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SURFACE OF LIMIT STATE IN NONLINEAR MATERIAL AND ITS RELATION WITH PLASTICITY CONDITION

In this work, the author implements concepts and methods of analysis of nonlinear elasticity theory in a simplified description of elastic-plastic properties of materials. Taking the principle of conservation of energy as the theoretical basis, the author formulates a criterion that makes it possible to examine the stability of internal equilibrium in deformed material whose nonlinear properties are defined by strain energy density function. The formulae allowing for assessment of complex states of strain in the aspect of material strength were derived on the assumption of small deformation. These formulae can replace mathematical relationships traditionally known as strength hypotheses. The example included in the paper presents the method of determining, in the space of strain state components, the areas where permanent deformation or destruction of material is possible because of strain state stability. Characteristic parameters used in the example are obtained in a static tensile test on specimen of constructional carbon steel of ordinary grade. The results of the analysis, based on the formulated strength hypothesis on stability of strain state, are compared with those resulting from the Huber's hypothesis on energy of non-dilatational strain.

1. Introduction

The most widely known and commonly accepted strength hypothesis is the Huber's hypothesis on energy of non-dilatational strain. Formulated by Maksymilian Tytus Huber in the work "Strain work per unit as a measure of effort of material" first published in Lvov in 1904, the hypothesis remains superior to any other one till nowadays. According to the hypothesis, the amount of non-dilatational strain energy decides on material effort, and consequently on its permanent deformation or destruction. The hypothesis has been confirmed by experiments, and it proved to be the one that describes actual

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properties of many materials in a most accurate way. It has been used by design engineers in strength analysis of structures since Huber's days [1] till nowadays [2, and other sources], although it has never been theoretically proven. The creator of the hypothesis, M. T. Huber himself, 27 years after the hypothesis had been formulated, wrote in a quarterly report [1] of the Institute of Aviation Research (Warsaw, 1930): "Although we can not say that our hypothesis has firm theoretical bases, it nevertheless provides, and will provide, important benefits in deriving theoretical strength formulae. (...) The superiority of the hypothesis over those previously used consists not only in a better consistence with experimental results, but also in the simplicity of derivation of the formulae in almost all the cases that practically matter". The author confirmed then that he was not able to find theoretical validation for his hypothesis, in spite of its undeniable advantages. In the cited work, M. T. Huber also quotes extreme opinions of some scientists who asserted that theoretical validation of strength hypothesis was not possible at all. He wrote [1], among other things, that: "W. Voight was then right in uttering the presumption that strength-related effects can not be incorporated into a strict theoretical scheme by means of parameters characteristic for the material, as it is the case in elasticity theory".

As it turned out, these pessimistic views fortunately did not prove right. The author formulated in his book [6] theoretical bases for a new strength hypothesis applicable to arbitrary nonlinear materials. The hypothesis was developed on the grounds of a fundamental physical law, the principle of conservation of energy. The consequence of it is the criterion of stability of the material subjected to strain. It assumes that the loss of stability of internal equilibrium in deformed material is the direct cause of damage or destruction of the material. If one considers in the analysis also this part of energy that is involved in the irreversible process of plastic deformations within the material, then the method proposed in [6] can be used for evaluation of surface of limit state of elastic-plastic material. The work also analyses some of more complex aspects of the internal equilibrium stability hypothesis connected with the problem's mathematical formalism, in phenomenological formulation, in the case of material subjected to high strains. Among the conclusions of the work, one can find the following statements:

- The knowledge of material's physical model that gives a relevant characterisation of material properties, together with the knowledge of material constants, is a sufficient condition for theoretical description of the areas where the components of strain state in the body pertain to a critical state of deformation in which coherence of the material may be lost.

- The role of strength hypothesis is played by the assertion that the material is in the state that threatens to evoke damage or destruction of material, because internal equilibrium in the material is in a critical state meaning the loss of stability.

- Verification of a detailed strength criterion, in application to a particular kind of material, consists in examining the conditions of stability of equilibrium state in the material, and applying the obtained results for assessment of the criterion.

- On the basis of linear theory of elasticity one can not theoretically prove, without introducing additional assumptions, the validity of any strength hypothesis.

The last statement could perhaps justify pessimistic opinions once expressed by the author of one of the best strength hypotheses ever created – undeniably the one most frequently used in practice by mechanical engineers [2] and implemented in numerous contemporary computer programs of strength analysis. Nonlinear theory of elasticity, along with the methods of testing of the so-called material stability, was created only in the mid-forties, soon before the death of the great Polish scientist – Professor Maksymilian Tytus Huber.

In this work, we introduce the formulae that would enable us to examine stability of an arbitrary state of strain in a material of nonlinear characteristics. Physical properties of the material will be defined by the function of strain energy, under the assumption of small deformation in material, and the properties will be predicted based on material's strength. We will also formulate mathematical criteria that will facilitate determining the states of strain conforming to the necessary conditions that decide whether, for energy reasons, plasticization of the material or a rupture would take place. The criteria can be applied to such materials like steel or alloys of non-ferrous metals. The formulae useful in examining stability of great deformations in materials such as for example rubber, and the resulting strength criteria were derived in a previous work by the author [6]. The results, obtained on the basis of the formulated strength hypothesis concerning the stability of strain state, will be illustrated by a numerical example for the steel St3S, and compared with those obtained on the grounds of the Huber's hypothesis of non-dilatational strain energy.

