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Extremal problems for second order hyperbolic systems involving multiple time delays

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Extremal problems for multiple time delay hyperbolic systems are presented. The optimal boundary control problems for hyperbolic systems in which multiple time delays appear both in the state equations and in the Neumann boundary conditions are solved. The time horizon is fixed. Making use of Dubovicki-Milutin scheme, necessary and sufficient conditions of optimality for the Neumann problem with the quadratic performance functionals and constrained control are derived.

Key words: optimal control theory, boundary control, second order hyperbolic systems, multiple time delays, electric long lines

1. Introduction

It is known by now that the Pontryagin maximum principle, the Bellman dynamic programming method and the Kalman optimal linear regulator theory are three milestones of modern (finite dimensional) optimal control theory. The study of optimal control theory for infinite dimensional systems can be tracked back to the begininng of the 1960s.

A main goal of such a theory is to establish the infinite dimensional version of the above-mentioned three fundamental theories. Consequently, many mathematicians and control theorities have made great contributions in this research area. They have been involved in the study of optimal control theory for infinite dimensional systems.

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Compared with the works of other mathematicians, they found that ours have their own flavor, and some of the methods might interest other people working in this area or in some related fields.

A unified presentation of optimal control theory for infinite dimensional systems is presented.

This includes the existence theory, the necessary conditions (Pontryagin type maximum principle) the dynamic programming method (involving the viscosity solution infinite dimensional Hamilton-Jacobi-Bellman equations) and the linear-quadratic optimal control problems.

Extremal problems are now playing an ever-increasing role in applications of mathematical control theory. It has been discovered that notwithstanding the great diversity of these problems, they can be approached by a unified functional-analytic approach, first suggested by Dubovicki and Milutin. The general theory of extremal problems has been developed so intensely recently that its basic concepts may now be considered complete.

Igor V. Girsanov, was one of the first mathematicians to study general extremum problems and to realize the feasibility and desirability of a unified theory of extremal problems, based on a functional–analytic approach.

His book [5] was apparently the first systematic exposition of a unified approach to the theory of extremal problems. This approach was based on the ideas of Dubovicki and Milutin concerning extremum problems in the presence of constraints. Dubovicki and Milutin found a necessary condition for an extremum in the form of an equation set down in the language of functional analysis.

For instance, in the paper [6], the Dubovicki-Milutin method was applied for solving optimal control problems for parabolic-hyperbolic systems. The existence and uniqueness of solutions of such parabolic-hyperbolic systems with the Dirichlet boundary conditions are discussed. Making use of the Dubovicki-Milutin method necessary and sufficient conditions of optimality for the Dirichlet problem with the quadratic performance functional and constrained control are derived.

In the papers [11–16], the Dubovicki-Milutin method was applied for solving boundary optimal control problems for the case of time lag parabolic equations [11] and for the case of parabolic equations involving time-varying lags [12–14], multiple time-varying lags [15], and integral time lags [16] respectively. Sufficient conditions for the existence of a unique solution of such parabolic equations [11–16] are presented.

Consequently, in the papers [11–14, 16], the linear quadratic problems of parabolic systems with time lags given in various forms (constant time lags [11], time-varying lags [12–14], multiple time-varying lags [15], integral time lags [16] etc.) were solved.

In the paper [17] the linear quadratic problems of optimal boundary control for hyperbolic systems in which constant time lags appear both in the state equations and in the Neumann boundary conditions are solved.

Sufficient conditions for the existence of a unique solution of such hyperbolic equations with the Neumann boundary conditions involving constant time delays are presented. Making use of Milutin-Dubovicki method [13], necessary and sufficient conditions of optimality with the quadratic cost functions and constrained boundary control are derived for the Neumann problem.

2. Purpose and scope of the research

Distributed parameter systems with delays can be used to describe many phenomena in the real world. As is well known, heat conduction, properties of elastic-plastic material, fluid dynamics, convection-reaction processes, diffusion-reaction processes, transmission of the signals at the certain distance by using electric long lines, etc., all lie within this area. The object that we are studying (temperature, displacement, concentration, velocity, etc.) is usually referred to as the state.

We are interested in the case where the state satisfies proper differential equations that are derived from basic physical laws, such as Newton's law, Fourier's law etc. The space in which the state exists is called the state space, and the equation that the state satisfies is called the state equation. In particular, we are interested in the cases where the state equations are one of the following types: partial differential equations, integro-differential equations, or abstract evolution equations.

For example, control and robustness problems of quasi-linear first-order hyperbolic partial differential equations (PDEs) systems including nonlinear controller design problems have been investigated in [1,2]. These equations constitute mathematical models of many convection-reaction processes. The distinct feature of hyperbolic PDEs is that all the eigenmodes of the spatial differential operator contain the same amount of energy, and thus an infinite number of modes is required to accurately describe their dynamic behavior. Therefore, this feature suggests addresing the control problem on the basis of the infinite-dimensional model itself. Following this framework, control methods were recently proposed for the synthesis of nonlinear distributed feedback controllers for quasi-linear hyperbolic PDEs utilizing geometric control [1] and Lyapunov-based control [2]. In turn, the distributed output feedback control problem of two-time-scale hyperbolic partial differential equations systems has been considered in [3]. Such systems constitute mathematical models of representative convection-reaction processes with time-scale multiplicity e.g. fixed-bed reactors, pressure swing absorption processes, etc. The objective is to synthesize distributed output feedback controllers that guarantee closed-loop stability and enforce output tracking, provided that the speed ratio of the fast versus the slow dynamical phenomena of the two-time scale system is sufficiently large. Initially, singular perturbation methods are used to derive two separate PDE models which describe the fast and slow dynamics of the original system. These models are then used as a basis for the synthesis of well-conditioned distributed state feedback controllers that guarantee stability and enforce output tracking. Then, two distributed state observers which incorporate well-conditioned observer gains are designed to prove estimates of the fast and slow states of the system. These state observers are coupled with the distributed state feedback controllers to yield distributed output feedback controllers that enforce the desired objectives in the closed-loop system.

Extremal problems for multiple time lag hyperbolic systems are investigated. The purpose of this paper is to show the use of Dubovicki-Milutin theorem [13] in solving optimal control problems for hyperbolic systems.

As an example, an optimal boundary control problem for a system described by a linear partial differential equation of hyperbolic type in which different multiple time delays appear both in the state equation and in the Neumann boundary condition is considered.

Such equations constitute, in a linear approximation, a universal mathematical model for many processes in which transmission signals at a certain distance with electric, hydraulic and other long lines take place.

In the processes mentioned above time–delayed feedback signals are introduced at the boundary of a system's spatial domain. Then the signal at the boundary of a system's spatial domain at any time depends on the signal emitted earlier. This leads to the boundary conditions involving time delays.

Sufficient conditions for the existence of a unique solution of such hyperbolic equation with the Neumann boundary condition are presented.

The performance functionals have the quadratic form. The time horizon is fixed. Finally, we impose some constraints on the boundary control. Making use of the Dubovicki-Milutin theorem [13], necessary and sufficient conditions of optimality with the quadratic performance functionals and constrained control are derived for the Neumann problem.

3. Research methodology

3.1. The Dubovicki-Milutin method

The Dubovicki-Milutin theorem [13] arises from the geometric form of the Hahn-Banach theorem (a theorem about the separation of convex sets).

We shall show it on the example.

Let us assume that
E – a linear topological space, locally convex,
$I(x)$ – a functional defined on E,
A_i, i=$1,2,\ldots,n$ – sets in E with inner points which represent inequality constraints,
B – a set in E without inner points representing equality constraint.

We must find some conditions necessary for a local minimum of the functional $I(x)$ on the set $Q = \bigcap\limits_{i=1}^{n} A_i \cap B$, or find a point $x_0 \in E$, so that $I(x_0) = \min\limits_{Q \cap U} I(x)$, where U means a certain environment of the point x_0.

