can be determined by uniaxial tests, are related directly to the applied load, and can reflect the material anisotropy.

With Eq. (17), Eq. (12) gives

$$\sum_i \left(\frac{\sigma_i}{F_i} \right)^2 - \frac{\sigma_x \sigma_y}{F_x F_y} - \frac{\sigma_y \sigma_z}{F_y F_z} - \frac{\sigma_z \sigma_x}{F_z F_x} - 1 = 0 \tag{18}$$

Apparently, Eq. (18) is analogous to the Hill's yield criterion, and can be regarded as the fatigue failure criterion, because it defines the fatigue failure surface that separates all possible combinations of design variables responsible for fatigue damage from those combinations that do not cause this effect. Thus, the dimensionless fatigue safety margin in the safe region of the basic variable space can be also defined by

$$\overline{m} = f^2 - 1 \tag{19}$$

The corresponding design criterion reads

$$\overline{\mu} \geq 0 \tag{20}$$

where

$$\overline{\mu} = \frac{\overline{m}}{f^2} = 1 - \frac{1}{f^2} \tag{21}$$

is the relative fatigue safety margin associated with the fatigue failure surface (18). At a given design point, the values of μ and $\overline{\mu}$ are different, but the limiting states $\mu = 0$ and $\overline{\mu} = 0$ coincide as far as the fatigue safety is concerned.

The values of F_i and σ_i in Eqs (17) through (21) are referenced to a particular coordinate system X_1, X_2, X_3. If, however, known stress components are related to another coordinate system, say $\overline{X}_1, \overline{X}_2, \overline{X}_3$, their amplitudes, $\overline{\sigma}_i$, have to be transformed to the X_1, X_2, X_3 system. For example, if the fatigue failure criterion is considered, Eq. (18) can be rewritten in the matrix form

$$S^T Q\, S - 1 = 0 \tag{22}$$

where

$$S = \left[\sigma_x \; \sigma_y \; \sigma_z \; \sigma_{xy} \; \sigma_{yz} \; \sigma_{zx} \right]^T$$

$$Q=\begin{bmatrix} \frac{1}{F_x^2} & -\frac{1}{2F_xF_y} & -\frac{1}{2F_xF_z} & 0 & 0 & 0 \\ -\frac{1}{2F_yF_x} & \frac{1}{F_y^2} & -\frac{1}{2F_yF_z} & 0 & 0 & 0 \\ -\frac{1}{2F_zF_x} & -\frac{1}{2F_zF_y} & \frac{1}{F_z^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{F_{xy}^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{F_{yz}^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{F_{zx}^2} \end{bmatrix}$$

and the column matrix

$$\bar{S}=\left[\bar{\sigma}_x\ \bar{\sigma}_y\ \bar{\sigma}_z\ \bar{\sigma}_{xy}\ \bar{\sigma}_{yz}\ \bar{\sigma}_{zx}\right]^T$$

can be transformed to the X_1, X_2, X_3 coordinate system by means of well-known matrix relationships [6]. For instance, when only the normal and shear stress components in a single plane are recognized, Eq. (22) reduces to

$$S_1^T Q_1 S_1 - 1 = 0 \tag{23}$$

where

$$S_1=\left[\sigma_x\sigma_y\sigma_{xy}\right]^T,\quad Q_1=\begin{bmatrix} \frac{1}{F_x^2} & -\frac{1}{2F_xF_y} & 0 \\ -\frac{1}{2F_yF_x} & \frac{1}{F_y^2} & 0 \\ 0 & 0 & \frac{1}{F_{xy}^2} \end{bmatrix}$$