The aim of this work is to direct reader's attention on the need of searching for theoretical validation of strength criteria, including those resulting from Huber's hypothesis – the one commonly known, accepted, verified experimentally and confirmed in vast areas of application. Intention of the author also consists in creating the possibilities of improving strength criteria in the case of many materials for which the existing strength criteria give only an approximate description of material's properties. Another goal of this study is to facilitate the formulation of strength criteria for newly developed composites and constructional materials.

Theoretical substantiation of strength criteria means proving their validity on the grounds of fundamental physical laws, such as the principle of conservation of energy. The consequence of this principle is the criterion of stability of deformed material, described in detail in Section 2, and used in further part of the work. The author emphasises the fact that the structure of correctly developed nonlinear models of materials should meet the demands resulting from energy conservation conditions. In particular, the areas of stable deformation determined on the basis of the model should be consistent with similar areas found experimentally in strength tests. In the model applied in this work, one takes an additional assumption. It says that, according to the hypothesis of internal equilibrium stability [6], the limit state, connected with the effect of the loss of stability, can be identified with the state dangerous for the material for the reason of possible destruction. The limit state may be determined based on the assumed properties of nonlinear model of material. The assumption taken in the quoted numerical example of modelling physical properties of St3S steel is that in limit state there exists the possibility of plastic flow appearing as a result of slip in layers of material, or the possibility of a rupture. In this way, one obtained a physically correct, nonlinear model of elastic-plastic properties of the material, theoretically validating the strength criteria that have been created on its basis.

2. Examining stability of internal equilibrium in material

In examining stability of the state of internal equilibrium in deformed material one should determine whether the increase of internal energy in an isolated segment of the body, caused by a small, permissible disturbance of its state, is equal to the work of the forces acting on the segment in the state of equilibrium on displacements caused by this disturbance. If the increase of work of forces acting on the isolated segment of the body, on displacements resulting from the change in configuration of the segment, is lower than the required increase of body's internal energy, then it would mean that it is not possible to change configuration of the body without delivering an additional amount of energy into it, so that the equilibrium state is stable. This way of thinking is adopted in nonlinear elasticity theory [7], in searching for bounds on material functions. The postulate of absolute minimum of total energy in an isolated segment of deformed body is used in theoretical validation of Coleman-Noll and strong ellipticity conditions. These have to ensure correct form of the function of strain energy density that defines physical properties of material. The conditions were used, among other things, in works [4], [5] in order to substantiate generalisation of Mooney's strain energy function for nearly incompressible materials. Until recently, nobody has noticed, however, that this problem has some wider aspect, and an analogous method can be applied in strength analysis of material. For the first time it was done in the work by the author [6], who developed an appropriate mathematical formalism and formulated the strength hypothesis. The same hypothesis will be used hereunder in the analysis of the material strength problem considered in this paper.

The analysis of stability of internal equilibrium in a material can be reduced to examining the sign of second-order variation of strain energy density function. The grounds for this assertion one can find in work [6]. However, for the sake of continuity of reasoning, the considerations will be briefly repeated here.

Let internal energy of the body be defined by energy density function W of n mutually independent quantities e_i, where n is equal to the number of degrees of freedom of the system. The quantities concerned are for example components of strain state. The increment of internal energy due to an arbitrary small change of these quantities, δe_i, can be found from the formula

$$\delta W = \sum_{i=1}^{n} \frac{\partial W}{\partial e_i} \delta e_i . \tag{2.1}$$

In order to keep the body in equilibrium, the increment of work of generalised external forces f_i, related to generalised displacements e_i on increments of these displacements δe_i

$$\delta L = \sum_{i=1}^{n} f_i \delta e_i \tag{2.2}$$

must be equal to the increment of internal energy. Then, the equilibrium condition takes the form

$$\delta W = \delta L \tag{2.3}$$

or

$$f_i = \frac{\partial W}{\partial e_i} \tag{2.4}$$

Equilibrium is stable when, for any series of increment values of generalised displacements $(\delta e_1, \delta e_2, ..., \delta e_n)$ of which at least one is not zero, the following inequality holds:

$$W(e_1 + \delta e_1, e_2 + \delta e_2, ..., e_n + \delta e_n) - W(e_1, e_2, ..., e_n) > \delta L \tag{2.5}$$

If we expand the left side of inequality (2.5) into a series, and truncate the series to second-order terms, the condition takes the form

$$\delta W + \frac{1}{2}\delta^2 W > \delta L \tag{2.6}$$

or, taking into account equilibrium condition (2.3)

$$\delta^2 W > 0 \tag{2.7}$$

Internal equilibrium is then stable, when second-order variation of strain energy density function is positive definite.