We define the set $A_0 = \{x\colon\ I(x) < I(x_0)\}$.

Then we formulate the necessary condition of optimality as follows: in the environment of the local minimum point, the intersection of system of sets (the set on which the functional attains smaller values than $I(x_0)$ and the sets representing constraints) is empty or $\bigcap\limits_{i=0}^{n} A_i \cap B = \emptyset$.

The condition $\bigcap\limits_{i=0}^{n} A_i \cap B = \emptyset$ is also equivalent to the one in which there are approximations of the sets $A_i, i = 1,2,\ldots,n$ and B instead of A_i or B ones. These approximations are cones with the vertex in a point $\{0\}$.

We shall approximate the inequality constraints by the regular admissible cones $RAC(A_i, x_0) (i = 1,2,\ldots,n)$, the equality constraint by the regular tangent cone $RTC(B, x_0)$ and for the performance functional we shall construct the regular improvement cone $RFC(I, x_0)$.

Then we have the necessary condition of the optimality $I(x)$ on the set $Q = \bigcap\limits_{i=1}^{n} A_i \cap B$ in the form of Euler-Langrange's equation $\sum\limits_{i=1}^{n+1} f_i = 0$, where f_i $(i = 1,2,\ldots,n+1)$ are linear, continuous functionals, all of them are not equal to zero at the same time and they belong to the adjoint cones

$$f_i \in [RAC(A_i, x_0)]^*, \quad i = 1,2,\ldots,n,$$

$$f_{n+1} \in [RTC(B, x_0)]^*, \quad f_0 \in [RFC(I, x_0)]^*,$$

$$\{f_i \in [RAC(A_i, x_0)]^* \ \Leftrightarrow\ f_i(x) \geqslant 0 \ \ \forall x \in RAC(A_i, x_0)\}.$$

For convex problems, i.e. problems in which the constraints are convex sets and the performance functional is convex, the Euler-Lagrange equation is the necessary and sufficient condition of optimality if only certain additional assumptions are fulfilled (the so-called Slater's condition).

3.2. Transposition method

Let us consider the following linear hyperbolic equation

$$\frac{\partial^2 y}{\partial t^2} + A(t)y = u, \qquad x \in \Omega, \quad t \in (0,T), \tag{1}$$

$$y(x,0) = y_0(x), \qquad x \in \Omega, \tag{2}$$

$$y'(x,0) = y_I(x), \qquad x \in \Omega, \tag{3}$$

$$\frac{\partial y}{\partial \eta_A}(x,t) = q, \qquad (x,t) \in \Gamma \times (0,T), \tag{4}$$

where: $\Omega \subset R^n$ is a bounded, open set with boundary Γ, which is a C^∞-manifold of dimension $(n-1)$. Locally, Ω is totally on one side of Γ.

$$Q = \Omega \times (0,T), \qquad \bar{Q} = \bar{\Omega} \times [0,T], \qquad \Sigma = \Gamma \times (0,T).$$

The operator $A(t)$ is given by

$$A(t)y = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial y(x,t)}{\partial x_j}\right) \tag{5}$$

and the coefficients $a_{ij}(x,t)$ are real C^∞ functions on $\bar{Q}$ (closure of Q) satisfying the following condition

$$\left.\begin{array}{l} \displaystyle\sum_{i,j=1}^{n} a_{ij}(x,t)\Phi_i\Phi_j \geqslant \nu \sum_{i=1}^{n} \Phi_i^2 \quad \nu > 0, \quad \forall (x,t) \in \bar{Q}, \ \forall \Phi_i \in R \\ \qquad\qquad a_{ij} = a_{ji} \quad \forall i,j \end{array}\right\}. \tag{6}$$

The equations (1)–(4) constitute a Neumann problem. The left hand side of (1) is written in the following form.

$$\frac{\partial y}{\partial \eta_A} = \sum_{i,j=1}^{n} a_{i,j}(x,t)\cos(n,x_i)\frac{\partial y(x,t)}{\partial x_j} = q(x,t), \tag{7}$$

where: $\dfrac{\partial y}{\partial \eta_A}$ is a deriviative at Γ, directed towards the exterior of Ω, $\cos(n,x_i)$ is i-th direction cosine of n and n being the normal at Γ exterior to Ω.

Let us consider linear hyperbolic equations (1)–(4)

We shall prove the existence of a unique solution of the mixed initial-boundary value problem (1)–(4) defined by transposition, see [20, Vol. 2, pp. 105–108 and 130–133], i.e.

$$\langle y, \omega'' + A\omega \rangle = L(\omega) \qquad \forall \omega \in X^1(Q), \tag{8}$$

where

$$L(\omega) = \langle u, \omega\rangle + \langle q, \omega\rangle + \langle y_I, \omega(0)\rangle - \langle y_0, \omega'(0)\rangle \tag{9}$$

and we denote by $X^1(Q)$ the space described by the solutions ω of the following adjoint problem

$$\begin{aligned} \omega'' + A\omega &= \Phi, \quad x \in \Omega,\ t \in (0,T), \\ \omega(x,T) &= 0, \quad x \in \Omega, \\ \omega'(x,T) &= 0, \quad x \in \Omega, \\ \frac{\partial \omega}{\partial \eta_A} &= 0, \quad x \in \Gamma,\ t \in (0,T), \end{aligned} \tag{10}$$

where

$$\Phi \in H^{1,2}_{0,0}(Q) = \text{closure of } D(Q) \text{ in } H^{1,2}(Q). \tag{11}$$

Some properties and central theorems for the spaces $D(Q)$ and $H^{r,s}(Q)$ are given in [9, 10] and [20].

Let us define by

$$X^1(Q) = \text{space described by } \omega \text{ as } \Phi \text{ describes } H^{1,2}_{0,0}(Q) \tag{12}$$

and then ($X^1(Q)$ being provided with the “translated” topology) we have (done what was necessary to have):

$$\omega \longrightarrow \omega'' + A\omega \ \text{ is an isomorphism of } \ X^1(Q) \longrightarrow H^{1,2}_{0,0}(Q). \tag{13}$$

Consequently, using the results of Chapter 4, Section 8.3 of ([20]: Vol. 2, p. 41), we may define the dual space of $H^{1,2}_{0,0}(Q)$. We denote this dual space by $H^{-1,-2}(Q)$, i.e. we set

$$\left(H^{1,2}_{0,0}(Q)\right)' = H^{-1,-2}(Q). \tag{14}$$

Now, we choose in (9) u, q, y_i to be (suitable) distributions on Q, Σ and Ω.

Thus, we shall follow a procedure analogous to Chapter 5, Section 10, of ([20]: Vol. 2, pp. 130–132). Moreover, Q and Σ have the same properties as in the problem (1)–(4).

