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A comparative study of the sensitivity analysis for systems with viscoelastic elements

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This paper discusses the different methods used for calculating first- and second-order sensitivity: the direct differentiation method, the adjoint variables method, and the hybrid method. The solutions obtained allow determining the sensitivity of dynamic characteristics such as eigenvalues and eigenvectors, natural frequencies, and nondimensional damping ratios. The methods were applied for analyzing systems with viscoelastic damping elements, whose behavior can be described by classical and fractional rheological models. However, the derived formulas are general and can also be applied to systems with damping elements described by other models. Their advantage is a compact and easy to code form. The paper also presents a comparison of the computational costs of the discussed methods. The correctness of all the proposed methods has been illustrated with numerical examples.

1. Introduction

Sensitivity analysis is widely used for various purposes such as structural health monitoring, damage detection [1], model updating, and structural optimization [2]. A review of this tool has been previously presented by some authors [3, 4].

The systems subjected to dynamic impacts involve a large amount of work devoted to the calculations of the sensitivity of eigenvalues and eigenvectors. The sensitivity of eigenvalues and eigenvectors can be determined using three different approaches. The first one is the modal method [5, 6] in which all or almost all modes should be determined and is therefore considered less efficient.

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The second method was developed by Nelson [7], which was later extended to viscous [8] and nonviscous damped systems [9]. This method is more effective than the modal method because it requires only the eigenvalue and its associated eigenvector to be determined.

The third one is the algebraic method which includes the direct differentiation method (DDM) and the adjoint variables method (AVM). The authors who proposed DDM [7] used it to calculate the sensitivities of eigenvalues and eigenvectors by solving a system of equations formed from a differentiated eigenproblem and differentiated normalization condition. The method was subsequently extended to systems with repeated eigenvalues [10] as well as to viscous and nonviscous damped systems [11]. AVM consists in adopting the response function to an eigenproblem in terms of eigenvalue, eigenvector, and design parameters, and then differentiating the augmented function, which is written using the eigenproblem and the vector normalization condition. This method can be used for distinct [12] and repeated eigenvalues [13]. In [14] AVM is used to compute complex eigenvalues and eigenvectors derivatives and the reverse algorithmic differentiation formula is proposed when only the eigenvalue is required. AVM seems to be more effective than DDM in some cases, especially when the number of parameters is large. Both methods, DDM and AVM, have been described in [15], in which the authors discuss the analysis of systems with viscoelastic dampers using these methods. Sensitivity analysis of transient response for systems with viscoelastic elements based on both DDM [16] and AVM [17] can be found.

The use of the second-order sensitivity can contribute to improving convergence in the optimization design and to increasing the accuracy of the approximation. The discussed methods have been extended to second- or higher-order analysis in [15, 18–21]. Three methods were proposed for the calculation of second-order sensitivity in [4]: DDM, AVM, and the hybrid method (HM). However, these methods were used only for undamped systems, whereas this paper presents a comparison of the efficiency of these methods for systems with viscoelastic elements which can be described by classical and fractional rheological models.

A lot of work is devoted to the application of design sensitivity analysis. In [22], the combination of the AVM and complex variable method is proposed to calculate the shape and size sensitivity for structural optimization. In [2] two optimizing procedures based on eigensensitivity are presented to find the optimal distribution of viscoelastic layer. In [23] the authors used the eigensensitivity method to optimize the damper location in trusses. A comprehensive description of design sensitivity analysis and its applications can be found in [24].

The equation of motion for structures with viscoelastic elements can be expressed as follows:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \int_0^t \tilde{\mathbf{G}}(t-\tau)\dot{\mathbf{q}}(\tau)\,\mathrm{d}\tau + \mathbf{K}\,\mathbf{q}(t) = \mathbf{f}(t), \tag{1}$$

where $\mathbf{q}(t)$ is the vector of nodal displacement, $\mathbf{f}(t)$ is the vector of excitation forces, $\mathbf{M}$, $\mathbf{K}$, and $\mathbf{C}$ are mass, stiffness, and damping matrices, respectively, and $\tilde{\mathbf{G}}(t-\tau)$ is the matrix of damping kernel functions. The form of matrix $\tilde{\mathbf{G}}$ depends on the model adopted [25, 26]. Applying the Laplace transform with zero initial conditions, the eigenvalue problem can be written as (for $\mathbf{f}(t) = \mathbf{0}$):

$$\mathbf{D}(s)\bar{\mathbf{q}}(s) = \mathbf{0}, \qquad \mathbf{D}(s) = s^2\mathbf{M} + s\mathbf{C} + \mathbf{G}(s) + \mathbf{K}, \tag{2}$$

where s is the Laplace variable, $\mathbf{G}(s)$ is the matrix depending on the model of viscoelastic element in Laplace domain, and $\bar{\mathbf{q}}(s)$ is the Laplace transform of $\mathbf{q}(t)$. A solution to the eigenproblem (2) for under-critically damped system is obtained in the form of a set of complex, conjugate eigenvalues s_l and corresponding eigenvectors $\bar{\mathbf{q}}_l(s)$ for $l = 1, 2, \ldots, 2n$, where n is the number of degrees of freedom of the structure.

The AVM of the second order and HM are presented for the first time for structures with viscoelastic elements along with the comparison of three sensitivity calculation methods for such systems with respect to their computational cost.

The paper is organized as follows: Section 2 presents the methods of calculating the sensitivities; Section 3 describes their application to structures with viscoelastic elements; Section 4 provides examples; Section 5 shows a comparison with respect to the computational cost of the presented methods, and Section 6 presents conclusions.

2. Design sensitivity analysis of the first and second order

2.1. Direct differentiation method (DDM)

Differentiation of Eq. (2) with respect to the design parameter p_i results in the following formula (for simplicity, (s) will be omitted):

$$\mathbf{D}\frac{\partial\bar{\mathbf{q}}}{\partial p_i} + \frac{\partial\mathbf{D}}{\partial s}\bar{\mathbf{q}}\frac{\partial s}{\partial p_i} = -\frac{\partial\mathbf{D}}{\partial p_i}\bar{\mathbf{q}}. \tag{3}$$

Since Eq. (3) has $n+1$ unknowns (sensitivity of eigenvalue $\partial s/\partial p_i$ and sensitivities of n elements of corresponding eigenvector $\partial\bar{\mathbf{q}}/\partial p_i$), it is necessary to include additional eigenvector normalization equation:

$$\frac{1}{2}\bar{\mathbf{q}}^T\frac{\partial\mathbf{D}}{\partial s}\bar{\mathbf{q}} = 1. \tag{4}$$

Matrices $\mathbf{D}$, $\partial\mathbf{D}/\partial s$ and $\partial\mathbf{D}/\partial p_i$ have dimensions $n\times n$ and vectors $\bar{\mathbf{q}}$ and $\partial\bar{\mathbf{q}}/\partial p_i$– $n\times 1$.

After differentiating Eq. (4), the following formula is obtained:

$$\bar{\mathbf{q}}^T\frac{\partial\mathbf{D}}{\partial s}\frac{\partial\bar{\mathbf{q}}}{\partial p_i} + \frac{1}{2}\bar{\mathbf{q}}^T\frac{\partial^2\mathbf{D}}{\partial s^2}\bar{\mathbf{q}}\frac{\partial s}{\partial p_i} = -\frac{1}{2}\bar{\mathbf{q}}^T\frac{\partial^2\mathbf{D}}{\partial s\partial p_i}\bar{\mathbf{q}}. \tag{5}$$

Eqs. (3) and (5) can be rewritten as a set of equations as follows:

$$\begin{bmatrix} \mathbf{D} & \dfrac{\partial \mathbf{D}}{\partial s}\bar{\mathbf{q}} \\ \bar{\mathbf{q}}^T\dfrac{\partial \mathbf{D}}{\partial s} & \dfrac{1}{2}\bar{\mathbf{q}}^T\dfrac{\partial^2 \mathbf{D}}{\partial s^2}\bar{\mathbf{q}} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \bar{\mathbf{q}}}{\mathrm{d} p_i} \\ \dfrac{\partial s}{\mathrm{d} p_i} \end{bmatrix} = \begin{bmatrix} -\dfrac{\partial \mathbf{D}}{\partial p_i}\bar{\mathbf{q}} \\ -\dfrac{1}{2}\bar{\mathbf{q}}^T\dfrac{\partial^2 \mathbf{D}}{\partial p_i \partial s}\bar{\mathbf{q}} \end{bmatrix}. \tag{6}$$

The first-order sensitivities of eigenvalues $\partial s/\partial p_i$ and eigenvectors $\partial\bar{\mathbf{q}}/\partial p_i$ are a solution to (6).