Suppose that the matrices S_1 and Q_1 are referenced to a X_1, X_2 coordinate system, and that the column matrix of known amplitudes of stress components

$$\bar{S}_1=\left[\bar{\sigma}_x\ \bar{\sigma}_y\ \bar{\sigma}_{xy}\right]^T$$

is related to the $\bar{X}_1,\bar{X}_2$ coordinate system obtained by a positive rotation of the X_1, X_2 - axes through an angle Θ in the X_1, X_2 plane. Knowing that

$$S_1 = R_1\bar{S}_1 \tag{24}$$

where

$$R_1=\begin{bmatrix} \cos^2\Theta & \sin^2\Theta & -\sin 2\Theta \\ \sin^2\Theta & \cos^2\Theta & \sin 2\Theta \\ 0.5\sin 2\Theta & -0.5\sin 2\Theta & \cos 2\Theta \end{bmatrix}$$

one obtains

$$S_1^T = \overline{S}_1^T R_1^T \tag{25}$$

so that Eq. (23) becomes

$$\overline{S}_1^T R_1^T Q_1 R_1 \overline{S}_1 - 1 = 0 \tag{26}$$

On the other hand, the column matrix $\overline{S}_1$ and unknown fatigue limits $\overline{F}_x, \overline{F}_y$ and $\overline{F}_{xy}$ related to the $\overline{X}_1, \overline{X}_2$ coordinate system satisfy equation analogous to Eq. (23), i.e.,

$$\overline{S}_1^T \overline{Q}_1 \overline{S}_1 - 1 = 0 \tag{27}$$

where

$$\overline{Q}_1 = \begin{bmatrix} \dfrac{1}{\overline{F}_x^2} & -\dfrac{1}{2\overline{F}_x \overline{F}_y} & 0 \\ -\dfrac{1}{2\overline{F}_y \overline{F}_x} & \dfrac{1}{\overline{F}_y^2} & 0 \\ 0 & 0 & \dfrac{1}{\overline{F}_{xy}^2} \end{bmatrix}$$

Of course, Eqs (26) and (27) should lead to the same result which requires that

$$\overline{Q}_1 = R_1^T Q_1 R_1 \tag{28}$$

Eq. (28) makes it possible to determine fatigue limits from the following equations of transformation in the plane

$$\begin{aligned} \overline{F}_x &= \left(\frac{\cos^4\Theta}{F_x^2} - \frac{\sin^2 2\Theta}{4F_x F_y} + \frac{\sin^4\Theta}{F_y^2} + \frac{\sin^2 2\Theta}{4F_{xy}^2} \right)^{-1/2} \\ \overline{F}_y &= \left(\frac{\sin^4\Theta}{F_x^2} - \frac{\sin^2 2\Theta}{4F_x F_y} + \frac{\cos^4\Theta}{F_y^2} + \frac{\sin^2 2\Theta}{4F_{xy}^2} \right)^{-1/2} \\ \overline{F}_{xy} &= \left(\frac{\sin^2 2\Theta}{F_x^2} + \frac{\sin^2 2\Theta}{F_x F_y} + \frac{\sin^2 2\Theta}{F_y^2} + \frac{\cos^2 2\Theta}{F_{xy}^2} \right)^{-1/2} \end{aligned} \tag{29}$$

3. Multiaxial stress with random-amplitude synchronous components

When the stress amplitudes in Eq. (18) are random variables of known statistical parameters, Eqs (5), (7), (8) and (17) can be applied. To account for the randomness of the phase angles in this case, the signs „–" in Eq. (17) must be replaced with „+". If the relative fatigue safety margin $\overline{\mu}$ (21) is employed instead of μ (5), the fatigue design criterion becomes

$$E\{\overline{\mu}\} \geq 0 \tag{30}$$

or, similarly to Eq. (8),

$$E\{\bar{\mu}\} - js_{\bar{\mu}} \geq 0, \quad j > 0 \tag{31}$$

where $s_{\bar{\mu}}$ is the standard deviation of $\bar{\mu}$. Contrary to the deterministic case, the limiting states

$$E\{\mu\} = 0 \tag{32}$$

and

$$E\{\bar{\mu}\} = 0 \tag{33}$$

do not coincide. The same can be said of the limiting states resulting from Eqs (8) and (31). Therefore, the use of the relative fatigue safety margins μ and $\bar{\mu}$ under random loading conditions is a matter for practical consideration.