3.2.1. Choice of u

According to Theorem 7.1 of ([20]: Vol. 2, p. 122), we have

$$X^1(Q) \subset H^{3,3}(Q). \tag{15}$$

Subsequently, we introduce as in Chapter 4, Section 9, of ([20]: Vol. 2, p. 43) the space

$$\Xi^{3,3}(Q) = \left\{\omega \middle| d^j(t)\omega^{(j)} \in L^2(0,T;\Xi^{3-j}(\Omega)),\ 0 \leqslant j \leqslant 3\right\}, \tag{16}$$

where $d(t)$ be a fixed infinitely differentiable function on $[0, T]$ such that

$$d(t) = \begin{cases} t & \text{if } \; t \leqslant t_0, \\ T - t & \text{if } \; T - t_0 \leqslant t \leqslant T, \end{cases} \tag{17}$$

t_0 fixed with $0 < t_0 < T - t_0$, and consequently, using the results of ([20]: Vol.1, pp. 170-173) we may define the following space

$$\Xi^{\mu}(\Omega) = \left\{ \omega | \; \rho^{|\alpha|} D^{\alpha} \omega \in L^2(\Omega), \; |\alpha| \leqslant \mu \right\} \tag{18}$$

which is a Hilbert space with the norm

$$\|\omega\|_{\Xi^{\mu}(\Omega)} = \left(\sum_{|\alpha| \leqslant \mu} \|\rho^{|\alpha|} D^{\alpha} \omega\|^2_{L^2(\Omega)} \right)^{1/2}, \tag{19}$$

where integer $\mu \geqslant 1$, and the function ρ is infinitely differentiable on $\bar{\Omega}$, positive on Ω, vanishing on Γ of the order of $d(x, \Gamma)$ (= distance from x to Γ), i.e. such that

$$\lim_{x \to x_0} \frac{\rho(x)}{d(x, \Gamma)} = d \neq 0 \quad \text{if } \; x_0 \in \Gamma, \tag{20}$$

such functions do exist, since Γ is an infinitely differentiable variety. Moreover,

$$D^{\alpha} = \frac{\partial^{\alpha_1 + \ldots + \alpha_n}}{\partial x_1^{\alpha_1} + \ldots + \partial x_n^{\alpha_n}}, \quad \alpha = \{\alpha_1, \ldots, \alpha_n\}, \quad |\alpha| = \alpha_1 + \ldots + \alpha_n \,. \tag{21}$$

Moreover, we have

$$\Xi^0(\Omega) = L^2(\Omega), \quad H^{\mu}(\Omega) \subset \Xi^{\mu}(\Omega) \subset L^2(\Omega). \tag{22}$$

Then, using the Proposition 9.2 of ([20]: Vol. 2, p. 45), we have

$$H^{3,3}(Q) \subset \Xi^{3,3}(Q). \tag{23}$$

Remark 1 *In the case when μ is a non-integer, we define the space $\Xi^{\mu}(\Omega)$ by interpolation. Consequently, let real $\mu > 0$ not be an integer, $\mu = k + \Theta$, with integer $k \geqslant 0$ and $0 < \Theta < 1$; we set*

$$\Xi^{\mu}(\Omega) = \left[\Xi^{k+1}(\Omega), \; \Xi^{k}(\Omega) \right]_{1-\Theta} \,. \tag{24}$$

From this definition, (22) and the inclusion properties of $H^{\mu}(\Omega)$, there results that

$$H^{\mu}(\Omega) \subset \Xi^{\mu}(\Omega) \subset \Xi^{\mu'}(\Omega) \subset L^2(\Omega), \tag{25}$$

where μ, μ' are real and > 0, $\mu' < \mu$.

From the density theorem and Proposition 6.2 of ([20]: Vol. 1, p. 11 and p. 171), we also deduce that $D(\Omega)$ is dense in $\Xi^{\mu}(\Omega)$ for all real $\mu \geqslant 0$.

Therefore, we see that $\Xi^{\mu}(\Omega)$ is a normal space of distributions on Ω.

Its dual space may be identified to a space of distributions on Ω. We denote this dual space by $\Xi^{-\mu}(\Omega)$, i.e. we set

$$\Xi^{-\mu}(\Omega) = (\Xi^{\mu}(\Omega))', \quad \mu > 0. \tag{26}$$

As for Proposition 9.1 of Chapter 4 of ([20]: Vol. 2, p. 43), we verify that

$$D(Q) \text{ is dense in } \Xi^{3,3}(Q). \tag{27}$$

Therefore

$$\Xi^{-3,-3}(Q) = \left(\Xi^{3,3}(Q)\right)' \subset D'(Q). \tag{28}$$

For this space, we have a structure result analogous to Proposition 9.3 given in Chapter 4 of ([20]: Vol.2, p. 45) and, thanks to (15), we have:

$$\text{if } u \in \Xi^{-3,-3}(Q), \text{ then } \omega \to \langle u, \omega\rangle \tag{29}$$

is continuous antilinear on $X^1(Q)$.

3.2.2. The space $D^{-1}_{A+D_t^2}(Q)$

Taking, in (8), $\omega = \Phi \in D(Q)$, we have $L(\Phi) = \langle f, \Phi\rangle_Q$ and therefore, in the sense of $D'(Q)$:

$$y'' + Ay = u. \tag{30}$$

This leads to the following definition ([20]: Vol. 2, p. 131):

$$D^{-1}_{A+D_t^2}(Q) \overset{\text{df}}{=} \left\{y |\, y \in H^{-1,-2}(Q),\ y'' + Ay \in \Xi^{-3,-3}(Q)\right\} \tag{31}$$

provided with the norm of the graph, this is a Hilbert space.

Then the solution y of (8) belongs to $D^{-1}_{A+D_t^2}(Q)$.

3.2.3. Choice of q

Consequently, we choose q.

From (15) and the Trace Theorem ([9] and [20]: Vol. 2, p. 9), we deduce that $\omega \to \omega|_{\Sigma}$ is a continuous linear mapping of

$$X^1(Q) \to H^{5/2,5/2}(\Sigma). \tag{32}$$

Consequently, we introduce as in Chapter 4, Section 11 of ([20]: Vol. 2, pp. 57–59) the spaces $H^{\alpha}\Xi^{\alpha}(\Sigma)$, real $\alpha \geqslant 0$;

first for integer:

$$H^{\alpha}\Xi^{\alpha}(\Sigma) = \left\{\omega | d^{j}(t)\omega^{(j)} \in L^2(0,T,H^{\alpha-j/\alpha}(\Gamma)),\ 0 \leqslant j \leqslant \alpha\right\}, \tag{33}$$

where the function $d(t)$ is defined in (17), and then for non-integer $\alpha \in R_{+}$, we define by interpolation

$$\left.\begin{aligned} &H^{\alpha}\Xi^{\alpha}(\Sigma) = [H^{\alpha_0}\Xi^{\alpha_0}(\Sigma),\ H^{0,0}(\Sigma)]_{\Theta} \\ &\text{integer } \alpha_0,\ (1-\Theta)\alpha_0 = \alpha \end{aligned}\right\}. \tag{34}$$

The space defined in this way depends only on α. Then:

$$H^{5/2,5/2}(\Sigma) \subset H^{5/2}\Xi^{5/2}(\Sigma) \tag{35}$$

As for Proposition 11.1, Chapter 4 of ([20]: Vol. 2, p. 58), we verify that

$$D(\Sigma) \text{ is dense in } H^{\alpha}\Xi^{\alpha}(\Sigma) \tag{36}$$

and therefore

$$H^{-\alpha}\Xi^{-\alpha}(\Sigma) = (H^{\alpha}\Xi^{\alpha}(\Sigma))' \subset D'(\Sigma). \tag{37}$$

According to (35), we have:

$$\left.\begin{aligned} &\text{if } q \in H^{-5/2}\Xi^{-5/2}(\Sigma),\ \text{then } \omega \longrightarrow \langle q,\ \omega\rangle \\ &\text{is continuous on } X^1(Q) \end{aligned}\right\}. \tag{38}$$

3.2.4. Choice of y_0 and y_I

Subsequently, we make choice of y_0 and y_I respectively.
Then, from the Theorem 2.1 of ([20]: Vol. 2, p. 9), it follows that

$$\begin{aligned} &\omega \longrightarrow \{\omega(0),\ \omega'(0)\} \text{ is a continuous mapping} \\ &\qquad\text{of } H^{3,3}(Q) \longrightarrow H^{5/2}(\Omega) \times H^{3/2}(\Omega). \end{aligned} \tag{39}$$

Using the spaces $\Xi^{\mu}(\Omega), \Xi^{-\mu}(\Omega)$ defined in (24) and (26), it follows that:

$$\left.\begin{aligned} &\text{if } \{y_0, y_I\} \in \Xi^{-3/2}(\Omega) \times \Xi^{-5/2}(\Omega), \\ &\text{then the form } \omega \longrightarrow \langle y_I, \omega(0)\rangle - \langle y_0, \omega'(0)\rangle \\ &\text{is continuous on } X^1(Q) \end{aligned}\right\}. \tag{40}$$

3.2.5. Central result

Consequently, the preceding results may be summarized by

Theorem 1 *Let y_0, y_I, q and u be given, with $y_0 \in \Xi^{-3/2}(\Omega)$, $y_I \in \Xi^{-5/2}(\Omega)$, $q \in H^{-5/2}\Xi^{-5/2}(\Sigma)$ and $u \in \Xi^{-3,-3}(Q)$. Then, there exists a unique solution $y \in D^{-1}_{A+D_{t^2}}(Q)$ for the problem (1)–(4) defined by transposition (8).*

This theorem will be the starting point for our considerations.