The second-order sensitivity can be determined by re-differentiating Eqs. (3) and (5) with respect to the parameter p_j. This will lead to the following set of equations:

$$\begin{bmatrix} \mathbf{D} & \dfrac{\partial \mathbf{D}}{\partial s}\bar{\mathbf{q}} \\ \bar{\mathbf{q}}^T\dfrac{\partial \mathbf{D}}{\partial s} & \dfrac{1}{2}\bar{\mathbf{q}}^T\dfrac{\partial^2 \mathbf{D}}{\partial s^2}\bar{\mathbf{q}} \end{bmatrix} \begin{bmatrix} \dfrac{\partial^2 \bar{\mathbf{q}}}{\partial p_i \partial p_j} \\ \dfrac{\partial^2 s}{\partial p_i \partial p_j} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \\ r_2 \end{bmatrix}, \tag{7}$$

where

$$\begin{aligned} \mathbf{r}_1 = &-\left[\left(\frac{\partial^2 \mathbf{D}}{\partial p_i \partial p_j} + \frac{\partial^2 \mathbf{D}}{\partial s^2}\frac{\partial s}{\partial p_i}\frac{\partial s}{\partial p_j} + \frac{\partial^2 \mathbf{D}}{\partial s \partial p_j}\frac{\partial s}{\partial p_i} + \frac{\partial^2 \mathbf{D}}{\partial s \partial p_i}\frac{\partial s}{\partial p_j}\right)\bar{\mathbf{q}} \right. \\ &\left. + \left(\frac{\partial \mathbf{D}}{\partial p_i} + \frac{\partial \mathbf{D}}{\partial s}\frac{\partial s}{\partial p_i}\right)\frac{\partial \bar{\mathbf{q}}}{\partial p_j} + \left(\frac{\partial \mathbf{D}}{\partial p_j} + \frac{\partial \mathbf{D}}{\partial s}\frac{\partial s}{\partial p_j}\right)\frac{\partial \bar{\mathbf{q}}}{\partial p_i}\right], \end{aligned}$$

$$\begin{aligned} r_2 = &-\left[\frac{1}{2}\bar{\mathbf{q}}^T\left(\frac{\partial^3 \mathbf{D}}{\partial s \partial p_i \partial p_j} + \frac{\partial^3 \mathbf{D}}{\partial^3 s}\frac{\partial s}{\partial p_i}\frac{\partial s}{\partial p_j} + \frac{\partial^3 \mathbf{D}}{\partial^2 s \partial p_i}\frac{\partial s}{\partial p_j} + \frac{\partial^3 \mathbf{D}}{\partial^2 s \partial p_j}\frac{\partial s}{\partial p_i}\right)\bar{\mathbf{q}} \right. \\ &\left. + \bar{\mathbf{q}}^T\left(\frac{\partial^2 \mathbf{D}}{\partial s \partial p_j} + \frac{\partial^2 \mathbf{D}}{\partial s^2}\frac{\partial s}{\partial p_j}\right)\frac{\partial \bar{\mathbf{q}}}{\partial p_i} + \bar{\mathbf{q}}^T\left(\frac{\partial^2 \mathbf{D}}{\partial s \partial p_i} + \frac{\partial^2 \mathbf{D}}{\partial s^2}\frac{\partial s}{\partial p_i}\right)\frac{\partial \bar{\mathbf{q}}}{\partial p_j} + \frac{\partial \bar{\mathbf{q}}^T}{\partial p_i}\frac{\partial \mathbf{D}}{\partial s}\frac{\partial \bar{\mathbf{q}}}{\partial p_j}\right]. \end{aligned}$$

The solution of (7) is the second-order sensitivities of eigenvalues $\partial^2 s/(\partial p_i \partial p_j)$ and eigenvectors $\partial^2\bar{\mathbf{q}}/(\partial p_i \partial p_j)$.

It is worth noting that the left sides of Eqs. (6) and (7) are the same. Therefore, higher-order sensitivities will only require the calculation of the right-hand side vector.

2.2. Adjoint variables method (AVM)

For AVM, the following objective function is assumed:

$$f = f\left(s, \bar{\mathbf{q}}, p_i\right) \tag{8}$$

which is understood as a response function to eigenproblem in terms of eigenvalue s, eigenvector $\bar{\mathbf{q}}$, and design parameter p_i. By using the variational principle applied for the analysis of design sensitivity [27, 28], the augmented function can be defined based on eigenproblem (2) and normalization condition (4), as follows:

$$F_1^A = f + \mathbf{x}_1^T \mathbf{D}\bar{\mathbf{q}} + y_1 \left(\frac{1}{2} \bar{\mathbf{q}}^T \frac{\partial \mathbf{D}}{\partial s} \bar{\mathbf{q}} - 1 \right), \tag{9}$$

where $\mathbf{x}_1$ and y_1 are adjoint variables (the so-called Lagrange multipliers) for eigenvalues and the associated eigenvectors, respectively. After differentiating Eq. (9) with respect to the design parameter p_i and considering Eqs. (2) and (4), the following formula is obtained:

$$\begin{aligned} \frac{\partial F_1^A}{\partial p_i} = {} & \frac{\partial f}{\partial p_i} + \mathbf{x}_1^T \frac{\partial \mathbf{D}}{\partial p_i} \bar{\mathbf{q}} + \frac{1}{2} y_1 \bar{\mathbf{q}}^T \frac{\partial^2 \mathbf{D}}{\partial s \partial p_i} \bar{\mathbf{q}} \\ & + \left(\frac{\partial f}{\partial s} + \mathbf{x}_1^T \frac{\partial \mathbf{D}}{\partial s} \bar{\mathbf{q}} + \frac{1}{2} y_1 \bar{\mathbf{q}}^T \frac{\partial^2 \mathbf{D}}{\partial s^2} \bar{\mathbf{q}} \right) \frac{\partial s}{\partial p_i} \\ & + \left[\left(\frac{\partial f}{\partial \bar{\mathbf{q}}} \right)^T + \mathbf{x}_1^T \mathbf{D} + y_1 \bar{\mathbf{q}}^T \frac{\partial \mathbf{D}}{\partial s} \right] \frac{\partial \bar{\mathbf{q}}}{\partial p_i} . \end{aligned} \tag{10}$$

Eq. (10) has two unknowns, which can be understood as the sensitivity of the eigenvalues $\partial s/\partial p_i$ and the associated eigenvectors $\partial \mathbf{q}/\partial p_i$. These can be eliminated by formulating two adjoint equations:

$$\frac{\partial f}{\partial s} + \mathbf{x}_1^T \frac{\partial \mathbf{D}}{\partial s} \bar{\mathbf{q}} + \frac{1}{2} y_1 \bar{\mathbf{q}}^T \frac{\partial^2 \mathbf{D}}{\partial s^2} \bar{\mathbf{q}} = 0, \tag{11}$$

$$\left(\frac{\partial f}{\partial \bar{\mathbf{q}}} \right)^T + \mathbf{x}_1^T \mathbf{D} + y_1 \bar{\mathbf{q}}^T \frac{\partial \mathbf{D}}{\partial s} = \mathbf{0}^T \tag{12}$$

which give rise to the following set of equations:

$$\begin{bmatrix} \mathbf{D} & \dfrac{\partial \mathbf{D}}{\partial s} \bar{\mathbf{q}} \\ \bar{\mathbf{q}}^T \dfrac{\partial \mathbf{D}}{\partial s} & \dfrac{1}{2} \bar{\mathbf{q}}^T \dfrac{\partial^2 \mathbf{D}}{\partial s^2} \bar{\mathbf{q}} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ y_1 \end{bmatrix} = - \begin{bmatrix} \dfrac{\partial f}{\partial \bar{\mathbf{q}}} \\ \dfrac{\partial f}{\partial s} \end{bmatrix} . \tag{13}$$

After solving (13), Lagrange multipliers are obtained. Substitution of these into Eq. (10), will reduce it to the following form:

$$\frac{\partial F_1^A}{\partial p_i} = \frac{\partial f}{\partial p_i} + \mathbf{x}_1^T \frac{\partial \mathbf{D}}{\partial p_i} \bar{\mathbf{q}} + \frac{1}{2} y_1 \bar{\mathbf{q}}^T \frac{\partial^2 \mathbf{D}}{\partial s \partial p_i} \bar{\mathbf{q}}, \tag{14}$$

where the explicit derivative $\partial f/\partial p_i$ is equal to 0.