According to the hypothesis of internal equilibrium stability [6], a state of deformation does not threaten damage or destruction of the material when internal equilibrium in the deformed material is stable. A dangerous state exists then in such a point of the body, where internal equilibrium reaches the critical state, what means the loss of stability of equilibrium. For such a state of deformation, second-order variation of strain energy density function is not positive definite.

3. Nonlinear model of physical properties of material

Experimental examination of many materials of great practical usability, such as for example steel, iron, nonferrous metals and their alloys, etc., confirmed linear elasticity of volumetric strain in a very wide range of stress. Non-dilatational strain does not have this property, at least not in such a wide range. Then, in the model we intend to develop, we must take into account experimentally determined properties of actual materials, measured by strength tests. One of the methods of defining physical properties of a material is determining the strain energy density as function of invariants of strain or components of the state of strain [4, 5 and other references].

Let us assume that physical properties of the material are defined by the strain energy function in the form of the following sum

$$W = W^{(v)} + \tilde{W}^{(s)} \tag{3.1}$$

where energy of volumetric strain is given by the formula

$$W^{(v)}(\varepsilon_x,\varepsilon_y,\varepsilon_z) = \frac{K}{2}(\varepsilon_x + \varepsilon_y + \varepsilon_z)^2 \tag{3.2}$$

while energy of non-dilatational strain is

$$\tilde{W}^{(s)} = W^{(s)} - \beta W_0^{\,1-\alpha} W^{(s)\alpha} \tag{3.3}$$

where

$$W^{(s)}(\varepsilon_x,\varepsilon_y,\varepsilon_z,\gamma_{xy},\gamma_{yz},\gamma_{zx}) = \frac{G}{3}[(\varepsilon_x-\varepsilon_y)^2 + (\varepsilon_y-\varepsilon_z)^2 + (\varepsilon_z-\varepsilon_x)^2 + \frac{3}{2}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2)] \tag{3.4}$$

Material constants α and β are numbers, and W_0 is an arbitrarily chosen reference value. Besides, α is a positive number not equal to one, and W_0 is a physical quantity expressed in the same units of measure as the energy density function. In further part of the work we will take W_0=1 MN/m^2. The other constants are known from linear theory of elasticity: K – Helmholtz modulus, also known as the modulus of volume elasticity, and G – Kirchhoff's modulus, also called the modulus of elasticity in shear, or the Lame's constant μ. These constants can be related to Young's modulus E and Poisson ratio v by the following formulae

$$K = \frac{E}{3(1-2v)}, \qquad G = \frac{E}{2(1+v)} \tag{3.5}$$

In this way we have defined the new model of material of nonlinear properties. Strain energy in the material can be expressed in a short form as

$$W = W^{(v)} + W^{(s)} - \eta W^{(s)^{\alpha}} \tag{3.6}$$

where

$$\eta = \beta W_0^{\ 1-\alpha} \tag{3.7}$$

The above function depends on four parameters: α, β, ν, E. When β approaches zero the function converts into the strain energy function of linearly elastic material.

Because internal energy of the body is defined by the energy density function W depending on six mutually independent quantities e_i – the components of strain state, where the index i denotes the number of component of the vector

$$\vec{e} = (\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}) \tag{3.8}$$

then the components of stress state f_i, where the index i denotes the number of component of the vector

$$\vec{f} = (\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}) \tag{3.9}$$

can be found from formula (2.4). In this way we obtain

$$\begin{aligned} \sigma_x &= KJ_1 + (1-\kappa)\frac{2G}{3}\vartheta_x, & \tau_{xy} &= (1-\kappa)G\gamma_{xy}, \\ \sigma_y &= KJ_1 + (1-\kappa)\frac{2G}{3}\vartheta_y, & \tau_{yz} &= (1-\kappa)G\gamma_{yz}, \\ \sigma_z &= KJ_1 + (1-\kappa)\frac{2G}{3}\vartheta_z, & \tau_{zx} &= (1-\kappa)G\gamma_{zx}, \end{aligned} \tag{3.10}$$

where the invariant of strain state is

$$J_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z, \tag{3.11}$$

while

$$\vartheta_x = 3\varepsilon_x - J_1, \qquad \vartheta_y = 3\varepsilon_y - J_1, \qquad \vartheta_z = 3\varepsilon_z - J_1, \tag{3.12}$$

and

$$\kappa = \eta\alpha W^{(s)^{\alpha-1}}. \tag{3.13}$$

The above formulae will be used in the next section for identification of material constants.

The simple model of nonlinear properties of material can be applied to elastic materials and, if we assume that the function of strain energy W given by formula (3.6) is a sum of energy of elastic strain and plastic strain, the model can also be used to describe elastic-plastic properties of materials under monotonic load. For the case of the release of load, one should create a similar model, taking into account the fact that the characteristic of the process is determined only by that part of strain energy, which has not been accumulated in plastic deformation in the material. Because it was found in experiments that the process of the release of load in the material runs according to the linear-elastic model, and in work [6] the author shows that in such a case the loss of stability of the material does not take place, then it does not seem necessary to develop any analytical formula for the model of the release of load process. Therefore, the created nonlinear model can be used as an approximate representation of elastic-plastic properties of the material.