4. Analysis of multiple time delay hyperbolic systems

4.1. Existence and uniqueness of solutions: $v \in L^2(\Sigma)$

Consider now the distributed-parameter system described by the following hyperbolic delay equation

$$\frac{\partial^2 y}{\partial t^2} + A(t)y + \sum_{i=1}^{m} y(x,t-h_i) = u, \qquad x \in \Omega,\ t \in (0,T), \tag{41}$$

$$y(x,t') = \Phi_0(x,t'), \qquad x \in \Omega,\ t' \in [-\xi,0), \tag{42}$$

$$y(x,0) = y_0(x), \qquad x \in \Omega, \tag{43}$$

$$y'(x,0) = y_I(x), \qquad x \in \Omega, \tag{44}$$

$$\frac{\partial y}{\partial \eta_A} = \sum_{s=1}^{l} y(x,t-k_s) + Gv, \qquad x \in \Gamma,\ t \in (0,T), \tag{45}$$

$$y(x,t') = \Psi_0(x,t'), \qquad x \in \Gamma,\ t' \in [-\xi,0), \tag{46}$$

where: Ω has the same properties as in the problem (1)–(4), h_i and k_s are specified positive number representing multiple time delays, such that $0 \leqslant h_1 < h_2 < \ldots < h_m$ for $i = 1,\ldots,m$ and $0 \leqslant k_1 < k_2 < \ldots < k_l$ for $s = 1,\ldots,l$ respectively, Φ_0, Ψ_0 are initial functions defined on Q_0 and Σ_0 respectively.

Moreover,

$$\xi \overset{\text{df}}{=} \max\{h_m, k_l\}.$$

The operator $A(t)$ has the form given by (5)–(6).

$$y \equiv y(x,t;v), \quad u \equiv u(x,t), \quad v \equiv v(x,t), \quad Q = \Omega \times (0,T),$$

$$\bar{Q} = \bar{\Omega} \times [0,T], \quad Q_0 = \Omega \times [-\xi,0), \quad \Sigma = \Gamma \times (0,T), \quad \Sigma_0 = \Gamma \times [-\xi,0),$$

G is a linear continuous operator on $L^2(\Sigma)$ into

$$\left(H^{5/2}\Xi^{5/2}(\Sigma)\right)' \quad \text{with } v \in L^2(\Sigma) \quad \text{and} \quad Gv \in H^{-5/2}\Xi^{-5/2}(\Sigma).$$

The operator $A(t)$ is given by the formula (5)–(6).

It is easy to notice that the equations (41)–(46) constitute the Neumann problem. The left-hand side of the Neumann boundary condition (45) is written in the following form

$$\frac{\partial y}{\partial \eta_A} = q(x,t) \qquad x \in \Gamma,\ t \in (0,T), \tag{47}$$

where

$$q(x,t) = \sum_{s=1}^{l} y(x,t-k_s) + Gv(x,t) \qquad x \in \Gamma,\ t \in (0,T). \tag{48}$$

We shall prove the existence of a unique solution of the mixed initial-boundary value problem (41)–(46) defined by transposition, i.e.

$$\langle y, \omega'' + A\omega \rangle = L(\omega) \qquad \forall \omega \in X^1(Q), \tag{49}$$

where

$$L(\omega) = \langle l, \omega \rangle + \langle q, \omega \rangle + \langle y_I, \omega(0) \rangle - \langle y_0, \omega'(0) \rangle \tag{50}$$

and

$$l = \left[u - \sum_{i=1}^{m} y(x, t - h_i) \right] \Bigg|_Q \tag{51}$$

and $X^1(Q)$ is the space described by the solutions ω of the adjoint problem (10).

We shall restrict our considerations to the case where $v \in L^2(\Sigma)$. For simplicity, we shall introduce the following notations

$$I_j \stackrel{\text{df}}{=} ((j-1)\lambda, j\lambda)\,,\ Q_j = \Omega \times I_j,\ \Sigma_j = \Gamma \times I_j \quad \text{for } j = 1, \ldots, K,$$

where

$$\lambda \stackrel{\text{df}}{=} \min\{h_1, k_1\}.$$

The existence of a unique solution for the mixed initial-boundary value problem (41)–(46) on the cylinder Q can be proved using a constructive method, i.e. first, solving problem (49) in the subcylinder Q_1 and in turn in Q_2 etc. until the procedure covers the whole cylinder Q. In this way the solution in the previous step determines the next one.

Using the Theorem 1 we can prove the following lemma.

Lemma 1 *[18]: Let*

$$u \in \Xi^{-3,-3}(Q), \tag{52}$$

$$l_j \in \Xi^{-3,-3}(Q_j), \tag{53}$$

where

$$l_j = \left[u - \sum_{i=1}^{m} y_{j-1}(x, t - h_i) \right] \Bigg|_{Q_j},$$

$$q_j \in H^{-5/2}\Xi^{-5/2}(\Sigma_j), \tag{54}$$

where

$$q_j = \left[\sum_{s=1}^{l} y_{j-1}(x, t - k_s) + Gv(x, t] \right] \Bigg|_{\Sigma_j}$$

and

$$G \in \alpha(\mathcal{L}^2(\Sigma),\ H^{-5/2}\ \Xi^{-5/2}(\Sigma)) \text{ with } v \in L^2(\Sigma),$$

$$y_{j-1}(\cdot,(j-1)\lambda) \in \Xi^{-3/2}(\Omega), \tag{55}$$

$$y'_{j-1}(\cdot,(j-1)\lambda) \in \Xi^{-5/2}(\Omega). \tag{56}$$

Then, there exists a unique solution $y_j \in D^{-1}_{A+D_t^2}(Q_j)$ *for the mixed initial-boundary value problem (41), (45), (55), (56) defined by transposition, i.e.*

$$\langle y_j, \omega''_j + A\omega_j\rangle = L(\omega_j), \qquad \forall \omega_j \in X^1(Q_j), \tag{57}$$

where

$$\begin{aligned} L(\omega_j) &= \langle l_j, \omega_j\rangle + \langle q_j, \omega_j\rangle \\ &\quad + \langle y'_{j-1}[(j-1)\lambda],\ \omega_j[(j-1)\lambda]\rangle - \langle y_{j-1}[(j-1)\lambda],\ \omega'_j[(j-1)\lambda]\rangle. \end{aligned}$$

Proof.