For determining the sensitivity of eigenvalue and the l-th element of the eigenvector, Eq. (13) has to be solved separately. If the sensitivity of the eigenvalues is of interest, the expression $f = s$ is substituted into Eq. (13) and $\partial f/\partial \bar{\mathbf{q}} = \mathbf{0}$ and $\partial f/\partial s = 1$. On the other hand, if the sensitivity of the l-th element of eigenvector is of interest, then the expression $f = q_l$ is substituted and $\partial f/\partial s = 0$ and $\partial f/\partial \bar{\mathbf{q}} = \mathbf{1}\delta_{ml}$, where δ_{ml} is the Kronecker delta.

For second-order sensitivity, the new augmented function should be written using eigenproblem (2), normalization condition (4), and two adjoint equations (11) and (12):

$$\frac{\partial F_2^A}{\partial p_i} = \frac{\partial F_1^A}{\partial p_i} + \mathbf{x}_2^T \mathbf{D}\bar{\mathbf{q}} + y_2\left(\frac{1}{2}\bar{\mathbf{q}}^T\frac{\partial \mathbf{D}}{\partial s}\bar{\mathbf{q}} - 1\right) + \mathbf{x}_3^T\left(\frac{\partial f}{\partial \bar{\mathbf{q}}} + \mathbf{D}\,\mathbf{x}_1 + y_1\frac{\partial \mathbf{D}}{\partial s}\bar{\mathbf{q}}\right)$$
$$+\ y_3\left(\frac{\partial f}{\partial s} + \mathbf{x}_1^T\frac{\partial \mathbf{D}}{\partial s}\bar{\mathbf{q}} + \frac{1}{2}y_1\bar{\mathbf{q}}^T\frac{\partial^2 \mathbf{D}}{\partial s^2}\bar{\mathbf{q}}\right). \tag{15}$$

There are four Lagrange multipliers: $\mathbf{x}_2$, $\mathbf{x}_3$, y_2, and y_3. After substituting Eq. (14), differentiating Eq. (15) with respect to the design parameter p_j, and fulfilling conditions (2) and (4), the following formula is obtained:

$$\frac{\partial^2 F_2^A}{\partial p_i \partial p_j} = \frac{\partial^2 f}{\partial p_i \partial p_j} + \mathbf{x}_3^T\frac{\partial^2 f}{\partial \bar{\mathbf{q}}\partial p_j} + y_3\frac{\partial^2 f}{\partial s \partial p_j} + \frac{1}{2}y_2\bar{\mathbf{q}}^T\frac{\partial^2 \mathbf{D}}{\partial s \partial p_j}\bar{\mathbf{q}}$$
$$+\ \mathbf{x}_3^T\left(\frac{\partial \mathbf{D}}{\partial p_j}\mathbf{x}_1 + y_1\frac{\partial^2 \mathbf{D}}{\partial s \partial p_j}\bar{\mathbf{q}}\right) + y_3\left(\mathbf{x}_1^T\frac{\partial^2 \mathbf{D}}{\partial s \partial p_j}\bar{\mathbf{q}} + \frac{1}{2}y_1\bar{\mathbf{q}}^T\frac{\partial^3 \mathbf{D}}{\partial s^2 \partial p_j}\bar{\mathbf{q}}\right)$$
$$+\ \left(\frac{\partial \mathbf{x}_1}{\partial p_j}\right)^T\left(\frac{\partial \mathbf{D}}{\partial p_i}\bar{\mathbf{q}} + \mathbf{D}\,\mathbf{x}_3 + y_3\frac{\partial \mathbf{D}}{\partial s}\bar{\mathbf{q}}\right) + \left(\frac{\partial \bar{\mathbf{q}}}{\partial p_j}\right)^T\left[\frac{\partial^2 f}{\partial p_i \partial \bar{\mathbf{q}}} + \frac{\partial \mathbf{D}}{\partial p_i}\mathbf{x}_1 + y_1\frac{\partial^2 \mathbf{D}}{\partial s \partial p_i}\bar{\mathbf{q}}\right.$$
$$\left. +\ \mathbf{D}\,\mathbf{x}_2 + y_2\frac{\partial \mathbf{D}}{\partial s}\bar{\mathbf{q}} + y_1\frac{\partial \mathbf{D}}{\partial s}\mathbf{x}_3 + y_3\left(\frac{\partial^2 f}{\partial s \partial \bar{\mathbf{q}}} + \frac{\partial \mathbf{D}}{\partial s}\mathbf{x}_1 + y_1\frac{\partial^2 \mathbf{D}}{\partial s^2}\bar{\mathbf{q}}\right)\right]$$
$$+\ \frac{\partial y_1}{\partial p_j}\left(\frac{1}{2}\bar{\mathbf{q}}^T\frac{\partial^2 \mathbf{D}}{\partial s \partial p_i}\bar{\mathbf{q}} + \mathbf{x}_3^T\frac{\partial \mathbf{D}}{\partial s}\bar{\mathbf{q}} + \frac{1}{2}y_3\bar{\mathbf{q}}^T\frac{\partial^2 \mathbf{D}}{\partial s^2}\bar{\mathbf{q}}\right)$$
$$+\ \frac{\partial s}{\partial p_j}\left[\frac{\partial^2 f}{\partial p_i \partial s} + \mathbf{x}_1^T\frac{\partial^2 \mathbf{D}}{\partial s \partial p_i}\bar{\mathbf{q}} + \frac{1}{2}y_1\bar{\mathbf{q}}^T\frac{\partial^3 \mathbf{D}}{\partial s^2 \partial p_i}\bar{\mathbf{q}} + \mathbf{x}_2^T\frac{\partial \mathbf{D}}{\partial s}\bar{\mathbf{q}} + \frac{1}{2}y_2\bar{\mathbf{q}}^T\frac{\partial^2 \mathbf{D}}{\partial s^2}\bar{\mathbf{q}}\right.$$
$$\left. +\ \mathbf{x}_3^T\left(\frac{\partial^2 f}{\partial s \partial \bar{\mathbf{q}}} + \frac{\partial \mathbf{D}}{\partial s}\mathbf{x}_1 + y_1\frac{\partial^2 \mathbf{D}}{\partial s^2}\bar{\mathbf{q}}\right) + y_3\left(\mathbf{x}_1^T\frac{\partial^2 \mathbf{D}}{\partial s^2}\bar{\mathbf{q}} + \frac{1}{2}y_1\bar{\mathbf{q}}^T\frac{\partial^3 \mathbf{D}}{\partial s^3}\bar{\mathbf{q}}\right)\right]. \tag{16}$$

If the sensitivity of the eigenvalue s ($f = s$) or the element of the eigenvector q_l ($f = q_l$) is of interest, then the relationships:

$$\frac{\partial^2 f}{\partial s \partial p_j} = 0, \qquad \frac{\partial^2 f}{\partial \bar{\mathbf{q}} \partial p_j} = \mathbf{0}, \qquad \frac{\partial^2 f}{\partial p_i \partial s} = 0, \qquad \frac{\partial^2 f}{\partial p_i \partial \bar{\mathbf{q}}} = \mathbf{0} \tag{17}$$

are always true.

The unknown derivatives $\partial \mathbf{x}_1/\partial p_j$ and $\partial y_1/\partial p_j$ are eliminated from Eq. (16) using the following two new adjoint equations:

$$\frac{\partial \mathbf{D}}{\partial p_i}\bar{\mathbf{q}} + \mathbf{D}\,\mathbf{x}_3 + y_3\frac{\partial \mathbf{D}}{\partial s}\bar{\mathbf{q}} = \mathbf{0}, \tag{18}$$

$$\frac{1}{2}\bar{\mathbf{q}}^T\frac{\partial^2 \mathbf{D}}{\partial s\partial p_i}\bar{\mathbf{q}} + \mathbf{x}_3^T\frac{\partial \mathbf{D}}{\partial s}\bar{\mathbf{q}} + \frac{1}{2}y_3\bar{\mathbf{q}}^T\frac{\partial^2 \bar{\mathbf{D}}}{\partial s^2}\bar{\mathbf{q}} = 0. \tag{19}$$

The above conditions can be written as a set of equations:

$$\begin{bmatrix} \mathbf{D} & \dfrac{\partial \mathbf{D}}{\partial s}\bar{\mathbf{q}} \\ \bar{\mathbf{q}}^T\dfrac{\partial \mathbf{D}}{\partial s} & \dfrac{1}{2}\bar{\mathbf{q}}^T\dfrac{\partial^2 \mathbf{D}}{\partial s^2}\bar{\mathbf{q}} \end{bmatrix}\begin{bmatrix}\mathbf{x}_3\\ y_3\end{bmatrix} = \begin{bmatrix} -\dfrac{\partial \mathbf{D}}{\partial p_i} \\ -\dfrac{1}{2}\bar{\mathbf{q}}^T\dfrac{\partial^2 \mathbf{D}}{\partial s\partial p_i}\bar{\mathbf{q}} \end{bmatrix}. \tag{20}$$

