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On modelling and description of the output signal of a sampling device

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Abstract: The problem of an inconsistent description of an "interface" between the A/D converter and the digital signal processor that implements, for example, a digital filtering (described by a difference equation) – when a sequence of some hypothetical weighted Dirac deltas occurs at its input, instead of a sequence of numbers – is addressed in this paper. Digital signal processors work on numbers, and there is no "interface" element that converts Dirac deltas into numbers. The output of the A/D converter is directly connected to the input of the signal processor. Hence, a clear conclusion must follow that sampling devices do not generate Dirac deltas. Not the other way around. Furthermore, this fact has far-reaching implications in the spectral analysis of discrete signals, as discussed in other works referred to in this paper.

Key words: signal sampling, modelling of ideal or non-ideal sampling via averaging operation, Dirac and Kronecker deltas

1. Introduction

In the current model of signal sampling operations, the sampled signal (considered as a function of continuous time) is modelled as a sequence of weighted Dirac impulses. See, for example, such prominent references as [3–11]. Here, this description is confronted with the fact that sampling devices that produce Dirac deltas in fact do not exist.

A model developed in this paper takes into account the fact that because of physical and technological constraints carrying out the signal sampling at a zeroth time is not possible. The time for getting a sample is always greater than zero. Only in an abstract idealized case, it can be assumed to be equal to zero. Furthermore, the signal samples are then considered to be ideal.

The latter case is discussed here, and, as shown, it can be perceived as a boundary of the former. Moreover, both can be treated in a consistent manner.

A basic mathematical tool used in developing our model is a local averaging operation, applied also in [4, 9], and [19]. However, our results presented here differ substantially from those presented in these references. Why this is the case is explained in detail in this paper.

2. Modelling of ideal or non-ideal signal sampling via averaging

Consider a signal $x(t)$ of a continuous time t and the operation of its sampling. The latter, as it is known, cannot be performed in an ideal way because A/D converters that sample a signal "pointwise on the time axis" simply do not exist. Any real A/D converter needs a finite amount of time for each sample it produces in a sequence of signal values; let us denote it here as τ. Furthermore, a value of the parameter τ depends upon the design principles and electronics that are used in construction of an given A/D converter.

Now we would like to know how to model the above in a fairly simple way, while leaving out all the physical details?

To this end, note that a manner in which similar problems have been handled in physics [14, 18] can be helpful. That is by treating them as involving a local averaging of physical quantities; or, more generally, processing which can be described as an application of a local convolution. We note further that this approach is already present in the area of signal processing; see, for example, [4, 9, 19]. However, in these cited publications, it is not fully correctly applied. Explanation of this fact, as already mentioned in the introduction, is the subject of this paper.

Further, in light of what was said above, it is legitimate to treat the values that appear as outputs of A/D converters as "deformed" or "smeared" (with reference to the desired perfect samples). "Deformed" or "smeared" by a local signal averaging [9] or as a result of a local convolution [4]. (We note that the first can also be expressed as a convolution; see, for instance, [12].)

In other words, we speak about a non-ideal sampling and a sequence of non-ideal samples. Obviously, in theory, an abstract (idealized) case, in which $\tau = 0$, can be also considered. Then, we say that we have to do with an ideal sampling. So, the samples are called ideal in the latter case.

Observe further that the case of $\tau = 0$ can be perceived as a boundary instance, when no deformation (smearing) of the value of a signal sample occurs. And, as is used in this sense in the current theory of (ideal) signal sampling.

As already said in the introduction, a description of the non-ideal signal sampling that follows from modelling it by a signal averaging is used here. Therefore, the reader's attention is drawn to the many convincing arguments for doing so that can be found in the literature; see, for example, [12–14].

On the other hand, note that the signal averaging operation as described, for example, in [9] cannot be applied in the same way (that is directly) in the case of signal sampling. Then, it must be modified due to the smearing effect. This fact has already been pointed out in the literature in [12] and [13], in discussing the possibility of modelling any measurement process via signal sampling.

Further, in the context of the above remark, we note that a direct application of the signal averaging (or convolution) to the non-ideal signal sampling, as it has been done in [4, 9, 19], is not fully correct. We will try to explain why not in our step-by-step analysis that follows.