For $j = 1$, conditions (53)–(56) can be satisfied if we assume that

$$\Phi_0 \in H^{-1,-2}(Q_0),\ \Psi_0 \in H^{-5/2}\Xi^{-5/2}(\Sigma_0)$$

and

$$Gv \in H^{-5/2}\Xi^{-5/2}(\Sigma),\ y_0 \in \Xi^{-3/2}(\Omega) \text{ and } y_I \in \Xi^{-5/2}(\Omega).$$

These assumptions are sufficient to ensure the existence of a unique solution $y_1 \in D^{-1}_{A+D_t^2}(Q_1) \subset H^{-1,-2}(Q_1)$. In order to extend the result to $Q_j,\ 1 < j \leqslant K$ it is sufficient to verify that

$$l_2 \in \Xi^{-3,-3}(Q_2), \tag{58}$$

$$q_2 \in H^{-5/2}\Xi^{-5/2}(\Sigma_2), \tag{59}$$

$$y_1(\cdot,\lambda) \in \Xi^{-3/2}(\Omega), \tag{60}$$

$$y'_1(\cdot,\lambda) \in \Xi^{-5/2}(\Omega). \tag{61}$$

According to (15) and (23) of Subsubsection 3.2.1 we have $X^1(Q_2) \subset H^{3,3}(Q_2)$ and obviously $H^{3,3}(Q_2) \subset \Xi^{3,3}(Q_2)$. Using the results of Subsubsection 3.2.1, we have:

if $y_1 \in H^{-1,-2}(Q_1)$ and $l_2 \in \Xi^{-3,-3}(Q_2)$, then $\omega_2 \to \langle l_2, \omega_2\rangle$ is continuous antilinear on $X^1(Q_2)$ and the condition (58) is fulfilled.

To verify (59) we use the results of Subsubsection 3.2.3. Then, from the inclusion $X^1(Q_2) \subset H^{3,3}(Q_2)$, and the Theorem 2.1 ([20]: Vol.2, p.9) we deduce that $\omega_2 \to \omega_2|_{\Sigma_2}$ is a linear continuous mapping of $X^1(Q_2) \to H^{5/2,5/2}(\Sigma_2) \subset H^{5/2}\Xi^{5/2}(\Sigma_2)$.

According to the fact mentioned above we have if $y_1|_{\Sigma_1} \in H^{-5/2}\Xi^{-5/2}(\Sigma_1)$ and $q_2 \in H^{-5/2}\Xi^{-5/2}(\Sigma_2)$, then $\omega_2 \to \langle q_2, \omega_2\rangle$ is continuous on $X^1(Q_2)$.

Consequently, using the results of the Subsubsection 3.2.4, we shall verify the conditions (60) and (61). From the Theorems 3.1 and 9.6 ([20]: Vol.1,pp. 19 and 43) $\omega_2 \in X^1(Q_2) \subset H^{3,3}(Q_2)$ implies that the mappings $t \to \omega_2(\cdot, t)$ and $t \to \omega_2'(\cdot, t)$ are continuous from $[\lambda, 2\lambda] \to H^{5/2}(\Omega)$ and $[\lambda, 2\lambda] \to H^{3/2}(\Omega)$ respectively. Hence $\omega_2(\cdot, \lambda) \in H^{5/2}(\Omega)$ and $\omega_2'(\cdot, \lambda) \in H^{3/2}(\Omega)$. Then, from (39) of Subsubsection 3.2.4, it follows that $\omega_2 \to (\omega_2(\lambda), \omega_2'(\lambda))$ is a continuous mapping of $H^{3,3}(Q_2) \to H^{5/2}(\Omega) \times H^{3/2}(\Omega)$.

Using the spaces $\Xi^{\alpha}(\Omega), \Xi^{-\alpha}(\Omega)$ defined in (24) and (26) we deduce that if $(y_1(\lambda), y_1'(\lambda)) \in \Xi^{-3/2}(\Omega) \times \Xi^{-5/2}(\Omega)$, then the form $\omega_2 \to \langle y_1'(\lambda), \omega_2(\lambda)\rangle - \langle y_1(\lambda), \omega_2'(\lambda)\rangle$ is continuous on $X^1(Q_2)$. The conditions (60) and (61) are fulfilled. Then, there exists a unique solution $y_2 \in D^{-1}_{A+D_t^2}(Q_2)$.

The foregoing result is now summarized.

Theorem 2 *[18] Let $y_0, y_I, \Phi_0, \Psi_0, v$ and u be given with*

$$y_0 \in \Xi^{-3/2}(\Omega), \quad y_I \in \Xi^{-5/2}(\Omega), \quad \Phi_0 \in H^{-1,-2}(Q_0),$$

$$\Psi_0 \in H^{-5/2}\Xi^{-5,2}(\Sigma_0), \quad v \in L^2(\Sigma) \text{ and } u \in \Xi^{-3,-3}(Q).$$

Then, there exists a unique solution $y \in D^{-1}_{A+D_t^2}(Q)$ for the problem (41)–(46) defined by transposition (49). Moreover, $y(\cdot, j\lambda) \in \Xi^{-3/2}(\Omega)$ and $y'(\cdot, j\lambda) \in \Xi^{-5/2}(\Omega)$ for $j = 1, \ldots, K$.

4.2. Problem formulation. Optimization theorems

We shall now formulate the optimal boundary control problem in the context of the case where $v \in L^2(\Sigma)$. Let us denote by $U = L^2(\Sigma)$ the space of controls. The time horizon T is fixed in our problem. The performance functional is given by

$$I(v) = \lambda_1 \parallel y(v) - z_d \parallel^2_{H^{-1,-2}(Q)} + \lambda_2 \langle Nv, v\rangle_{L^2(\Sigma)} \tag{62}$$

where: $\lambda_i \geqslant 0$ and $\lambda_1 + \lambda_2 > 0$; z_d is a given element in $H^{-1,-2}(Q)$, and N is a positive linear operator on $L^2(\Sigma)$ into $L^2(\Sigma)$. Finally, we assume the following constraint on controls

$$v \in U_{ad} \tag{63}$$

where: U_{ad} is a closed, convex set with non-empty interior, a subset of U.

Let $y(x, t, v)$ denote the solution of (41)–(46), (62), (63) at (x, t) corresponding to a given control $v \in U_{ad}$. We note from the Theorem 2 that for any $v \in U_{ad}$ the cost function (62) is well defined since $y \in D^{-1}_{A+D_t^2}(Q) \subset H^{-1,-2}(Q)$.

The optimal control problem (41)–(46), (62), (63) will be solved as the optimization one in which the function v is the unknown function. Making use of Dubovicki-Milutin theorem [13] we shall derive the necessary and sufficient conditions of optimality for the optimization problem (41)–(46), (62), (63).

The solution of the stated optimal control problem is equivalent to seeking a pair $(y^0, v^0) \in E = D^{-1}_{A+D_t^2}(Q) \times L^2(\Sigma)$ which satisfies the equation (41)–(46) and minimizing the performance functional (62) with the constraints on boundary control (63).

We formulate the necessary and sufficient conditions of the optimality in the form of Theorem 2.