The above set of equations is identical to (6) in the DDM. Hence, it is not necessary to solve Eq. (20) because it is known that $\mathbf{x}_3 = \partial\bar{\mathbf{q}}/\partial p_i$ and $y_3 = \partial s/\partial p_i$ are the first-order sensitivities. Next, the other unknown derivatives $\partial\bar{\mathbf{q}}/\partial p_j$ and $\partial s/\partial p_j$ should be eliminated from Eq. (16) by introducing the following conditions:

$$\begin{aligned}&\frac{\partial \mathbf{D}}{\partial p_i}\mathbf{x}_1 + y_1\frac{\partial^2 \mathbf{D}}{\partial s\partial p_i}\bar{\mathbf{q}} + \mathbf{D}\,\mathbf{x}_2 + y_2\frac{\partial \mathbf{D}}{\partial s}\bar{\mathbf{q}} + y_1\frac{\partial \mathbf{D}}{\partial s}\mathbf{x}_3 \\ &\qquad + y_3\left(\frac{\partial \mathbf{D}}{\partial s}\mathbf{x}_1 + y_1\frac{\partial^2 \mathbf{D}}{\partial s^2}\bar{\mathbf{q}}\right) = \mathbf{0},\end{aligned} \tag{21}$$

$$\begin{aligned}&\mathbf{x}_1^T\frac{\partial^2 \mathbf{D}}{\partial s\partial p_i}\bar{\mathbf{q}} + \frac{1}{2}y_1\bar{\mathbf{q}}^T\frac{\partial^3 \mathbf{D}}{\partial s^2\partial p_i}\bar{\mathbf{q}} + \mathbf{x}_2^T\frac{\partial \mathbf{D}}{\partial s}\bar{\mathbf{q}} + \frac{1}{2}y_2\bar{\mathbf{q}}^T\frac{\partial^2 \mathbf{D}}{\partial s^2}\bar{\mathbf{q}} \\ &\qquad + \mathbf{x}_3^T\left(\frac{\partial \mathbf{D}}{\partial s}\mathbf{x}_1 + y_1\frac{\partial^2 \mathbf{D}}{\partial s^2}\bar{\mathbf{q}}\right) + y_3\left(\mathbf{x}_1^T\frac{\partial^2 \mathbf{D}}{\partial s^2}\bar{\mathbf{q}} + \frac{1}{2}y_1\bar{\mathbf{q}}^T\frac{\partial^3 \mathbf{D}}{\partial s^3}\bar{\mathbf{q}}\right) = 0.\end{aligned} \tag{22}$$

These form the following set of equations:

$$\begin{bmatrix} \mathbf{D} & \dfrac{\partial \mathbf{D}}{\partial s}\bar{\mathbf{q}} \\ \bar{\mathbf{q}}^T\dfrac{\partial \mathbf{D}}{\partial s} & \dfrac{1}{2}\bar{\mathbf{q}}^T\dfrac{\partial^2 \mathbf{D}}{\partial s^2}\bar{\mathbf{q}} \end{bmatrix}\begin{bmatrix}\mathbf{x}_2\\ y_2\end{bmatrix} = \begin{bmatrix}\mathbf{r}_3\\ r_4\end{bmatrix}, \tag{23}$$

where

$$\mathbf{r}_3 = -\left[\frac{\partial \mathbf{D}}{\partial p_i}\mathbf{x}_1 + y_1\frac{\partial^2 \mathbf{D}}{\partial s\partial p_i}\bar{\mathbf{q}} + y_1\frac{\partial \mathbf{D}}{\partial s}\mathbf{x}_3 + y_3\left(\frac{\partial \mathbf{D}}{\partial s}\mathbf{x}_1 + y_1\frac{\partial^2 \mathbf{D}}{\partial s^2}\bar{\mathbf{q}}\right)\right],$$

$$\begin{aligned} r_4 = -\Bigg[&\mathbf{x}_1^T\frac{\partial^2 \mathbf{D}}{\partial s\partial p_i}\bar{\mathbf{q}} + \frac{1}{2}y_1\bar{\mathbf{q}}\frac{\partial^3 \mathbf{D}}{\partial s^2\partial p_i}\bar{\mathbf{q}} + \mathbf{x}_3^T\left(\frac{\partial \mathbf{D}}{\partial s}\mathbf{x}_1 + y_1\frac{\partial^2 \mathbf{D}}{\partial s^2}\bar{\mathbf{q}}\right) \\ &+ y_3\left(\mathbf{x}_1^T\frac{\partial^2 \mathbf{D}}{\partial s^2}\bar{\mathbf{q}} + \frac{1}{2}y_1\bar{\mathbf{q}}^T\frac{\partial^3 \mathbf{D}}{\partial s^3}\bar{\mathbf{q}}\right)\Bigg].\end{aligned}$$

Eq. (23) is solved with respect to the Lagrange multipliers $\mathbf{x}_2$ and y_2. Then, considering conditions (2), (4), (11), and (12), Eq. (16) is reduced to the following form:

$$\begin{aligned}\frac{\partial^2 F_2^A}{\partial p_i \partial p_j} = {} & \mathbf{x}_1^T \frac{\partial^2 \mathbf{D}}{\partial p_i \partial p_j}\bar{\mathbf{q}} + \frac{1}{2} y_1 \bar{\mathbf{q}}^T \frac{\partial^3 \mathbf{D}}{\partial s \partial p_i \partial p_j}\bar{\mathbf{q}} + \mathbf{x}_2^T \frac{\partial \mathbf{D}}{\partial p_j}\bar{\mathbf{q}} \\ & + \frac{1}{2} y_2 \bar{\mathbf{q}}^T \frac{\partial^2 \mathbf{D}}{\partial s \partial p_j}\bar{\mathbf{q}} + \mathbf{x}_3^T \left(\frac{\partial \mathbf{D}}{\partial p_j}\mathbf{x}_1 + y_1 \frac{\partial^2 \mathbf{D}}{\partial s \partial p_j}\bar{\mathbf{q}}\right) \\ & + y_3 \left(\mathbf{x}_1^T \frac{\partial^2 \mathbf{D}}{\partial s \partial p_j}\bar{\mathbf{q}} + \frac{1}{2} y_1 \bar{\mathbf{q}}^T \frac{\partial^3 \mathbf{D}}{\partial s^2 \partial p_j}\bar{\mathbf{q}}\right). \end{aligned} \tag{24}$$

(the explicit derivative $\partial^2 f / \partial p_i \partial p_j = 0$). The obtained formula (24) allows determining the second-order sensitivities with the use of AVM. Similar to DDM, the matrices on the left side of Eqs. (13) and (23) are the same.

2.3. Hybrid method (HM)

HM is also based on determining the objective function, but it can be written as the first derivative of function $f(s, p_i, \bar{\mathbf{q}})$ with respect to design parameter $(\partial f(s, p_i, \bar{\mathbf{q}}) / \partial p_i)$. First, the direct differentiation technique is used for computing the derivatives $\partial s / \partial p_i$ and $\partial \bar{\mathbf{q}} / \partial p_i$ and then the augmented function is determined as a combination of the differentiated eigenproblem (3) and the differentiated normalization condition (5):

$$\begin{aligned}\frac{\partial F_1^H}{\partial p_i} = {} & \frac{\partial f}{\partial p_i} + \mathbf{x}_4^T \left(\mathbf{D}\frac{\partial \bar{\mathbf{q}}}{\partial p_i} + \frac{\partial \mathbf{D}}{\partial s}\bar{\mathbf{q}}\frac{\partial s}{\partial p_i} + \frac{\partial \mathbf{D}}{\partial p_i}\bar{\mathbf{q}}\right) \\ & + y_4 \left(\bar{\mathbf{q}}^T \frac{\partial \mathbf{D}}{\partial s}\frac{\partial \bar{\mathbf{q}}}{\partial p_i} + \frac{1}{2}\bar{\mathbf{q}}^T \frac{\partial^2 \mathbf{D}}{\partial s^2}\bar{\mathbf{q}}\frac{\partial s}{\partial p_i} + \frac{1}{2}\bar{\mathbf{q}}^T \frac{\partial^2 \mathbf{D}}{\partial s \partial p_i}\bar{\mathbf{q}}\right). \end{aligned} \tag{25}$$