Let us determine now the second-order variation of strain energy density function

$$\delta^2 W = \sum_{i=1}^{6}\sum_{j=1}^{6}\frac{\partial^2 W}{\partial e_i \partial e_j}\delta e_i \delta e_j \tag{3.14}$$

It is a quadratic form of six variables – components of the vector

$$\delta\vec{e} = (\delta\varepsilon_x, \delta\varepsilon_y, \delta\varepsilon_z, \delta\gamma_{xy}, \delta\gamma_{yz}, \delta\gamma_{zx}) \tag{3.15}$$

The variation can be expressed in matrix notation, with respect to formula (3.1), as

$$\delta^2 W = \lambda\delta\vec{e}\mathbf{X}\delta\vec{e}^T \tag{3.16}$$

where elements x_{ij} of the symmetric matrix $\mathbf{X}$ of quadratic form, determined from the relation

$$x_{ij} = \frac{1}{\lambda}\frac{\partial^2 W}{\partial e_i \partial e_j} \tag{3.17}$$

are given by formulae

$$x_{ij} = 1 + \frac{2}{3}\chi + \frac{2}{3}\chi(1-\kappa)(3\delta_{ij} - 1) - \frac{4}{9}\bar{\kappa}\vartheta_i\vartheta_j, \qquad i, j \in \{1,2,3\}, \tag{3.18}$$

$$x_{ij} = -\frac{2}{3}\bar{\kappa}\vartheta_i e_j, \qquad j \in \{4,5,6\},\ i \in \{1,2,3\}, \tag{3.19}$$

$$x_{ij} = \chi(1-\kappa)\delta_{ij} - \bar{\kappa}e_i e_j, \qquad i, j \in \{4,5,6\}. \tag{3.20}$$

One should assume

$$\vartheta_1 = \vartheta_x, \quad \vartheta_2 = \vartheta_y, \quad \vartheta_3 = \vartheta_z \tag{3.21}$$

and take the Lame's constant and the remaining quantities given by formulae

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad \chi = \frac{G}{\lambda} = \frac{1-2\nu}{2\nu}, \qquad \bar{\kappa} = \frac{\kappa(\alpha-1)\lambda\chi^2}{W^{(s)}}, \tag{3.22}$$

where δ_{ij} is the Kronecker delta.

Internal equilibrium at a certain point of deformed material is unstable if there exist such a non-zero vector $\delta\vec{e}$, given by formula (3.15), for which

$$\delta^2 W \le 0 \tag{3.23}$$

Evaluation of the state of strain in the material can then be reduced to examining the sign of expression (3.16), that is a quadratic form of six variables, of symmetrical matrix **X**.

According to Sylvester's theorem, the necessary and sufficient condition for quadratic form of matrix **X** to be positive definite is that all principle minors of the matrix are positive. It can be symbolically written as

$$\underset{k}{\forall} \det \mathbf{X}^{(k)} > 0 , \tag{3.24}$$

where symbol k in parentheses indicates that the choice of elements of the matrix was made to create a principal minor of rank k of the matrix **X**.

Internal equilibrium at a point within the deformed material is then unstable when

$$\underset{k}{\exists} \det \mathbf{X}^{(k)} \le 0 , \tag{3.25}$$

and the above inequalities determine areas of instability in the space of co-ordinates of the components of strain state.

Let us assume that the directions of co-ordinate axes coincide with principal directions of strain field. Then, the vector (3.8) of components of strain state is

$$\vec{e} = (\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3,\ 0,\ 0,\ 0) , \tag{3.26}$$

and the matrix of quadratic form can be reduced to the following one

$$[x_{i\,j}] = \begin{bmatrix} x_{11} & x_{12} & x_{13} & 0 & 0 & 0 \\ x_{12} & x_{22} & x_{23} & 0 & 0 & 0 \\ x_{13} & x_{23} & x_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & x_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & x_{44} \end{bmatrix} \tag{3.27}$$

where

$$x_{44} = \chi(1-\kappa) , \tag{3.28}$$

and the remaining non-zero elements of the matrix can be derived from formula (3.18). In this case, the conditions (3.25) can be reduced to the following inequalities

$$x_{11} \le 0, \quad \det \begin{bmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{bmatrix} \le 0, \quad \det \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{bmatrix} \le 0, \quad x_{44} \le 0 \tag{3.29}$$

The sufficient condition of instability of internal equilibrium in deformed material is that at least one of the above inequalities is satisfied. Using these inequalities, one can determine the states of strain for which destruction of the material becomes possible.