Theorem 3 *The solution of the optimization problem (41)–(46), (62), (63) exists and it is unique with the assumptions mensioned above; the necessary and sufficient conditions of the optimality are characterized by the following system of partial differential equations and inequalities.*

State equation:

$$\frac{\partial^2 y^0}{\partial t^2} + A(t)y^0 + \sum_{i=1}^{m} y^0(x, t - h_i) = f, \qquad x \in \Omega,\ t \in (0, T), \tag{64}$$

$$y^0(x, t') = \Phi_0(x, t'), \qquad x \in \Omega,\ t' \in [-\xi, 0), \tag{65}$$

$$y^0(x, 0) = y_1(x), \qquad x \in \Omega, \tag{66}$$

$$\frac{\partial y^0}{\partial t} = y_2(x), \qquad x \in \Omega, \tag{67}$$

$$\frac{\partial y^0}{\partial \eta_A} = \sum_{s=1}^{l} y^0(x, t - k_s) + Gv^0, \qquad x \in \Gamma,\ t \in (0, T), \tag{68}$$

$$y^0(x, t') = \Psi_0(x, t'), \qquad x \in \Gamma,\ t' \in [-\xi, 0). \tag{69}$$

Adjoint equation:

$$\frac{\partial^2 p}{\partial t^2} + A(t)p + \sum_{i=1}^{m} p(x, t + h_i) = \lambda_1 \Lambda_1 (y^0 - z_d), \quad x \in \Omega,\ t \in (0, T - \xi), \tag{70}$$

$$\frac{\partial^2 p}{\partial t^2} + A(t)p = \lambda_1 \Lambda_1 (y^0 - z_d), \qquad x \in \Omega,\ t \in (T - \xi, T), \tag{71}$$

$$\frac{\partial p}{\partial \eta_A} = \sum_{s=1}^{l} p(x, t + k_s), \qquad x \in \Gamma,\ t \in (0, T - \xi), \tag{72}$$

$$\frac{\partial p}{\partial \eta_A} = 0, \qquad x \in \Gamma,\ t \in (T-\xi, T), \tag{73}$$

$$p(x,T) = 0, \qquad x \in \Omega, \tag{74}$$

$$\frac{\partial p(x,T)}{\partial t} = 0, \qquad x \in \Omega, \tag{75}$$

where Λ_1 is a canonical isomorphism of $H^{-1,-2}(Q)$ onto $H^{1,2}_{0,0}(Q)$.
Maximum condition:

$$\left\langle G^* p(v^0) + \lambda_2 N v^0, v - v^0 \right\rangle_{L^2(\Sigma)} \geqslant 0 \quad \forall v \in U_{ad} \tag{76}$$

We can also notice that

$$\frac{\partial p}{\partial \eta_A} = \sum_{i,j=1}^{n} a_{ji}(x,t) \cos(n, x_i) \frac{\partial p}{\partial x_j}. \tag{77}$$

Proof.
According to the Dubovicki-Milutin theorem [13], we approximate the set representing the inequality constraints by the regular admissible cone, the equality constraints by the regular tangent cone and the performance functional by the regular improvement cone.

a) Analysis of the equality constraint

The set Q_1 representing the equality constraint has has the form

$$Q_1 = \begin{cases} \dfrac{\partial^2 y}{\partial t^2} + A(t)y + \sum\limits_{i=1}^{m} y(x, t-h_i) = u, & x \in \Omega,\ t \in (0,T), \\ y(x,t') = \Phi_0(x,t'), & x \in \Omega,\ t' \in [-\xi, 0), \\ y(x,0) = y_1(x), & x \in \Omega, \\ \dfrac{\partial y(x,0)}{\partial t} = y_2(x), & x \in \Omega, \\ \dfrac{\partial y}{\partial \eta_A} = \sum\limits_{s=1}^{l} y(x, t-k_s) + Gv, & x \in \Gamma,\ t \in (0,T), \\ y(x,t') = \Psi_0(x,t'), & x \in \Gamma,\ t' \in [-\xi, 0). \end{cases} \tag{78}$$

We construct the regular tangent cone of the set Q_1 using the Lusternik theorem (Theorem 9.1 [5]). For this purpose, we define the operator P in the form

$$P(y,v) = \Bigg(\frac{\partial^2 y}{\partial t^2} + Ay + \sum_{i=1}^{m} y(x, t-h_i) - u, \\ y(x,t') - \Phi_0(x,t'), y(x,0) - y_1(x), \frac{\partial y(x,0)}{\partial t} - y_2(x), \\ \frac{\partial y}{\partial \eta_A} - \sum_{s=1}^{l} y(x,t-k_s) - Gv, y(x,t') - \Psi_0(x,t') \Bigg). \tag{79}$$

The operator P is the mapping from the space $D^{-1}_{A+D_t^2}(Q) \times L^2(\Sigma)$ into the space $\Xi^{-3,-3}(Q) \times D^{-1}_{A+D_t^2}(Q_0) \times \Xi^{-3/2}(\Omega) \times \Xi^{-5/2}(\Omega) \times H^{-5/2}\Xi^{-5/2}(\Sigma) \times H^{-5/2}\Xi^{-5/2}(\Sigma_0)$.

The Fréchet differential of the operator P can be written in the following form:

$$P'(y^0,v^0)(\overline{y},\overline{v}) = \Bigg(\frac{\partial^2 \overline{y}}{\partial t^2} + A\overline{y} + \sum_{i=1}^{m} \overline{y}(x,t-h_i),\ \overline{y}\big|_{Q_0}(x,t'),\ \overline{y}(x,0), \\ \frac{\partial \overline{y}(x,0)}{\partial t},\ \frac{\partial \overline{y}}{\partial \eta_A} - \sum_{s=1}^{l} \overline{y}(x,t-k_s) - G\overline{v},\ \overline{y}\big|_{\Sigma_0}(x,t') \Bigg). \tag{80}$$

Really, $\frac{\partial^2}{\partial t^2}$ (Theorem 2.8 [21]), $A(t)$ (Theorem 2.1 [19]) and $\frac{\partial}{\partial \eta_A}$ (Theorem 2.1 [20]; vol. 2, p. 9) are linear and bounded mappings. Using Theorem 2, we can prove that P' is the operator "one to one" from the space $D^{-1}_{A+D_t^2}(Q) \times L^2(\Sigma)$ onto the space $\Xi^{-3,-3}(Q) \times D^{-1}_{A+D_t^2}(Q_0) \times \Xi^{-3/2}(\Omega) \times \Xi^{-5/2}(\Omega) \times H^{-5/2}\Xi^{-5/2}(\Sigma) \times H^{-5/2}\Xi^{-5/2}(\Sigma_0)$.

Considering that the assumptions of the Lusternik's theorem are fulfilled, we can write down the regular tangent cone for the set Q_1 in a point (y^0, v^0) in the form

$$RTC(Q_1,(y^0,v^0)) = \Big((\overline{y},\overline{v}) \in E, P'(y^0,v^0)(\overline{y},\overline{v}) = 0 \Big). \tag{81}$$

It is easy to notice that it is a subspace. Therefore, using Theorem 10.1 [5] we know the form of the functional belonging to the adjoint cone

$$f_1\,(\overline{y},\overline{v}) = 0 \quad \forall\,(\overline{y},\overline{v}) \in RTC(Q_1,(y^0,v^0)). \tag{82}$$

b) Analysis of the constraint on controls

The set $Q_2 = Y \times U_{ad}$ representing the inequality constraints is a closed and convex one with non-empty interior in the space E.

Using Theorem 10.5 [5] we find the functional belonging to the adjoint regular admissible cone, i.e.

$$f_2\,(\overline{y},\overline{v}) \in [RAC(Q_2,(y^0,v^0))]^*.$$

We can note if E_1, E_2 are two linear topological spaces, then the adjoint space to $E = E_1 \times E_2$ has the form

$$E^* = \{f = (f_1, f_2);\; f_1 \in E_1^*, f_2 \in E_2^*\}$$

and

$$f(x) = f_1(x_1) + f_2(x_2)$$

So we note the functional $f_2(\overline{y},\overline{v})$ as follows

$$f_2\,(\overline{y},\overline{v}) = f_1'(\overline{y}) + f_2'(\overline{v}), \tag{83}$$

where:

$f_1'(\overline{y}) = 0\ \forall y \in Y$ (Theorem 10.1 [5]),

$f_2'(\overline{v})$ is a support functional to the set U_{ad} in a point v_0 (Theorem 10.5 [5]).