Eq. (25) has two Lagrange multipliers: $\mathbf{x}_4$ and y_4. After differentiating this equation with respect to parameter p_j and considering formulas (3), (5), and (17), the following equation is obtained:

$$\begin{aligned}\frac{\partial^2 F_1^H}{\partial p_i \partial p_j} = {} & \frac{\partial^2 f}{\partial p_i \partial p_j} + \mathbf{x}_4^T \left[\left(\frac{\partial \mathbf{D}}{\partial p_i \partial p_j} + \frac{\partial^2 \mathbf{D}}{\partial s \partial p_i}\frac{\partial s}{\partial p_j} + \frac{\partial^2 \mathbf{D}}{\partial s \partial p_j}\frac{\partial s}{\partial p_i}\right.\right. \\ & \left.\left. + \frac{\partial^2 \mathbf{D}}{\partial s^2}\frac{\partial s}{\partial p_i}\frac{\partial s}{\partial p_j}\right)\bar{\mathbf{q}} + \left(\frac{\partial \mathbf{D}}{\partial p_i} + \frac{\partial \mathbf{D}}{\partial s}\frac{\partial s}{\partial p_i}\right)\frac{\partial \bar{\mathbf{q}}}{\partial p_j} + \left(\frac{\partial \mathbf{D}}{\partial p_j} + \frac{\partial \mathbf{D}}{\partial s}\frac{\partial s}{\partial p_j}\right)\frac{\partial \bar{\mathbf{q}}}{\partial p_i}\right] \\ & + y_4 \left[\frac{1}{2}\bar{\mathbf{q}}^T \left(\frac{\partial^3 \mathbf{D}}{\partial s \partial p_i \partial p_j} + \frac{\partial^3 \mathbf{D}}{\partial s^2 \partial p_i}\frac{\partial s}{\partial p_j} + \frac{\partial^3 \mathbf{D}}{\partial s^2 \partial p_j}\frac{\partial s}{\partial p_i} + \frac{\partial^3 \mathbf{D}}{\partial s^3}\frac{\partial s}{\partial p_i}\frac{\partial s}{\partial p_j}\right)\bar{\mathbf{q}}\right. \\ & + \bar{\mathbf{q}}^T \left(\frac{\partial^2 \mathbf{D}}{\partial s \partial p_j} + \frac{\partial^2 \mathbf{D}}{\partial s^2}\frac{\partial s}{\partial p_j}\right)\frac{\partial \bar{\mathbf{q}}}{\partial p_i} + \bar{\mathbf{q}}^T \left(\frac{\partial^2 \mathbf{D}}{\partial s \partial p_i} + \frac{\partial^2 \mathbf{D}}{\partial s^2}\frac{\partial s}{\partial p_i}\right)\frac{\partial \bar{\mathbf{q}}}{\partial p_j} \end{aligned} \tag{26}$$

[(26) cont.]

$$+\left(\frac{\partial\bar{\mathbf{q}}}{\partial p_i}\right)^T\frac{\partial\mathbf{D}}{\partial s}\frac{\partial\bar{\mathbf{q}}}{\partial p_j}\Bigg]+\left[\left(\frac{\partial f}{\partial\bar{\mathbf{q}}}\right)^T+\mathbf{x}_4^T\mathbf{D}+y_4\bar{\mathbf{q}}^T\frac{\partial\mathbf{D}}{\partial s}\right]\frac{\partial^2\bar{\mathbf{q}}}{\partial p_i\partial p_j}$$
$$+\left(\frac{\partial f}{\partial s}+\mathbf{x}_4^T\frac{\partial\mathbf{D}}{\partial s}\bar{\mathbf{q}}+\frac{1}{2}y_4\bar{\mathbf{q}}^T\frac{\partial^2\mathbf{D}}{\partial s^2}\bar{\mathbf{q}}\right)\frac{\partial^2 s}{\partial p_i\partial p_j}\,.$$

Similar to AVM, the unknown sensitivities of the second order $\partial^2\bar{\mathbf{q}}/\partial p_i\partial p_j$ and $\partial^2 s/\partial p_i\partial p_j$ are eliminated in HM. Two adjoint equations form a set of equations which is the same as (13) obtained for first-order sensitivities. Thus, the Lagrange multipliers are those obtained in the case of the first order ($\mathbf{x}_4\equiv\mathbf{x}_1$ and $y_4\equiv y_1$).

The second-order derivative can be written as follows:

$$\frac{\partial^2 F_1^H}{\partial p_i\partial p_j}=-\mathbf{x}_1^T\mathbf{r}_5-y_1 r_6\,,\tag{27}$$

where

$$\mathbf{r}_5=-\left[\left(\frac{\partial\mathbf{D}}{\partial p_i\partial p_j}+\frac{\partial^2\mathbf{D}}{\partial s\partial p_i}\frac{\partial s}{\partial p_j}+\frac{\partial^2\mathbf{D}}{\partial s\partial p_j}\frac{\partial s}{\partial p_i}+\frac{\partial^2\mathbf{D}}{\partial s^2}\frac{\partial s}{\partial p_i}\frac{\partial s}{\partial p_j}\right)\bar{\mathbf{q}}\right.$$
$$\left.+\left(\frac{\partial\mathbf{D}}{\partial p_i}+\frac{\partial\mathbf{D}}{\partial s}\frac{\partial s}{\partial p_i}\right)\frac{\partial\bar{\mathbf{q}}}{\partial p_j}+\left(\frac{\partial\mathbf{D}}{\partial p_j}+\frac{\partial\mathbf{D}}{\partial s}\frac{\partial s}{\partial p_j}\right)\frac{\partial\bar{\mathbf{q}}}{\partial p_i}\right],$$

$$r_6=-\left[\frac{1}{2}\bar{\mathbf{q}}^T\left(\frac{\partial^3\mathbf{D}}{\partial s\partial p_i\partial p_j}+\frac{\partial^3\mathbf{D}}{\partial s^2\partial p_i}\frac{\partial s}{\partial p_j}+\frac{\partial^3\mathbf{D}}{\partial s^2\partial p_j}\frac{\partial s}{\partial p_i}+\frac{\partial^3\mathbf{D}}{\partial s^3}\frac{\partial s}{\partial p_j}\frac{\partial s}{\partial p_j}\right)\bar{\mathbf{q}}\right.$$
$$+\bar{\mathbf{q}}^T\left(\frac{\partial^2\mathbf{D}}{\partial s\partial p_j}+\frac{\partial^2\mathbf{D}}{\partial s^2}\frac{\partial s}{\partial p_j}\right)\frac{\partial\bar{\mathbf{q}}}{\partial p_i}+\bar{\mathbf{q}}^T\left(\frac{\partial^2\mathbf{D}}{\partial s\partial p_i}+\frac{\partial^2\mathbf{D}}{\partial s^2}\frac{\partial s}{\partial p_i}\right)\frac{\partial\bar{\mathbf{q}}}{\partial p_j}$$
$$\left.+\left(\frac{\partial\bar{\mathbf{q}}}{\partial p_i}\right)^T\frac{\partial\mathbf{D}}{\partial s}\frac{\partial\bar{\mathbf{q}}}{\partial p_j}\right].$$

It should be noted that only one-time calculation of the set of equations is required for determining the first- and second-order sensitivities by HM.

Both AVM and HM allow simultaneous calculation of the sensitivities of eigenvalues and the associated eigenvectors.

3. Application to structures with viscoelastic elements

Eq. (2) describes an eigenproblem for systems with viscoelastic elements, in which, as mentioned in Section 1, the matrix $\mathbf{G}(s)$ depends on the type of system considered and the rheological model of viscoelastic elements. It can be written in the following general form:

$$\mathbf{G}(s)=\sum_{k=1}^{r}\mathbf{G}_k(s)=\sum_{k=1}^{r}\mathbf{K}_{v,k}g_k(s),\tag{28}$$

where r denotes the number of viscoelastic elements, $\mathbf{K}_{v,k}$ is a location matrix of viscoelastic elements and $g_k(s)$ is a function describing the viscoelastic properties of element k. Structures with viscoelastic elements have already been described in detail, including frame with viscoelastic dampers [29], beam with viscoelastic layers [30], and plates with viscoelastic layers [31] or with viscoelastic dampers [32].

Dynamic behavior of structures with viscoelastic elements can be described by the natural frequency ω_l and the nondimensional damping ratio γ_l (for $l = 1, 2, \ldots, n$) expressed as:

$$\omega_l = \sqrt{\mu_l^2 + \eta_l^2}\,, \qquad \gamma_l = -\frac{\mu_l}{\omega_l}\,, \tag{29}$$

where $\mu_l = \mathrm{Re}(s_l)$ and $\eta_l = \mathrm{Im}(s_l)$, $(s_l = \mu_l + \mathrm{i}\eta_l,\ \mathrm{i} = \sqrt{-1})$.