Service loadings usually create more complex states of stress than those considered in this paper. However, the fatigue safety criteria presented above may be used also in the cases of uniaxial and multiaxial periodic or stationary random stress if suitable stress models are found. One of the possible solutions of this problem can be based on the theory of energy transformation systems [7] that links the lifetime of dynamically loaded systems, as well as any volume of material in a system, with the energy dissipated internally and externally by a system (volume). Such an approach is presented for the high-cycle regime in [1], where for the stress components given in the form of Fourier series or, respectively, characterized by their power spectral densities, adequate stress models are developed. The models are equivalent, in terms of fatigue lifetime, to the original stress patterns. Thereby, the cycle counting is avoided, and the application of conventional fatigue damage accumulation rules is not required.

The influence of multiaxial mean stress on fatigue safety is considered in [8].

4. Example

Let us calculate the probability of an infinite fatigue life at the limiting states (32) and (33) if the load is uniaxial and the stress amplitude follows Rayleigh distribution. Let us compare the results with those at the limiting states

$$E\{\mu\} - s_{\mu} = 0 \tag{34}$$

$$E\{\bar{\mu}\} - s_{\bar{\mu}} = 0 \tag{35}$$

Solution. In the case of uniaxial stress state, Eqs (32) and (33) give

$$\eta_1 = F \tag{36}$$

$$\eta_2 = F^2 \tag{37}$$

where at Rayleigh distribution is [9]

$$\eta_1 = E\{\sigma\} = (0.5\pi)^{1/2} s, \quad \eta_2 = E\{\sigma^2\} = 2s^2 \tag{38}$$

and the formula for the distribution function reads

$$D_{\sigma} = 1 - \exp\left(-\frac{\sigma^2}{2s^2}\right) \tag{39}$$

Here s is the standard deviation of the stress process. The probability of an infinite fatigue life becomes

$$P = 1 - \exp\left(-\frac{F^2}{2s^2}\right) \tag{40}$$

so that

$$P = 1 - \exp\left(-\frac{0.5\pi s^2}{2s^2}\right) = 0.54$$

in the limiting state (32), and

$$P = 1 - \exp\left(-\frac{2s^2}{2s^2}\right) = 0.63$$

in the limiting state (33).

Referring now to the limiting states (34) and (35), we have

$$s_\mu = \left[E\left\{(\mu - E\{\mu\})^2\right\}\right]^{1/2} = \left[E\left\{\left(1 - \frac{\sigma}{F} - 1 + \frac{\eta_1}{F}\right)^2\right\}\right]^{1/2} =$$
$$= \frac{1}{F}\left(\eta_2 - \eta_1^2\right)^{1/2} = 0.655\frac{s}{F}$$

$$E\{\mu\} - s_\mu = 1 - \frac{\eta_1}{F} - 0.655\frac{s}{F} = 1 - 1.908\frac{s}{F}$$

$$s_{\bar{\mu}} = \left[E\left\{(\bar{\mu} - E\{\bar{\mu}\})^2\right\}\right]^{1/2} = \left[E\left\{\left(1 - \frac{\sigma^2}{F^2} - 1 + \frac{\eta_2}{F^2}\right)^2\right\}\right]^{1/2} =$$
$$= \frac{1}{F^2}\left(E\{\sigma^4\} - 4s^4\right)^{1/2}$$

Since the probability density function of the stress amplitude is that of Rayleigh type, its statistical moments are given by [9]

$$E\{\sigma^k\} = 2^{k/2}\Gamma(1 + 0.5k)s^k \tag{41}$$

where Γ is the gamma function. At k=4 is $\Gamma(1+2) = 2$. So, $E\{\sigma^4\} = 8s^4$. Hence

$$s_{\bar{\mu}} = 2\frac{s^2}{F^2}, \qquad E\{\bar{\mu}\} - s_{\bar{\mu}} = 1 - \frac{\eta_2}{F^2} - 2\frac{s^2}{F^2} = 1 - 4\frac{s^2}{F^2}$$