Let us start first with a remark that time for an A/D converter at its input and its output is the same, which is very significant. While we are speaking about a continuous time at the input and a discrete time at the output, but this regards exclusively elements of two time spaces (one regarded in the literature as a continuous time and the second as a discrete time). This does not absolutely mean that the physical time on both sides of an A/D converter is different.

Because we have to do with only one physical time space, there is rather something artificial in its treatment differently in the case of the signals of a continuous time, and differently when considering the so-called discrete-time signals.

We previously noted the inconsistency mentioned above in [20] as well as in [21]. However, many researchers working in the area of signal processing still believe that it does not matter. It has already been shown, in a couple of recently published papers (see, for example, [16] and [17]), that just the opposite holds. This article is the next in a series, which shows how the inconsistency mentioned above affects correctness of results – in case of solving "mixed problems" that involve signals taken from both the continuous and discrete time spaces. Here, it will be shown that the ideal signal sampling models presented in [4, 9, 19] are not fully correct. They need some improvements, which we develop in this paper.

In our approach, the signals considered in the literature separately as continuous and discrete time series, that is as if they came from "two different worlds" (or spaces): consisting of only continuous time (analog) signals and of only discrete-time signal waveforms, are regarded uniformly. In [20] and [21], a new signal notion has been introduced into the theory, and defined as a signal object. This concept means that every series (signal) of a continuous time together with its infinite number of discrete-time equivalents, where the discrete-time equivalent (of a given continuous time signal) stands for any of its sampled versions, which can be reconstructed perfectly using one of the known reconstruction formulas as, for instance, the Shannon's [2] or Vetterli's formula [9]. Obviously, by definition, the signal object possesses an infinite number of elements. Furthermore, all of them are functions of a continuous time variable t.

So, invoking the above two properties (expressed here in a descriptive way, so as not to complicate the development presented here), it is clear that the elements of the signal object are unique. Admittedly, they differ visually from each other, but that does not matter (in the sense defined above).

Finally, it must be strongly emphasized that in our approach the sampled signals (i.e. the discrete-time ones) are not simply sets of indexed values (samples), or something like "bags of indexed elements". They are always considered as functions of a continuous time variable t. Further, this is consistent with what was said above about the physical time at the input and output of an A/D converter.

Inspired by [14], modelling of the operation of averaging has been tailored in [12] to the needs of processing signals. And, this way is adopted here, however, in a slightly modified version. The modification regards the instant of "delivering" a "smeared" value of a sample. Namely, in modelling a measurement process, this instant is the moment of completion of the local signal averaging operation. But, unlike this, the result of a local signal averaging has to be "clipped" to the instant of the beginning of the averaging process in the non-ideal signal sampling. In other words, it is attributed to this instant – however – only virtually. Actually, this value is available at the output of an A/D converter for processing by a microprocessor a one sampling period, T, later.

For illustration, consider Fig. 1 below.

In Fig. 1, $x_{S,T}(t)$ means a "smeared" sampled signal in which every sample does not represent a single value, but it forms an impulse of width τ. So, it can be viewed as "a kind of smearing of a discrete sample value on an interval τ". And, these "smeared" impulses are "clipped" to the instants of virtual appearances of the ideal samples. Moreover, the smearing process mentioned is modelled here as a local signal averaging.

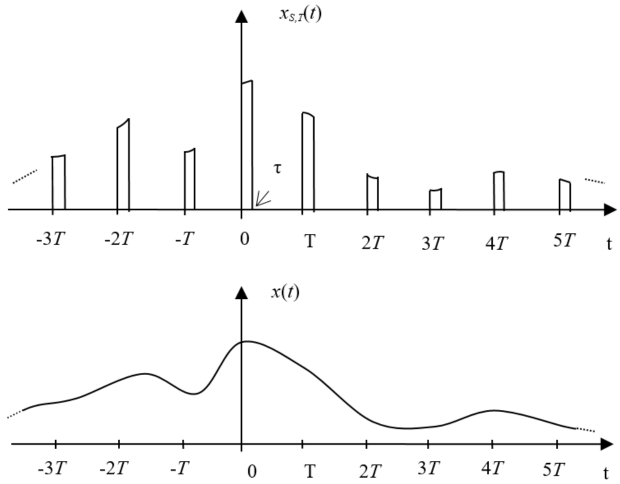

Fig. 1. Illustration of a not ideally sampled signal (upper curve) in form of a series of smeared samples (forming narrow impulses), an example signal used in this sampling is shown below (lower curve), figure taken from [1]