4. Examples

4.1. System with four degrees of freedom

The first example is the analysis of a system with four degrees of freedom described previously in [10] (Fig. 1). The matrix $\mathbf{D}(s)$ is expressed as:

$$\mathbf{D}(s) = s^2\mathbf{M} + \mathbf{K} + \mathbf{K}_v g(s), \tag{30}$$

where

$$\mathbf{K}_v = \mathrm{diag}\begin{bmatrix} 4c_1 & 4c_2 & 4cl & 6c \end{bmatrix}, \qquad g(s) = s,$$

$$\mathbf{K} = \begin{bmatrix} 4k + k_1 & -k_1 & 0 & 0 \\ -k_1 & 5k_1 & 0 & 0 \\ 0 & 0 & 4k & 0 \\ 0 & 0 & 0 & 6k \end{bmatrix}, \qquad \mathbf{M} = \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{bmatrix}.$$

In the above example, $k = k_1 = 1000$ N/m, $c_1 = c_2 = c = 10$ Ns/m and $m = 1$ kg.

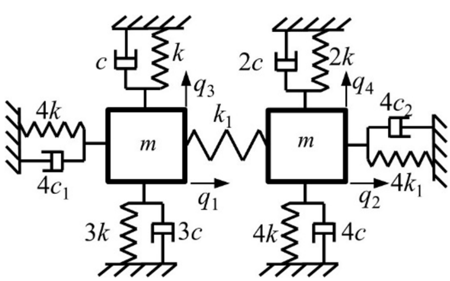


Fig. 1. System with four degrees of freedom

Sensitivity was calculated with respect to the change of parameter k. Only distinct eigenvalues were considered in the analysis, because the presented method cannot be applied to repeated eigenvalues (Table 1).

The values obtained here are the same as in [10].

Table 1. Comparison of results

Eigenvalue	Sensitivity [10]	Sensitivity DDM
$s_1 = -20.0 + 74.8333\mathrm{i}$	$1.3363 \cdot 10^{-2}\mathrm{i}$	$1.3363 \cdot 10^{-2}\mathrm{i}$
$s_2 = -30.0 + 71.414\mathrm{i}$	$4.2008 \cdot 10^{-2}\mathrm{i}$	$4.2008 \cdot 10^{-2}\mathrm{i}$

The system in consideration was also analyzed in terms of the second-order sensitivity. The analysis was carried out for the first eigenvalue with respect to the change of parameters c_1 and c_2. The obtained results are summarized in Table 2.

Table 2. Comparison of sensitivities of the first- and second-order

Parameter	Sensitivity of the first order	Sensitivity of the second order
c_1	$-1.0 - 0.2673\mathrm{i}$	$0.08 - 0.1533\mathrm{i}$
c_2	$-1.1546 - 0.3086\mathrm{i}$	$0.0564 - 0.1072\mathrm{i}$

For the imaginary part, the second-order sensitivity value for the parameter c_1 is about 57% of the first-order values, and for the parameter c_2 it is about 35%. This indicates that for the case under consideration, it is advisable to take into account the second-order sensitivity since the first-order sensitivity may give an erroneous approximation.

4.2. Frame with viscoelastic dampers

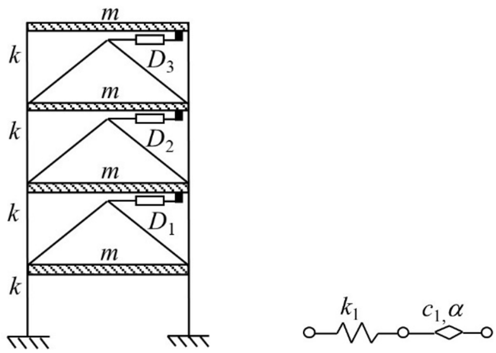
The second example is the analysis of a frame with built-in viscoelastic dampers (Fig. 2a). The structure was modeled as a shear frame assuming the following parameters: mass of the floor $m = 10\,000$ kg and stiffness of the storey $k = 1.0$ MN/m.

The behavior of the dampers is described using the fractional Maxwell model (Fig. 2b). The same parameters were assumed for all dampers: stiffness $k_{1,k} = 1.0$ MN/m, damping coefficient $c_{1,k} = 0.1$ MNs/m and fractional parameter $\alpha_k = 0.6$. The function $g_k(s)$ in Eq. (28) takes the following form:

$$g_k(s) = \frac{k_{1,k} c_{1,k} s^{\alpha_k}}{k_{1,k} + c_{1,k} s^{\alpha_k}} . \tag{31}$$

A model of such a structure has been previously described in detailed [29].

First, the results of the first- and second-order sensitivity obtained using DDM, AVM, and HM were compared. The sensitivity of the dynamic characteristics was calculated with respect to the change of the first damper parameter $c_{1,1}$.



(a) (b)

Fig. 2. A frame with viscoelastic dampers(a) and a model of damper (b)

This was followed by a comparison of the sensitivity of natural frequency and nondimensional damping ratio. Using the following formulas:

$$\frac{\partial s}{\partial p_i} = \frac{\partial \mu}{\partial p_i} + \mathrm{i}\frac{\partial \eta}{\partial p_i}, \qquad \frac{\partial^2 s}{\partial p_i \partial p_j} = \frac{\partial^2 \mu}{\partial p_i \partial p_j} + \mathrm{i}\frac{\partial^2 \eta}{\partial p_i \partial p_j} \tag{32}$$

they can be computed as:

$$\frac{\partial \omega}{\partial p_i} = \frac{1}{\omega}\left(\mu\frac{\partial \mu}{\partial p_i} + \eta\frac{\partial \eta}{\partial p_i}\right), \qquad \frac{\partial \gamma}{\partial p_i} = -\frac{1}{\omega}\frac{\partial \mu}{\partial p_i} - \frac{\gamma}{\omega}\frac{\partial \omega}{\partial p_i} \tag{33}$$

for the first-order sensitivity and

$$\frac{\partial^2 \omega}{\partial p_i \partial p_j} = \frac{1}{\omega}\left(\frac{\partial \mu}{\partial p_i}\frac{\partial \mu}{\partial p_j} + \frac{\partial \eta}{\partial p_i}\frac{\partial \eta}{\partial p_j} + \mu\frac{\partial^2 \mu}{\partial p_i \partial p_j} + \eta\frac{\partial^2 \eta}{\partial p_i \partial p_j} - \frac{\partial \omega}{\partial p_i}\frac{\partial \omega}{\partial p_j}\right), \tag{34}$$

$$\frac{\partial^2 \gamma}{\partial p_i \partial p_j} = -\frac{1}{\omega}\left(\gamma\frac{\partial^2 \omega}{\partial p_i \partial p_j} + \frac{\partial^2 \mu}{\partial p_i \partial p_j} + \frac{\partial \gamma}{\partial p_i}\frac{\partial \omega}{\partial p_j} + \frac{\partial \gamma}{\partial p_j}\frac{\partial \omega}{\partial p_1}\right) \tag{35}$$

for the second-order sensitivity.

The results of the comparison are presented in Table 3.

Table 3. Comparison of the results obtained by DDM, AVM, and HM

Methods	Sensitivities			
	$\partial\omega/\partial c_{1,1}$ $\cdot 10^{-7}$	$\partial^2\omega/\partial c_{1,1}^2$ $\cdot 10^{-12}$	$\partial\gamma/\partial c_{1,1}$ $\cdot 10^{-7}$	$\partial^2\gamma/\partial c_{1,1}^2$ $\cdot 10^{-12}$
DDM	7.4970	–1.7382	1.0741	–1.3461
AVM	7.4970	–1.7382	1.0741	–1.3461
HM	7.4970	–1.7382	1.0741	–1.3461

The values of sensitivities determined by all three methods are the same. In most of the cases analyzed, the differences appeared only in the 15th significant place. This indicates that the results are not influenced by the method used.