Now the probabilities of interest are

$$P = 1 - \exp\left(-\frac{1.908^2 s^2}{2s^2}\right) = 0.838$$

$$P = 1 - \exp\left(-\frac{4s^2}{2s^2}\right) = 0.865$$

Consequently, in the considered case the following design criterion

$$E\{\bar{\mu}\} - js_{\bar{\mu}} \geq 0, \quad j \geq 1 \tag{42}$$

seems to be justified.

5. Conclusions

The presented fatigue safety criteria of design for an infinite fatigue life of machinery parts refer to uniaxial stress as well as to the multiaxial stress states with constant-amplitude in-phase components or random-amplitude synchronous components. As shown in the Example, they may lead to more or less conservative design solutions. The safety criteria concerned with multiaxial fatigue are based on the distortion energy theory, and include up to six fatigue limits of the material related to individual stress components. Therefore, their use should be confined to ductile materials under non-corrosive environmental conditions. In the cases of uniaxial and multiaxial periodic or stationary random stress, these criteria can not be applied unless adequate stress models are found (e.g., by means of the procedures described in [1]).

Manuscript received by Editorial Board, May 15, 2000;
final version, February 12, 2001

REFERENCES

[1] Kolenda J.: Analytical procedures of high-cycle fatigue assessment of structural steel elements. 2nd Ed., Polish Naval Academy, Gdynia 1999.

[2] Kocańda S., Szala J.: Fundamentals of fatigue calculations (in Polish). PWN, Warszawa 1997.

[3] Troost A., El-Magd E.: Schwingfestigkeit bei mehrachsiger Beanspruchung ohne und mit Phasenverschiebung. Konstruktion, 8, 1981, pp. 297÷304.

[4] Sonsino C.M.: Multiaxial fatigue of welded joints under in-phase and out-of-phase local strains and stresses. Int. J. Fatigue, 1, 1995, pp. 55÷70.

[5] Życzkowski M. (Ed.): Technical Mechanics (in Polish), Vol. 9, PWN, Warszawa 1988.

[6] Gere J.M., Weaver W., Jr.: Matrix Algebra for Engineers. Van Nostrand Reinhold Co., New York 1965.

[7] Cempel C.: Theory of energy transformation systems and their application in diagnostic of operating systems. Applied Mathematics and Computer Sciences, 2, 1993, pp. 533÷548

[8] Kolenda J.: A new design criterion of steel elements at multiaxial constant and in-phase loadings. Proc. of the 18-th Symp. on Fatigue and Fracture Mechanics, Bydgoszcz-Pieczyska, 23-26 May, 2000, ATR Bydgoszcz 2000, pp. 239÷243.

[9] Pacut A.: Probability. Theory. Probabilistic Modelling in Technology (in Polish). WNT, Warszawa 1985.

O kryteriach bezpieczeństwa zmęczeniowego części maszyn w ujęciu deterministycznym i probabilistycznym

Streszczenie

Rozpatrywane są kryteria nieograniczonej trwałości zmęczeniowej części maszyn przy jedno- i wieloosiowych stanach wahadłowych naprężeń o zgodnych w fazie stałoamplitudowych składowych oraz o synchronicznych składowych z losowo zmiennymi amplitudami. Zdefiniowano bezwymiarowe i względne marginesy bezpieczeństwa dla tych stanów. Przedstawione kryteria dotyczą materiałów ciągliwych posiadających granice zmęczenia. Dla granic zmęczenia odnoszących się do innego układu współrzędnych niż składowe płaskiego stanu naprężenia podano reguły transformacji na płaszczyźnie.