So, as a result of performing these two operations described above, it is obtained

$$x_{A,T}(t) = \mathrm{AV}\left(x_{S,T}(t)\right) = \sum_{k=-\infty}^{\infty} \mathrm{av}\left(\text{from } x_{S,T}(\lambda = kT)\right.$$
$$\left.\text{to } x_{S,T}(\lambda = kT + \tau)\right) \delta_{k,\frac{t}{T}}(t) = \sum_{k=-\infty}^{\infty} \overline{x}(kT)\delta_{k,\frac{t}{T}}(t), \tag{1}$$

where the symbol AV stands for an operator that transforms an infinite train of impulses as in Fig. 1 into an infinite train of single values as shown in Fig. 2 – according to these two rules given in a descriptive form above. The result of this operation is the signal $x_{A,T}(t)$. And, the next symbol, "small av", means performing a local averaging around an indicated "smeared" sample (that is on a given impulse of $x_{S,T}(t)$); the result of this operation is denoted here by $\overline{x}(kT)$. And finally, $\delta_{k,\frac{t}{T}}(t)$ in (2) means a time-shifted Kronecker time function [1, 15].

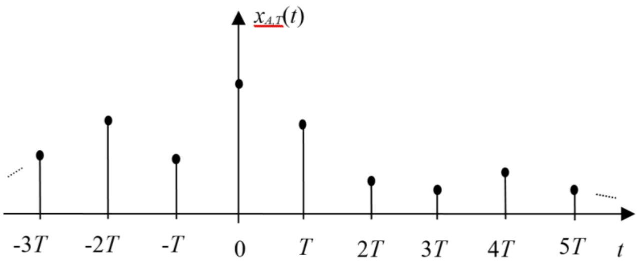

Fig. 2. Illustration to transformation of the signal $x_{S,T}(t)$ shown in Fig. 1 to the signal $x_{A,T}(t)$. Note that the lengths of "posts" in Fig. 2 are not equal to the values of the ideal signal samples $x(kT)$. They differ from them and equal the values of $\overline{x}(kT)$

Here, in order to model the local signal averaging, its description presented in detail in [12] and [14] is used. So, along these lines, the following can be written:

$$\overline{x}(kT) = \text{av}\,(\text{from } \ x(t = kT) \ \text{ to } \ x(t = kT + \tau)) = \int_{kT}^{kT+\tau} x(\lambda)a(kT + \tau - \lambda)\,\mathrm{d}\lambda, \tag{2}$$

where the function $a(t)$ is assumed to have the following form [12]:

$$a(t) = \frac{1}{\tau} \ \text{ for } \ 0 < t < \tau, \ \text{ and 0 elsewhere.} \tag{3}$$

Note further that because of a sifting character of the function $a(t)$ given by (3) (in the interval from 0 to τ) (2) can be rewritten as

$$\overline{x}(kT) = \int_{-\infty}^{\infty} x(\lambda)a(kT + \tau - \lambda)\,\mathrm{d}\lambda. \tag{4}$$

So, it can be concluded from (4) that the averaged (smeared sample) value $\overline{x}(kT)$ can be expressed as a convolution of the signal $x(t)$ with an impulse response $a(t + \tau)$, that is calculated for the instant kT. (For the needs of our further derivations, it can be assumed that this convolution exists for all $t \in \mathrm{R}$, where R denotes the set of real numbers.)

Now, let us check what happens when we approach the boundary case of $\tau = 0$ mentioned before. That is when $\tau \to 0$ in the model describing Formulas: (1) to (4), what means, as said above, that we approach then the case of an ideal sampling.

Note that using arguments presented, for example, in [14] for the case of $\tau \to 0$, it can be written $a(t) \to \delta(t)$, where $\delta(t)$ means a Dirac delta (called also a Dirac impulse or distribution). So, applying this in (4) leads to

$$\overline{x}(kT) \to x(kT) = \int_{-\infty}^{\infty} x(\lambda)\delta(kT - \lambda)\,\mathrm{d}\lambda = \int_{-\infty}^{\infty} x(kT)\delta(kT - \lambda)\,\mathrm{d}\lambda. \tag{5}$$

Accordingly, (1) reduces for the case of the ideal sampling to

$$x_{A,T}(t) \rightarrow x_{I,T}(t) = \sum_{k=-\infty}^{\infty} x(kT)\delta_{k,\frac{t}{T}}(t), \tag{6}$$

where $x_{I,T}(t)$ means the corresponding version of $x_{A,T}(t)$ (just for the ideal sampling).