The next analysis was a comparison of the exact results obtained by changing the values of the damper parameters and the results obtained from the first- and second-order sensitivity analysis. The values obtained from the sensitivity analysis were calculated from the Taylor series expansion [13]:

$$\tilde{f} = f + \frac{\partial f}{\partial p_i}\Delta p_i + \frac{\partial f}{\partial p_j}\Delta p_j \tag{36}$$

for the first order sensitivity and

$$\tilde{f} = f + \frac{\partial f}{\partial p_i}\Delta p_i + \frac{\partial f}{\partial p_j}\Delta p_j + \frac{1}{2}\left(\frac{\partial^2 f}{\partial p_i^2}\Delta p_i^2 + \frac{\partial^2 f}{\partial p_j^2}\Delta p_j^2 + 2\frac{\partial^2 f}{\partial p_i \partial p_j}\Delta p_i \Delta p_j\right) \tag{37}$$

for the second order sensitivity. f denotes the function calculated for the parameters p_i and p_j and $\tilde{f}$ the function after changing the parameters. For the analysis of the variability of one parameter, formulas (36) and (37) reduce to:

$$\tilde{f} = f + \frac{\partial f}{\partial p_i}\Delta p_i \tag{38}$$

for the first order sensitivity and

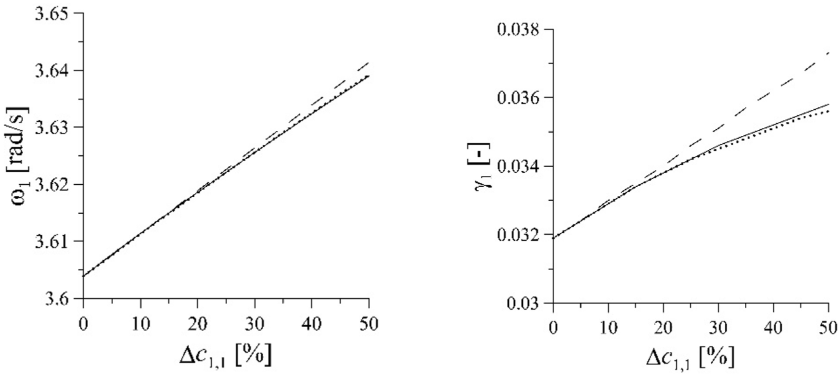
$$\tilde{f} = f + \frac{\partial f}{\partial p_i}\Delta p_i + \frac{1}{2}\frac{\partial^2 f}{\partial p_i^2}\Delta p_i^2 \tag{39}$$

for the second order sensitivity.

The results of the comparison are presented in Figs. 3 and 4. The influence of the variability of the damper parameters on the first natural frequency and nondimensional damping ratio was investigated. The below charts present a comparison of the exact results (solid line) and the results obtained from the first- (dashed line) and second-order (dotted line) sensitivity analysis. The comparisons suggest that the second-order sensitivity analysis allows predicting the correct value of dynamic characteristics even for changes in design parameters up to 50%.

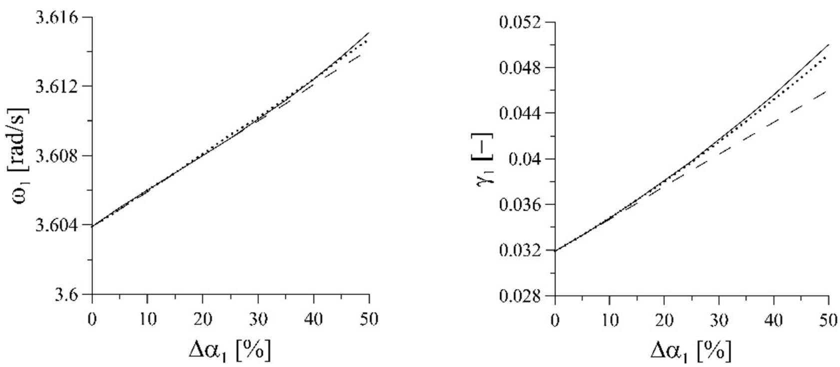
Then, the influence of the simultaneous change of two parameters, $c_{1,1}$ and α_1 was examined, and the exact results obtained by changing the values of the damper parameters were compared with those obtained from the first- and second-order sensitivity analyses. The results of selected changes are presented in Table 4.

The comparison proves that the sensitivity analysis gives good results when two parameters are changed simultaneously.



(a) (b)

Fig. 3. Comparison of the first natural frequency (a) and nondimensional damping ratio (b) with respect to the change of $c_{1,1}$



(a) (b)

Fig. 4. Comparison of the first natural frequency (a) and nondimensional damping ratio (b) with respect to the change of α_1

Table 4. Comparison of the first nondimensional damping ratio for the simultaneous change of two parameters

Results	Changes of $c_{1,1}$ and α_1		
	$\Delta c_{1,1}$ 30% $\Delta\alpha_1$ 10%	$\Delta c_{1,1}$ 30% $\Delta\alpha_1$ 20%	$\Delta c_{1,1}$ 30% $\Delta\alpha_1$ 30%
Exact	0.0378	0.0413	0.0452
First-order sensitivity	0.0379	0.0408	0.0436
Error [%]	0.26	1.21	3.54
Second-order sensitivity	0.0378	0.0414	0.0453
Error [%]	0.00	0.24	0.22

4.3. Three-layered composite beam

The third example is the analysis of a simply supported beam with a viscoelastic layer (Fig. 5). The bottom and top layers of the beam are assumed to be made of aluminum, and the core is made of a viscoelastic material described by the Zener model. The matrix $\mathbf{G}(s)$ is written in the following form:

$$\mathbf{G}(s) = \frac{\tau^\alpha s^\alpha}{1+\tau^\alpha s^\alpha}\mathbf{K}_\infty\,, \tag{40}$$

where τ and α are the parameters of the Zener model and $\mathbf{K}_\infty$ is a known matrix. The model has been previously described in detail [30]. The following formulas can be substituted in Eq. (28):

$$g(s) = \frac{\tau^\alpha s^\alpha}{1+\tau^\alpha s^\alpha}\,, \qquad \mathbf{K}_v = \mathbf{K}_\infty\,. \tag{41}$$

For elastic layers, the parameters assumed were as follows: $h_f = 0.001$ m, $\rho_f = 2690$ kg/m^3 and $E_f = 70.3$ GPa. For the viscoelastic layer, the following parameters were assumed: $h_c = 0.002$ m, $\rho_c = 1600$ kg/m^3, $E_0 = 1.5$ MPa, and $E_\infty = 69.9495$ MPa. The Poisson ratio was $\nu_c = 0.5$, and the parameters of the Zener material were the following: time relaxation $\tau = 1.4052 \cdot 10^{-5}$ s and fractional order $\alpha = 0.7915$. The length of the beam was 0.2 m, and the beam was divided into 10 elements.

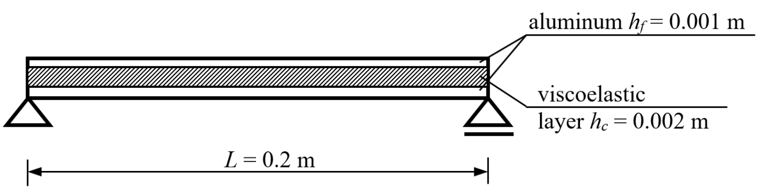


Fig. 5. Diagram of a composite beam

For the considered beam, the sensitivities of natural frequency and nondimensional damping ratio with respect to the change of time relaxation were determined for the first three modes. The obtained results are presented in Table 5.

Table 5. Sensitivities of natural frequency and nondimensional damping ratio with respect to the change of time relaxation

Sensitivities	Mode		
	1	2	3
$\partial\omega_i/\partial\tau$	$1.5251 \cdot 10^6$	-660.1300	$-1.1468 \cdot 10^3$
$\partial^2\omega_i/\partial\tau^2$	$7.7978 \cdot 10^9$	$3.6369 \cdot 10^3$	$4.4957 \cdot 10^3$
$\partial\gamma_i/\partial\tau$	$7.3608 \cdot 10^3$	-0.3805	-0.2318
$\partial^2\gamma_i/\partial\tau^2$	$-1.5802 \cdot 10^8$	-0.4372	-0.6142

The analysis of the sensitivity values revealed that the second-order sensitivity is significant in relation to the first-order sensitivity, and is even higher in some cases. This proves that the second-order sensitivity should be included in the calculations.

An advantage of the presented methods is that they allow simultaneous calculation of the sensitivity of eigenvalues and eigenvectors. The real and imaginary parts of the eigenvector elements corresponding to the vertical degrees of dynamic freedom are presented in Fig. 6. The exact values (solid line) were compared with those obtained for the first- (dashed line) and second-order sensitivity (dotted line). The change of the parameter α by 5% was analyzed.

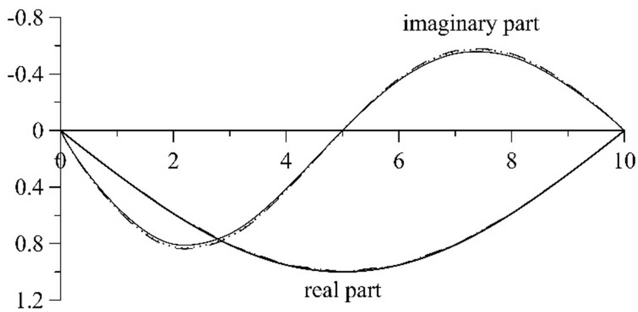


Fig. 6. Real and imaginary parts of the elements of eigenvector (the imaginary part is multiplied by 10^7)

The eigenvector elements are characterized by much smaller sensitivity than the previously analyzed dynamic characteristics. As a result, the compared values almost overlap on the graph, and no significant difference can be found between the results obtained using the first- and second-order sensitivity.