Let us ask under what conditions it becomes possible, in terms of energy relations, to initiate the effect of plastic flow in the material. Among assumptions of plasticity theory, one of the fundamental ones is that in plastic deformations only the form of the body can change [3]. Let us assume then that the characteristic feature of plastic flow, which is an effect of slip in layers of material, is deformation that keeps volume of the material constant. We assume then that in the initial phase of the process it runs with constant volume of the deformed material. On that assumption we have

$$\delta J_1 = \delta\varepsilon_x + \delta\varepsilon_y + \delta\varepsilon_z = 0 \tag{3.30}$$

The postulate (3.30) is not equivalent to the assumption that the material is incompressible. We only examine the possibility of taking, by a material of arbitrary properties, a disturbed configuration whose characteristic feature is the same volume as that in the original configuration. This condition imposes bounds on the variables $\delta\varepsilon_x, \delta\varepsilon_y, \delta\varepsilon_z$ of the quadratic form, and decreases by one the number of the system degrees of freedom. Then, we can eliminate variable $\delta\varepsilon_z$ that depends on the remaining two variables. Let us assume that the directions of the system co-ordinates are consistent with principal directions of strain. Basing on the trimmed quadratic form (3.16), out of which one of the variables was eliminated, we find that the necessary condition of plastic flow is satisfied when at least one of the following inequalities holds

$$y_{11} \le 0, \quad \det\begin{bmatrix} y_{11} & y_{12} \\ y_{12} & y_{22} \end{bmatrix} \le 0 \tag{3.31}$$

where

$$\begin{aligned} y_{11} &= x_{11} - 2x_{13} + x_{33}, \\ y_{12} &= x_{12} - x_{13} - x_{23} + x_{33}, \\ y_{22} &= x_{22} - 2x_{23} + x_{33}. \end{aligned} \tag{3.32}$$

From inequalities (3.31) one can determine the states of strain at which plasticization of the material is possible.

There is some analogy between the presented method of examining stability of internal equilibrium in deformed material, and well known Lagrange-Dirichlet theorem concerning stability of the equilibrium state in potential systems. In its classical formulation, the theorem applies to the systems whose potential energy is a quadratic form of generalised co-ordinates. In the considered problem this condition is not satisfied. However, second order variation of strain energy density function is a quadratic form, and that fact enables us to formulate stability conditions similar to those resulting from Lagrange-Dirichlet theorem.

The strength criteria introduced in this work, formulated as inequalities (3.31) or (3.29), define the states of strain allowing for plasticization or destruction of the material, and replace strength hypotheses formulated in a traditional way. The latter consisted in defining some function of the state of strain or stress components, accepted as a measure of material effort, and comparing the function value with a permissible value determined in uniaxial tensile tests. There is a difference, however, between the new and the traditional formulation. Once we know a physical model of the body, expressed by the strain energy function, we no longer need to perform any experiments in order to get some supplementary information about the value of measure of material effort at which the loss of material coherence could take place.

4. Numerical example

The strength conditions, derived in the previous section on the basis of hypothesis of internal equilibrium stability, will now be used to determine the dangerous areas in the space of strain state components. Danger is meant as a threat to material strength, when plastic deformation or permanent destruction of material becomes possible. The subject of analysis will be constructional carbon steel of ordinary grade, denoted with symbol St3S. The relation between the loading force and elongation in the direction of tension axis, determined in static tensile test in uniaxial state of stress, was used for identification of material constants α, β, ν, and E. The author used for this purpose nonlinear relations between the components of stress state and the components of strain state in three-axial state of stress. These can be taken from the first column of formula (3.10) on the assumption $\sigma_y = \sigma_z = 0$. The values of material constants α, β, ν, and E are chosen in such a way that the values of limit stresses from calculations are equal to experimental values, shown in Table 2, and the shape of the curve approximating the function of load vs. elongation was such that the two curves, theoretical and experimental one, are possibly close. The material constants of steel St3S, obtained in this way, are shown in Table 1.

Table 1

Segment of tension characteristics	Material constants for steel St3S			
	α	β	ν	E [MN/m^2]
A	2.3	0.827	0.3	$2.05 \cdot 10^5$
B	1.028	0.827	0.3	$6.5 \cdot 10^5$

The graph of tension for the steel St3S, obtained by approximating physical properties of the material with strain energy function (3.1) for the above material constants, is presented in Fig. 1.

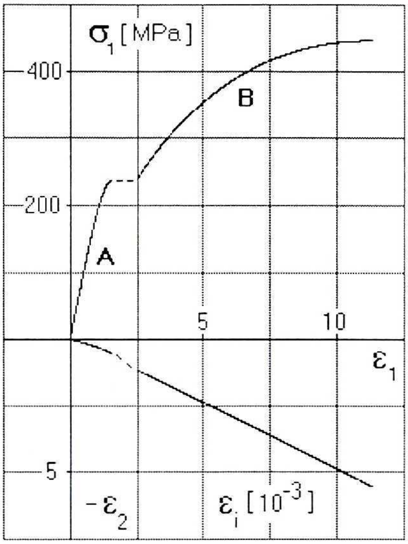

Fig. 1. Approximation of tension characteristics for steel St3S

The dotted line in the graph indicates the segment of the characteristics that was not approximated, as it pertains to plastic flow of the material. The graph also shows the tension trajectory in the space of strain state components, obtained by the numerical method. The co-ordinates of characteristic points of the graph are included in Table 2.