c) Analysis of the performance functional

Using Theorem 7.5 [5] we find the regular improvement cone of the performance functional (62)

$$RFC(I,(y^0,v^0)) = \left\{(\overline{y},\overline{v}) \in E, I'(y^0,v^0)\,(\overline{y},\overline{v}) < 0\right\}, \tag{84}$$

where: $I'(y^0,v^0)(\overline{y},\overline{v})$ is the Fréchet differential of the performance functional (62) and it can be written as

$$I'(y^0,v^0)(\overline{y},\overline{v}) = 2\lambda_0\lambda_1 \left\langle \Lambda_1(y^0 - z_d), \overline{y}\right\rangle_{H^{-1,-2}(Q)} + 2\lambda_0\lambda_2 \left\langle Nv^0, \overline{v}\right\rangle_{L^2(\Sigma)}. \tag{85}$$

On the basis of Theorem 10.2 [5] we find the functional belonging to the adjoint regular improvement cone, which has the form

$$f_3(\overline{y},\overline{v}) = -\lambda_0\lambda_1 \left\langle \Lambda_1(y^0 - z_d), \overline{y}\right\rangle_{H^{-1,-2}(Q)} - \lambda_0\lambda_2 \left\langle Nv^0, \overline{v}\right\rangle_{L^2(\Sigma)}, \tag{86}$$

where: $\lambda_0 > 0$.

d) Analysis of Euler-Lagrange's equation

The Euler-Lagrange's equation for our optimization problem has the form

$$\sum_{i=1}^{3} f_i = 0 \tag{87}$$

Let $p(x,t)$ be the solution of (70)–(75) for (v^0, y^0). Then, $p(v)$ is defined by transposition, i.e.

$$\langle p, y'' + Ay\rangle = M(y), \quad \forall y \in D^{-1}_{A+D_t^2}(Q), \tag{88}$$

where

$$M(y) = \langle p'' + Ap, y\rangle + \langle p, l\rangle - \langle p, q\rangle - \langle p(0), y_2\rangle + \langle p'(0), y_1\rangle$$

and y satisfies (41)–(46).

We observe that, for given z_d and v, equations (70)–(75) can be solved backward in time starting from $t = T$, i.e. first solving problem (70)–(75) in the subcylinder Q_1, and in turn in Q_{k-1} etc., until the procedure covers the whole cylinder Q. For this purpose, we may apply Theorem 2.

Lemma 2 *Let the hypothesis of Theorem 2 be satisfied. Then, for given $z_d \in H^{-1,-2}(Q)$, and any $v \in L^2(\Sigma)$, there exists a unique solution*

$$p(v) \in H^{3,3}(Q) \subset \Xi^{3,3}(Q)$$

to the problem (70)–(75) defined by transposition (88).

Next we denote by $\overline{y}$ the solution of $P'(\overline{y}, \overline{v}) = 0$ for any fixed $\overline{v}$. Then taking into account (82)–(83) and (86) we can express (87) in the form

$$f_2'(\overline{v}) = \lambda_0\lambda_1 \left\langle \Lambda_1(y^0 - z_d), \overline{y}\right\rangle_{H^{-1,-2}(Q)} + \lambda_0\lambda_2 \left\langle Nv^0, \overline{v}\right\rangle_{L^2(\Sigma)}$$
$$\forall\, (\overline{y}, \overline{v}) \in RTC\left(Q_1(\overline{y}, \overline{v})\right). \tag{89}$$

We transform the first component of the right-hand side of (89) using the formulae (70)–(75). For this purpose setting $v = v_0$ in (70)–(75) and then taking the scalar product of both sides of (70), (71) by an element $\overline{y}(v)$ respectively, and then adding both sides of (70), (71), we get

$$\lambda_0\lambda_1 \left\langle \Lambda_1(y^0 - z_d), \overline{y}\right\rangle_{H^{-1,-2}(Q)} = \left\langle \frac{\partial^2 p}{\partial t^2} + A(t)p, \overline{y}\right\rangle_{H^{-1,-2}(Q)}$$
$$+ \left\langle \sum_{i=1}^{m} p(x, t + h_i), \overline{y}\right\rangle_{H^{-1,-2}[\Omega\times(0,T-\xi)]} = \left\langle p, \frac{\partial^2 \overline{y}}{\partial t^2}\right\rangle_{H^{-3,-3}(Q)}$$
$$+ \langle A(t)p, \overline{y}\rangle_{H^{-1,-2}(Q)} + \sum_{i=1}^{m} \langle p(x, t + h_i), \overline{y}\rangle_{H^{-1,-2}[\Omega\times(0,T-\xi)]} \tag{90}$$

By using the equation (41), the first term on the right-hand side of (90) can be rewritten as

$$\begin{aligned}\left\langle p, \frac{\partial^2 \overline{y}}{\partial t^2}\right\rangle_{H^{-3,-3}(Q)} &= -\left\langle p, A(t)\overline{y}\right\rangle_{H^{-3,-3}(Q)} \\ &- \left\langle p, \sum_{i=1}^{m} \overline{y}(x, t-h_i)\right\rangle_{H^{-1,-2}(Q)} = -\left\langle p, A(t)\overline{y}\right\rangle_{H^{-3,-3}(Q)} \\ &- \sum_{i=1}^{m} \left\langle p(x, t'+h_i), \overline{y}(x,t)\right\rangle_{H^{-1.-2}[\Omega\times(-h_i, T-h_i)]} . \end{aligned} \tag{91}$$

The second integral on the right-hand side of (90) in view of Green's formula can be expressed as

$$\begin{aligned}\left\langle A(t)p,\ \overline{y}\right\rangle_{H^{-1,-2}(Q)} &= \left\langle p, A(t)\overline{y}\right\rangle_{H^{-3,-3}(Q)} \\ &+ \left\langle p, \frac{\partial \overline{y}}{\partial \eta_A}\right\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)} - \left\langle \frac{\partial p}{\partial \eta_A}, \overline{y}\right\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)} . \end{aligned} \tag{92}$$

By using the boundary condition (45), the second term on the right-hand side of (92) can be written as

$$\begin{aligned}\left\langle p,\ \frac{\partial \overline{y}}{\partial \eta_A}\right\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)} &= \left\langle p,\ \sum_{s=1}^{l} \overline{y}(x, t-k_s)\right\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)} + \left\langle p,\ G\overline{v}\right\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)} \\ &= \sum_{s=1}^{l} \left\langle p,\ \overline{y}(x, t-k_s)\right\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)} + \left\langle p, G\overline{v}\right\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)} \\ &= \sum_{s=1}^{l} \left\langle p(x, t'+k_s),\ \overline{y}(x,t')\right\rangle_{H^{-5/2}\Xi^{-5/2}[\Gamma\times(-k_s, T-k_s)]} \\ &+ \left\langle p,\ G\overline{v}\right\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)} . \end{aligned} \tag{93}$$

The last term in (92) can be written as

$$\begin{aligned}\left\langle \frac{\partial p}{\partial \eta_A},\ \overline{y}\right\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)} &= \left\langle \frac{\partial p}{\partial \eta_A},\ \overline{y}\right\rangle_{H^{-5/2}\Xi^{-5/2}[\Gamma\times(0, T-\xi)]} \\ &+ \left\langle \frac{\partial p}{\partial \eta_A},\ \overline{y}\right\rangle_{H^{-5/2}\Xi^{-5/2}[\Gamma\times(T-\xi, T)]} . \end{aligned} \tag{94}$$