3. Discussion

Let us summarize here findings and draw further conclusions from the developments of the previous section. Note that they are of a general nature and apply to all areas, where the digital processing is performed with the use of devices such as signal micpprocessors (as, for example, in telecommunications), or with the use of microcontrollers (which are used, for instance, in digital control of electrical power flows; see example applications discussed in [23, 24]).

First of all, let us recall once again that the signal finite value samples (that appear sequentially at the output of an A/D converter) form a signal of a continuous time [25–31]. They cannot be treated simply as a "bag of indexed elements"; for example, when it is tried to calculate its spectrum. And, there exists in signal processing a vehicle which enables performing this task. It is based on representing a sequence of signal finite value samples on the continuous time axis as a series of Dirac impulses multiplied by the corresponding sample values. Subsequently, one calculates the spectrum of the latter series treated as a function of a continuous time.

As known, there is a consensus in the signal processing community that the above means of modelling is correct [32–43]. However, there is something odd about it. Namely that the A/D signal converter is assumed to produce a continuous time signal (at its output or inside) in the form of a series of weighted Dirac impulses – which is not the case. Such A/D signal converters that produce series of weighted Dirac impulses do not and cannot exist.

We consulted, regarding the above issue with practitioners, and designers of A/D converters. They basically confirmed the signals at the outputs of the A/D converters (as functions of a continuous time) have a form of the signal $x_{S,T}(t)$ illustrated in Fig. 1 (upper curve) or a form of the signal $x_{A,T}(t)$ shown in Fig. 2. Whereby the first form is rather very rarely exploited in signal processing. (Calculation of its spectrum does not pose any problems – as shown in [1].) Primarily, in digital signal processing applying microprocessors results in the second case mentioned above. (And here, unlike in the first case, calculation of its spectrum is problematic; see, for example, [22].) Further, the opinion of these A/D practitioners is that neither of these two types of signals mentioned above can be perceived as sequences of weighted Dirac impulses.

Note now that our model fully reflects the above observations. And, in particular, those regarding the case we are most interested in. That is the "digital" one described by (1) (which describes a non-ideal "digital" sampling) or (6) (which refers to the ideal version of the latter, which can be called a "digital" ideal sampling). We also note that the non-ideal and ideal "digital" samplings in our model differ only in the values of samples. These are the $\overline{x}(kT)'s$ in the first case and $x(kT)'s$ in the second. And, obviously, they essentially differ from a modelling approach which uses a sampled signal description in form of a sequence of short impulses, as illustrated in Fig. 1 (upper curve) and discussed in detail in [1].

Finally here, let us explain (illustratively) the nature of the error which is committed in modelling the ideally sampled signal by

$$x_{D,T}(t) = \sum_{k=-\infty}^{\infty} x(t)\delta(t - kT) = \sum_{k=-\infty}^{\infty} x(kT)\delta(t - kT), \tag{7}$$

where $x_{D,T}(t)$ stands for a sampled signal (considered as a function of a continuous time) at the A/D output, which is modelled as a sequence of weighted time-shifted Dirac deltas [3–11]. To this end, rewrite (6) with $x(kT)'s$ replaced with the use of (5). As a result, one obtains then

$$x_{I,T}(t) = \sum_{k=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(kT)\delta(kT - \lambda)\,\mathrm{d}\lambda \right) \cdot \delta_{k,\frac{t}{T}}(t). \tag{8}$$

In the next step, see that introducing the function $\delta_{k,\frac{t}{T}}(t)$ in the sum on the right-hand side of (7) does not change values of the function $x_{D,T}(t)$. So, equivalently, (7) can be rewritten as

$$x_{D,T}(t) = \sum_{k=-\infty}^{\infty} \big(x(kT)\delta(t - kT)\big) \cdot \delta_{k,\frac{t}{T}}(t). \tag{9}$$

Comparison now of (9) with (8) shows that in fact the expressions

$$x(kT)\delta(t - kT)$$

in (9) are replaced by the expressions

$$\int_{-\infty}^{\infty} x(kT)\delta(kT - \lambda)\,\mathrm{d}\lambda = x(kT)$$

in (8).