Table 2

	$\varepsilon[10^{-3}]$	R [MN/m^2]
Yield point	1.55	237
Strength limit	11,3	445

The graph of tension shown in Fig. 1 clearly shows that for steel St3S there exist a distinct onset of yielding (dotted line in graph 1) that indicates the presence of the effect of slide in layers of material. The course of the process in this area is out of scope of this work. Because material properties change above the yield point, two different models, called model A and B, must be used for the approximation. However, for the materials that do not exhibit any distinct onset of yielding, the curve of tension can be approximated with one analytical expression, as it was shown in work [6].

Finally, let us present the areas of stable strain in the space of strain state components. In Fig. 2, elliptically-shaped curves limit the values of material strain components for which internal equilibrium in deformed material is stable. These were determined based on physical characteristics of the material, approximated by model A. The respective set of approximation parameters in

Table 1 is denoted with letter A. Fig. 3 also shows two curves that circumscribe the areas where, according to Huber's hypothesis, no risk of plastic flow would exist (areas H_1 and H_2). The latter were calculated on the assumption that the value of strain energy dangerous for the material is given by formula (3.4) for strain components in principal directions. The following relations were accommodated:

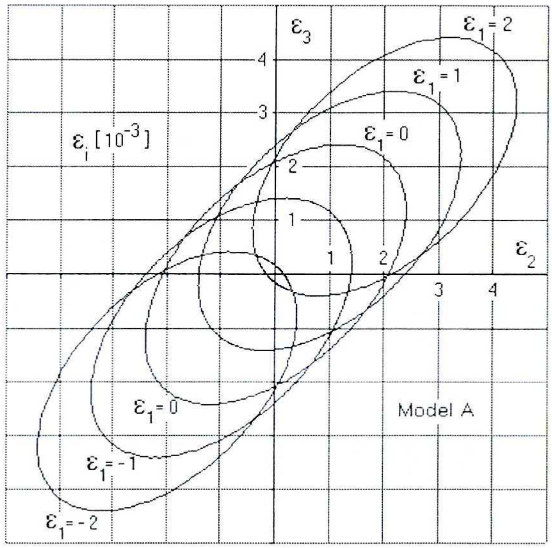

Fig. 2. Areas of stable internal equilibrium in deformed material

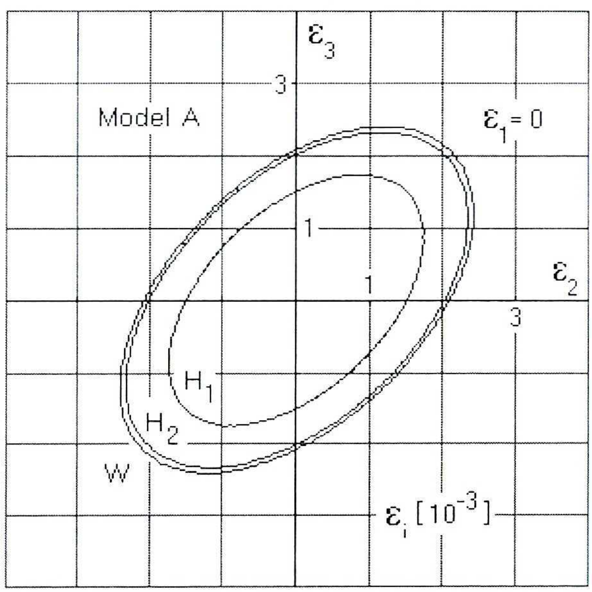

Fig. 3. Comparison of areas not endangered by plastic flow determined by means of different strength hypothesis

$$\varepsilon_2 = \varepsilon_3 = -\nu\varepsilon_1, \qquad \varepsilon_1 = \varepsilon_e \tag{4.1}$$

Then we have

$$W_e^{(s)} = \frac{1+\nu}{3} E\varepsilon_e^{\,2} \tag{4.2}$$

Taking the values ν, E from Table 1 for the model A and $\varepsilon_e = 1{,}55 \cdot 10^3$, we obtain $W_e^{(s)} = 0.213\ \mathrm{MN/m^2}$. This value was used to calculate the ellipse H_2 in Fig. 3. However, if we use the relation

$$W_e^{(s)} = \frac{1+\nu}{3E} \sigma_e^{\,2} \tag{4.3}$$

analogous to formula (4.2) that follows directly on the expression of non-dilatational strain energy in the function of stress state components, quoted by most of the source books on strength of materials, then taking $\sigma_e = R_e = 237\ \mathrm{MN/m^2}$ we obtain $W_e^{(s)} = 0.119\mathrm{MN/m^2}$. The latter was used to calculate the ellipse H_1 in Fig. 3. The elliptically-shaped curve denoted by W in the same graph was determined on the basis of the hypothesis of internal equilibrium stability in deformed material. To calculate this curve, one only needs to know the values of material constants defining physical properties of the material. Information of any of the physical quantities characterising the state of strain or stress in the material, such as, for example, energy of non-dilatational strain, is not necessary in this case. The curve pertains to $W_e^{(s)} = 0.230\ \mathrm{MN/m^2}$. One should bear in mind that this is not the whole energy of non-dilatational strain, but, in this case, only one of its components in formula (3.3). The energy of non-dilatational strain that refers to yield point in uniaxial tension equals $\tilde{W}_e^{(s)} = 0.202\ \mathrm{MN/m^2}$, due to the assumed approximation of physical characteristics of material given by formula (3.1).