5. Computational efficiency

To compare the methods used for calculating sensitivity, the number of operations to be performed was estimated for the systems whose eigenproblem is described by formula (2). Although the presented comparison is only a certain estimate, it clearly proves the effectiveness of the described methods for the considered systems. Only multiplication/division was assumed for the comparison because the time taken for addition/subtraction is significantly shorter. The following notations were used for the analysis: n is the number of degrees of freedom, p is the number of parameters, and r is the number of viscoelastic elements. It was assumed that sensitivity is calculated with respect to parameters describing viscoelastic elements because they generate more operations than structure related parameters. It was also considered that some operations are performed using complex numbers.

The number of operations performed for the first-order sensitivity when both eigenvalue and eigenvector are calculated, and when only the eigenvalue is calculated, is listed in Table 6. For DDM, the procedure is the same for both cases. The left-hand side of the set of equations was omitted because it is the same for all methods.

A similar comparison was also made for the second-order sensitivity. A comparison of DDM, AVM, and HM is presented in Table 7. When the sensitivity

Table 6. Number of operations for sensitivity of the first order

Operation	Number of operations	Complexity
Sensitivity of s and $\mathbf{q}$ simultaneously, DDM		
Setting up a set of equations	$4np\,(rn + 2n + 1)$	$O\left(rpn^2\right)$
Solving a set of equations	$4p\left(2n^3 + 15n^2 + 19n + 6\right)/6$	$O\left(pn^3\right)$
Sensitivity of s and $\mathbf{q}$ simultaneously, AVM		
Solving a set of equations	$4(n+1)\left(2n^3 + 15n^2 + 19n + 6\right)/6$	$O\left(n^4\right)$
Additional equation	$4p\left(rn^2 + 3n^2 + 3n + 1\right)$	$O\left(rpn^2\right)$
Sensitivity of s, AVM		
Solving a set of equations	$4\left(2n^3 + 15n^2 + 19n + 6\right)/6$	$O\left(n^3\right)$
Additional equation	$4p\left(rn^2 + 2n^2 + 2n + 1\right)$	$O\left(rpn^2\right)$

Table 7. Number of operations for sensitivity of the second order

Operation	Number of operations	Complexity
Sensitivity of s and $\mathbf{q}$ simultaneously, DDM		
Setting up a set of equations	$4p\left(2rn^2 + 11n^2 + 3n + 1\right)$	$O\left(rpn^2\right)$
Solving a set of equations	$4p\left(2n^3 + 15n^2 + 19n + 6\right)/6$	$O\left(pn^3\right)$
Sensitivity of s and $\mathbf{q}$ simultaneously, AVM		
Setting up a set of equations	$4p\left(2n^3 + 12n^2 + 12n + rn^2 + 3\right)$	$O\left(pn^3\right)$
Solving a set of equations	$4(n+1)\left(2n^3 + 15n^2 + 19n + 6\right)/6$	$O\left(n^4\right)$
Additional equation	$2p\left(3rn^2 + 12n^2 + 18n + 8\right)$	$O\left(rpn^2\right)$
Sensitivity of s and $\mathbf{q}$ simultaneously, HM		
Additional equation	$4p\left(2rn^2 + 12n^2 + 5n + 3\right)$	$O\left(rpn^2\right)$
Sensitivity of s, AVM		
Setting up a set of equations	$p\left(24n^2 + 36n + 4rn^2 + 12\right)$	$O\left(pn^2\right)$
Solving a set of equations	$4p\left(2n^3 + 15n^2 + 19n + 6\right)/6$	$O\left(pn^3\right)$
Additional equation	$2p\left(3rn^2 + 6n^2 + 10n + 8\right)$	$O\left(rpn^2\right)$
Sensitivity of s, HM		
Additional equation	$4p\left(2rn^2 + 11n^2 + 4n + 3\right)$	$O\left(rpn^2\right)$

of the eigenvalue is of interest, for AVM and HM, it is also necessary to find the first-order sensitivity of the associated eigenvector. It was assumed that it would be calculated using the first-order AVM.

Figs. 7 and 8 present a comparison of the number of operations for $r = 3$, $n = 20$, and the number of parameters varying from 1 to 50 (solid line DDM, dashed line AVM, dotted line HM). For calculating the first-order sensitivity of the eigenvalue and the eigenvector, AVM is better for a greater number of parameters, while for the second-order sensitivity AVM analysis is completely ineffective, and HM is the best. If the sensitivity of only eigenvalues is to be calculated, AVM has a definite advantage in the case of the first-order analysis, while for the second-order analysis, HM is more advantageous, but only for a larger number of parameters.

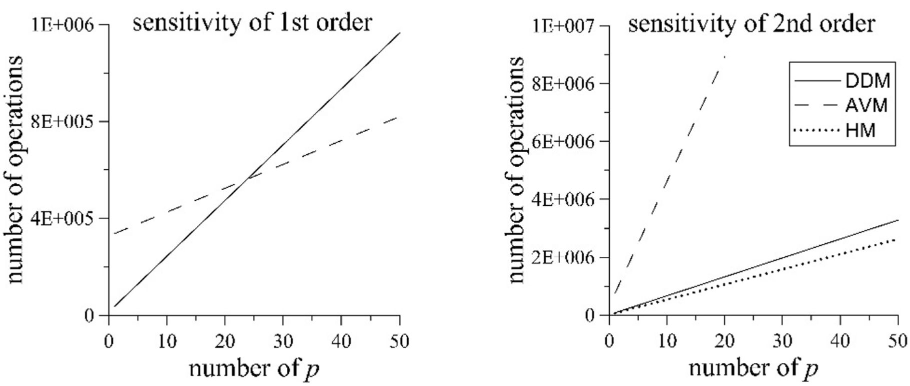


Fig. 7. Number of operations for sensitivity of eigenvalues and eigenvectors

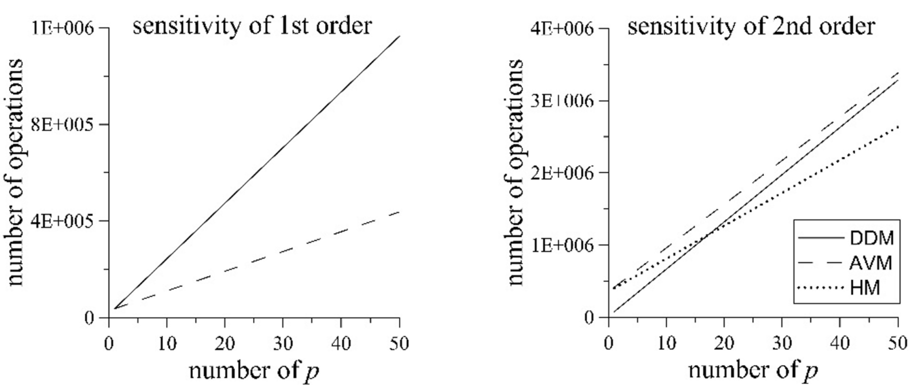


Fig. 8. Number of operations for sensitivity of eigenvalues

The cases where the degree of dynamic freedom was greater were also analyzed, and the conclusions obtained are similar.

6. Conclusions

The paper describes three methods that allow calculating the sensitivity of dynamic characteristics of systems with viscoelastic damping elements. A comparison of these methods with respect to their computational cost is also presented. The solutions obtained from these methods can be applied to systems with damping elements described by classical and fractional rheological models, which are illustrated with three examples. All three methods are simple to code and can be successfully used for various applications such as model updating, damage detection, and structural optimization.

The results for all three methods are the same, but the computational cost depends on:

I) number of degrees of freedom of the structure,
II) number of design parameters taken into account,
III) number of viscoelastic elements,
IV) calculations performed only for eigenvalues or for both eigenvalues and eigenvectors,
V) first-order sensitivity calculations or both first- and second-order sensitivity calculations.

In the case of the sensitivity of eigenvalues and eigenvectors, for first-order sensitivity, DDM is more advantageous with fewer design parameters. As the number of parameters increases, AVM becomes more useful. However, for the sensitivity of the second-order, regardless of the number of parameters, HM is the most advantageous, while AVM generates a very large number of operations.

In the case of calculating the sensitivity of eigenvalues, AVM is more favorable than DDM for first-order sensitivity, and its advantage increases with the increasing number of parameters. Whereas for the sensitivity of the second-order, for a small number of parameters, DDM is more favorable, while for a larger number of parameters, HM is more advantageous.

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