However, such a replacement does not lead to the same result because

$$x(kT)\delta(\cdot) \neq x(kT).$$

So, simply, the function $x_{D,T}(t)$ assumed in the literature a priori to model the sampled signal is not correct.

It has been shown in this paper that the discrepancy between the real signals output from A/D converters and their modelling as weighted Dirac impulses can be significant, and cannot be neglected from a scientific as well as a didactic point of view. So, certainly, this issue needs a re-discussion.

References

[1] Borys A., *Spectrum aliasing does occur only in case of non-ideal signal sampling*, International Journal of Electronics and Telecommunications, vol. 67, no. 1, pp. 79–85 (2021), DOI: 10.24425/ijet.2021.135947.

[2] Shannon C.E., *Communications in the presence of noise*, Proceedings of the IRE, vol. 37, pp. 10–21 (1949).

[3] Oppenheim A.V., Schafer R.W., Buck J.R., *Discrete-Time Signal Processing*, New Jersey: Prentice Hall (1998).

[4] Marks R.J. II, *Introduction to Shannon Sampling and Interpolation Theory*, New York: Springer-Verlag (1991).

[5] Bracewell R.N., *The Fourier Transform and Its Applications*, New York: McGraw-Hill (2000).

[6] McClellan J.H., Schafer R., Yoder M., *DSP First*, London, England: Pearson (2015).

[7] Brigola R., *Fourier-Analysis und Distributionen*, Hamburg: edition swk (2013).

[8] So H.C., *Signals and Systems – Lecture 6 – Discrete-Time Fourier Transform*, www.ee.cityu.edu.hk/~hcso/ee3210.html, accessed December 2019.

[9] Vetterli M., Kovacevic J., Goyal V.K., *Foundations of Signal Processing*, Cambridge, England: Cambridge University Press (2014).

[10] Wang R., *Introduction to Orthogonal Transforms with Applications in Data Processing and Analysis*, Cambridge, England: Cambridge University Press (2010).

[11] Ingle V.K., Proakis J.G., *Digital Signal Processing Using Matlab*, Stamford, CT, USA: Cengage Learning (2012).

[12] Borys A., *Further discussion on modeling of measuring process via sampling of signals*, International Journal of Electronics and Telecommunications, vol. 66, no. 3, pp. 507–513 (2020), DOI: 10.24425-ijet.2020.134006.

[13] Borys A., *Measuring process via sampling of signals, and functions with attributes*, International Journal of Electronics and Telecommunications, vol. 66, no. 2, pp. 309–314 (2020), DOI: 10.24425-ijet.2020.131879.

[14] Strichartz R., *A Guide to Distribution Theory and Fourier Transforms*, Boca Raton, USA: CRC Press (1994).

[15] Borys A., *Extended definitions of spectrum of a sampled signal*, International Journal of Electronics and Telecommunications, vol. 67, no. 3, pp. 395–401 (2021), DOI: 10.24425-ijet.2021.137825.

[16] Borys A., *Definition of sampled signal spectrum and Shannon's proof of reconstruction formula*, submitted for publication in the International Journal on Marine Navigation and Safety of Sea Transportation.

[17] Borys A., *On derivation of discrete time Fourier transform from its continuous counterpart*, Intwenational Journal of Electronics and Telecommunications, vol. 66, no. 2, pp. 355–368 (2020), DOI: 10.24425/ijet.2020.13188.

[18] Peters K.H., *Schönheit, Exaktheit, Wahrheit: Der Zusammenhang von Mathematik und Physik am Beispiel der Geschichte der Distributionen*, Diepholz, Germany: GNT-Verlag (2004).

[19] Vetterli M., Marziliano P., Blu T., *Sampling signals with finite rate of innovation*, IEEE Transactions on Signal Processing, vol. 50, no. 6, pp. 1417–1428 (2002), DOI: 10.1109/TSP.2002.1003065.

[20] Borys A., *Some topological aspects of sampling theorem and reconstruction formula*, International Journal of Electronics and Telecommunications, vol. 66, no. 2, pp. 301–307 (2020), DOI: 10.24425/ijet.2020.131878.

[21] Borys A., *Filtering property of signal sampling in general and under-sampling as a specific operation of filtering connected with signal shaping at the same time*, International Journal of Electronics and Telecommunications, vol. 66, no. 3, pp. 589–594 (2020), DOI: 10.24425-ijet.2020.134016.