Similarly as it was presented in Fig. 3, one can make a comparison between the hypothesis of non-dilatational strain energy and the hypothesis of internal equilibrium stability for an arbitrary cut in the space of strain state components. The relation between the areas determined by both hypotheses would be similar.

The material strengthens in the process of plastic flow, so that its physical characteristics change. In the further part we will approximate these altered characteristics with the model B, referring to the set of approximation parameters denoted with letter B in Table 1. Fig. 4 presents a juxtaposition of the areas of stable strain in the space of strain state components obtained for different models of material properties. The components of strain state that lie on the edge of the area circumscribed by the elliptically-shaped curve denoted with letter B refer to strength limit of steel St3S. The strain component values pertaining to the points outside of this area signify the threat of destruction of the material.

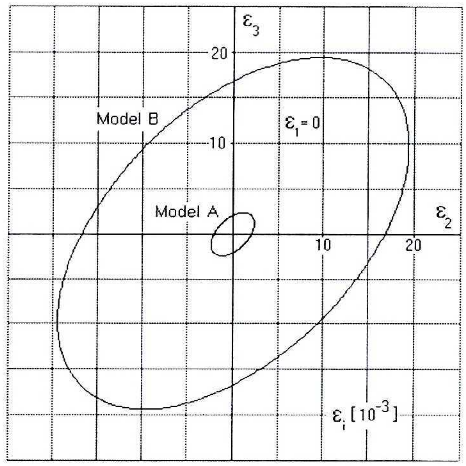

Fig. 4. Comparison of areas of stable strain for models A and B of physical properties of steel St3S

The lines shown in Figs 5 and 6 limit the areas in the space of strain state components where, according to Model B, the danger of material destruction does not exist. Additionally, Fig. 6 shows the area that is not endangered by plastic flow of the material (due to Model A).

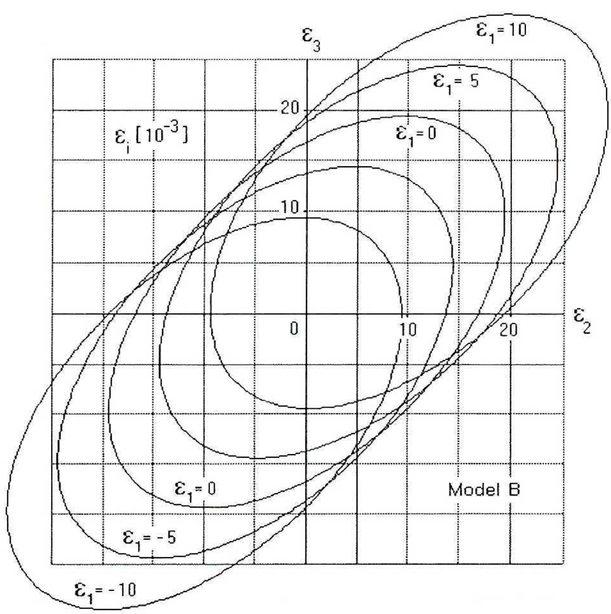

Fig. 5. Areas of stable strain of steel St3S where material is not endangered by damage

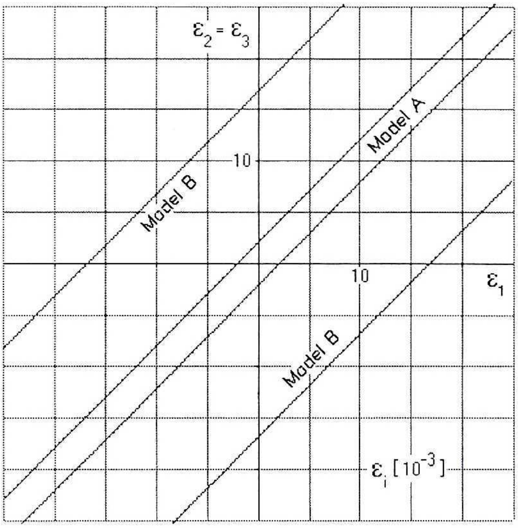

Fig. 6. Areas of stable strain before (Model A), and after exceeding yield point for steel St3S

The areas of stable strain for steel St3S in a selected cut of the space of strain components, before and after exceeding the yield point, are presented in Fig. 7.