[22] Borys A., *Spectrum aliasing does not occur in case of ideal signal sampling*, International Journal of Electronics and Telecommunications, vol. 67, no. 1, pp. 71–77 (2021), DOI: 10.24425/ijet.2021.135946.

[23] Dewantoro G., Rizky I., Murtianta B., *Frequency response analysis of microcontroller-based discrete-time lead-lag compensators*, Archives of Electrical Engineering, vol. 69, no. 4, pp. 937−950 (2020).

[24] Sozański K., *Digital Signal Processing in Power Electronics Control Circuits*, London: Springer-Verlag (2013).

[25] Wepman J.A., *Analog-to-digital converters and their applications in radio receivers*, IEEE Commun. Mag., vol. 33, no. 5, pp. 39–45 (1995).

[26] Walden R.H., *Analog-to-digital converter survey and analysis*, IEEE Journal on Selected Areas in Communications, vol. 17, no. 4, pp. 539–550 (1999).

[27] Kinniment D., Yakovlev A., Gao B., *Synchronous and asynchronous A-D conversion*, IEEE Transactions on VLSI Systems, vol. 8, no. 2, pp. 217–220 (2000), DOI: 10.1109/92.831441.

[28] Robert J., Temes G.C., Valencic V., Dessoulavy R., Deval P., *A 16-bit low-voltage CMOS A/D converter*, IEEE Journal of Solid-State Circuits, vol. 22, no. 2, pp. 157–163 (1987).

[29] Low T.S., Bi C., *Design of A/D converters with hierarchic networks*, IEEE Transactions on Industrial Electronics, vol. 43, no. 1, pp. 184–191 (1996).

[30] Miller M.R., Petrie C.S., *A multibit sigma-delta ADC for multimode receivers*, IEEE Journal Solid-State Circuits, vol. 38, no. 3, pp. 475–482 (2003), DOI: 10.1109/JSSC.2002.808321.

[31] Gulati K., Lee H.S., *A low-power reconfigurable analog-to-digital converter*, IEEE Journal Solid-State Circuits, vol. 36, no. 12, pp. 1900–1911 (2001), DOI: 10.1109/4.972140.

[32] Jerri A.J., *The Shannon sampling theorem-its various extensions and applications: A tutorial review*, Proc. IEEE, vol. 65, no. 11, pp. 1565–1597 (1977).

[33] Brown J.L., *On the error in reconstructing a nonbandlimited function by means of the band pass sampling theorem*, J. Math. Anal. Appl., vol. 18, pp. 75–84 (1967).

[34] Brown J.L., *Multi-channel sampling of lowpass-pass signals*, IEEE Trans. Circuits Syst., vol. 28, no. 2, pp. 101–106 (1981).

[35] Papoulis A., *Generalized sampling expansion*, IEEE Trans. Circuits Syst., vol. 24, no. 11, pp. 652−654 (1977).

[36] Unser M., Aldroubi A., *A general sampling theory for non-ideal acquisition devices*, IEEE Trans. Signal Processing, vol. 42, no. 11, pp. 2915–2925 (1994).

[37] Unser M., Zerubia J., *Generalized sampling: Stability and performance analysis*, IEEE Trans. Signal Processing, vol. 45, no. 12, pp. 2941–2950 (1997).

[38] Unser M., Zerubia J., *A generalized sampling without bandlimiting constraints*, IEEE Trans. Circuits Syst., vol. 45, no. 8, pp. 959–969 (1998).

[39] Xia G., Zhang Z., *On sampling theorem, wavelets, and wavelet transforms*, IEEE Trans. Signal Processing, vol. 41, no. 12, pp. 3524–3535 (1993).

[40] Djokovic I., Vaidyanathan P.P., *Generalized sampling theorems in multiresolution subspaces*, IEEE Trans. Signal Processing, vol. 45, no. 3, pp. 583–599 (1997).

[41] Cheung K.F., Marks R.J II., *Ill-posed sampling theorems*, IEEE Trans. Circuits Syst., vol. 32, no. 5, pp. 481–484 (1985).

[42] Butzer P.L., *A survey of the Whittaker–Shannon sampling theorem and some of its extensions*, J. Math. Res. Exposit., vol. 3, pp. 185–212 (1983).

[43] Butzer P.L., Stens R.L., *Sampling theory for not-necessarily band-limited functions: A historical overview*, SIAM Rev., vol. 34, pp. 40–53 (1992).