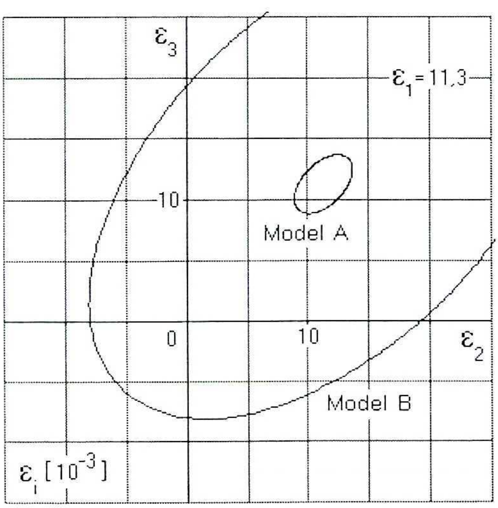

Fig. 7. Areas of stable deformation of steel St3S

5. Concluding remarks

In this work, the author presented a strength hypothesis of internal equilibrium stability in deformed material. The hypothesis is based on the premise that the fundamental reason for the existence of conditions involving the threat of damage or destruction of material is unstable state of internal equilibrium in the material. Evaluation of material strength done by this method follows on a theoretical analysis, based on the principle of conservation of energy, whose consequence is the criterion of stability of deformed material, described in Section 2 and consequently used in further part of the work.

The considerations were limited to homogenous, continuous, and isotropic materials. The nonlinear model of material, used in the analysis, assumes linearity of volumetric elasticity. Therefore, mathematical strength criteria formulated on this basis refer only to the assumed theoretical model of material.

The analysis presented in this work is an illustration of the thesis formulated earlier by the author [6]. It states that the knowledge of physical model of material and material constants adequately representing material's properties is a sufficient condition for theoretical determination of the areas where components of strain of the body represent the state of strain in which damage or destruction of the material could take place.

In the light of the results of this work, we can assert that the hypothesis of internal equilibrium stability confirms rightness, and gives a theoretical justification for the Huber's hypothesis, at least for the materials whose volumetric elasticity function is linear. At the same time, the comparative analysis presented in this work provides experimental confirmation for the hypothesis of internal equilibrium stability in the case of material of linear volumetric elasticity function. One should also bear in mind that Huber's hypothesis has been experimentally proven for various materials, among them for steels exhibiting a distinct onset of yielding, like it is in the examined steel type St3S.

The hypothesis of internal equilibrium stability, presented in this work, can be applied to materials of arbitrary nonlinear physical properties, including the materials whose volumetric elasticity function is not linear.

The method of simplified analysis based on nonlinear approximation of elastic-plastic properties of material, proposed in this work, can be useful in practical applications in mechanical engineering. It is especially convenient to apply the proposed method of numerical analysis to materials, for which strict, experimentally verified strength criteria have not yet been formulated. It refers to many new materials, such as polymers. The same method has previously been applied to some materials exhibiting big deformations, such as rubber and rubber-like materials, as it is described in work [6]. The method can be very helpful for design engineers who, because of scarcity of other simple methods, are often bound to use Huber's hypothesis even in the case when they have no guarantee that properties of material comply with this hypothesis.

It must not be forgotten that the commonly used Huber's hypothesis incorrectly describes strength characteristics of all constructional composites under high load in the state close to equal omnidirectional tension. The phenomenological method of description of materials, proposed in this work, makes it possible to formulate strength criteria free of the drawback of Huber's hypothesis. However, this goal could only be reached when nonlinear models of materials become even more perfect and allow for relevant description of properties of actual materials.

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Powierzchnia stanów granicznych materiału o nieliniowych właściwościach i jej związek z warunkiem plastyczności

Streszczenie

W pracy wykorzystano pojęcia i metody analizy nieliniowej teorii sprężystości do uproszczonego opisu właściwości sprężysto-plastycznych materiału. Przyjmując jako teoretyczną podstawę prawo zachowania energii, sformułowano kryterium umożliwiające badanie stateczności równowagi wewnętrznej w odkształconym materiale o nieliniowych właściwościach fizycznych zdefiniowanych funkcją gęstości energii odkształcenia. Założywszy małe odkształcenia materiału wyprowadzono wzory pozwalające na ocenę wytrzymałościową złożonych stanów odkształcenia. Wzory te zastępują formuły matematyczne tradycyjnie nazywane hipotezami wytrzymałościowymi. W charakterze przykładu – wykorzystując wielkości charakterystyczne uzyskane w statycznej próbie rozciągania węglowej stali konstrukcyjnej zwykłej jakości – wyznaczono w przestrzeni składowych stanu odkształcenia obszary, w których ze względu na stateczność stanu odkształcenia jest możliwe odkształcenie trwałe lub zniszczenie materiału. Wyniki tej analizy bazującej na sformułowanej wytrzymałościowej hipotezie stateczności stanu odkształcenia porównano z wynikami uzyskanymi na podstawie hipotezy energii odkształcenia postaciowego Hubera.