Substituting (93) and (94) into (92) and then (91) and (92) into (90) we obtain

$$
\begin{aligned}
&\lambda_0\lambda_1\left\langle \Lambda_1(y^0 - z_d),\ \overline{y}\right\rangle_{H^{-1,-2}(Q)}\\
&\quad= -\left\langle p, A(t)\overline{y}\right\rangle_{H^{-3,-3}(Q)} - \sum_{i=1}^{m}\left\langle p(x,t+h_i),\overline{y}\right\rangle_{H^{-1.-2}[\Omega\times(-h_i,0)]}\\
&\qquad- \sum_{i=1}^{m}\left\langle p(x,t+h_i),\ \overline{y}\right\rangle_{H^{-1.-2}[\Omega\times(0,T-h_i)]} + \left\langle p,\ A(t)\overline{y}\right\rangle_{H^{-3,-3}(Q)}\\
&\qquad+ \sum_{s=1}^{l}\left\langle p(x,t+k_s),\ \overline{y}\right\rangle_{H^{-5/2}\Xi^{-5/2}[\Gamma\times(-k_s,0)]}\\
&\qquad+ \sum_{s=1}^{l}\left\langle p(x,t+k_s),\ \overline{y}\right\rangle_{H^{-5/2}\Xi^{-5/2}[\Gamma\times(0,T-k_s)]}\\
&\qquad+ \left\langle p,\ G\overline{v}\right\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)} - \left\langle \frac{\partial p}{\partial\eta_A},\ \overline{y}\right\rangle_{H^{-5/2}\Xi^{-5/2}[\Gamma\times(0,T-\xi)]}\\
&\qquad- \left\langle \frac{\partial p}{\partial\eta_A},\ \overline{y}\right\rangle_{H^{-5/2}\Xi^{-5/2}[\Gamma\times(T-\xi,T)]}\\
&\qquad+ \sum_{i=1}^{m}\left\langle p(x,t+h_i),\ \overline{y}\right\rangle_{H^{-1.-2}[\Omega\times(0,T-\xi)]} = \left\langle p,\ G\overline{v}\right\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)}\\
&\quad= \left\langle G^* p,\ \overline{v}\right\rangle_{L^2(\Sigma)}. \qquad (95)
\end{aligned}
$$

Substituting (95) into (89) gives

$$
f_2'(\overline{v}) = \lambda_0\left\langle G^*p + \lambda_2 N v^0, \overline{v}\right\rangle_{L^2(\Sigma)}. \qquad (96)
$$

Using the definition of the support functional [5] and dividing both sides of the obtained inequality by λ_0, we finally get

$$
\left\langle G^*p + \lambda_2 N v^0, v - v^0\right\rangle_{L^2(\Sigma)} \geqslant 0 \quad \forall v \in U_{ad}. \qquad (97)
$$

The last inequality is equivalent to the maximum condition (76).

The uniqueness of the optimal control follows from the strict convexity of the performance functional (62).

This last remark finishes the proof of Theorem 3.

One may also consider analogous optimal control problem with the performance functional

$$
\widehat{I}(y,v) = \lambda_1\left\| y(v)\big|_\Sigma - z_{\Sigma d}\right\|^2_{H^{-5/2}\Xi^{-5/2}(\Sigma)} + \lambda_2\left\langle (Nv),\ v\right\rangle_{L^2(\Sigma)}, \qquad (98)
$$

where: $z_{\Sigma d}$ is a given element in $H^{-5/2}\Xi^{-5/2}(\Sigma)$; we assume that the space $H^{-5/2}\Xi^{-5/2}(\Sigma)$ is such that $y(v)\big|_{\Sigma} \in H^{-5/2}\Xi^{-5/2}(\Sigma)$. Then the solution of the formulated optimal control problem is equivalent to seeking a pair

$$(y^0, v^0) \in E = D^{-1}_{A+D_t}(Q) \times L^2(\Sigma)$$

that satisfies the equation (41)–(46) and minimizing the cost function (98) with the constraints on controls (63).

We can prove the following theorem:

Theorem 4 *The solution of the optimization problems (41)–(46), (98), (63) exists and it is unique with the assumptions mentioned above; the necessary and sufficient conditions of the optimality are characterized by the following system of partial differential equations and inequalities:*

State equations (41)–(46),

Adjoint equations

$$\frac{\partial^2 p}{\partial t^2} + A(t)p + \sum_{i=1}^{m} y(x, t+h_i) = 0, \qquad x \in \Omega,\ t \in (0, T-\xi), \tag{99}$$

$$\frac{\partial^2 p}{\partial t^2} + A(t)p = 0, \qquad x \in \Omega,\ t \in (T-\xi, T), \tag{100}$$

$$\frac{\partial p}{\partial \eta_A} = \sum_{s=1}^{l} p(x, t+k_s) + \lambda_0 \Lambda_2 \left(y^0\big|_{\Sigma} - z_{\Sigma d} \right), \quad x \in \Gamma, t \in (0, T-\xi), \tag{101}$$

$$\frac{\partial p}{\partial \eta_A} = \lambda_0 \Lambda_2 \left(y^0\big|_{\Sigma} - z_{\Sigma d} \right), \qquad x \in \Gamma,\ t \in (T-\xi, T), \tag{102}$$

$$p(x, T) = 0, \qquad z \in \Omega, \tag{103}$$

$$\frac{\partial p(x, T)}{\partial t} = 0, \qquad x \in \Omega, \tag{104}$$

where: Λ_2 is a canonical isomorphism of $H^{-5/2}\Xi^{-5/2}(\Sigma)$ into $H^{5/2}\Xi^{5/2}(\Sigma)$.

Maximum condition

$$\left\langle G^* p(v^0) + \lambda_2 N v^0,\ v - v^0 \right\rangle \geqslant 0 \ \ \forall v \in U_{ad}\,. \tag{105}$$

Moreover, it can be proved the following result.

Lemma 3 *Let the hypothesis of Theorem 2 be satisfied. Then, for given* $z_{\Sigma d} \in H^{-5/2}\Xi^{-5/2}(\Sigma)$ *and any* $v \in L^2(\Sigma)$*, there exists a unique solution* $p(v) \in H^{3,3}(Q) \subset \Xi^{3,3}(Q)$ *to the problem (99)–(104) defined by transposition (88).*

The idea of the proof of the Theorem 4 is the same as in the case of the Theorem 3.

We must notice that the conditions of optimality derived above (Theorems 3 and 4) allow us to obtain an analytical formula for the optimal control in particular cases only (e.g. there are no constraints on boundary control). It results from the following: the determining of the function $p(x,t)$ in the maximum condition from the adjoint equation is possible if and only if we know that $y^0(x,t)$ will suit the control $v^0(x,t)$. These mutual connections make the practical use of the derived optimization formulas difficult. Thus we resign from the exact determining of the optimal control and we use approximation methods.

In the case of performance functionals (62) and (98) with $\lambda_1 > 0$ and $\lambda_2 = 0$, the optimal control problem reduces to the minimizing of the functional on a closed and convex subset in a Hilbert space. Then, the optimization problem is equivalent to a quadratic programming one [7, 10, 18] which can be solved by the use of the well-known algorithms, e.g. Gilbert's [4, 7, 10, 18] ones.

The practical application of Gilbert's algorithm to optimal control problem for a parabolic system with the boundary condition involving a time lag is presented in [7]. Using of the Gilbert's algorithm a one dimensional numerical example of the plasma control process is solved.

5. Conclusions and perspectives

The derived conditions of the optimality (Theorems 3 and 4) are original from the point of view of application of the Dubovicki-Milutin theorem [13] in solving optimal boundary control problems for second order hyperbolic systems in which different multiple time lags appear both in the state equations and in the Neumann boundary conditions.

The existence and uniqueness of solutions for such hyperbolic systems was presented – Lemma 1 and Theorem 2. The optimal control was characterized by using the adjoint equations – Lemmas 2 and 3. Necessary and sufficient conditions of optimality with the quadratic performance functionals (62) and (98) and constrained control (63) are derived for the Neumann problem – Theorems 3 and 4.

The proved optimization results (Theorems 3 and 4) constitute a novelty of the paper with respect to the references [8, 18] concerning application of the Lions scheme [19] for solving linear quadratic hyperbolic problems of optimal control.

Moreover, the optimization problems presented here constitute a generalization of optimal control problems considered in [17] for hyperbolic systems with constant time lags appearing in the state equations and in the boundary conditions simultaneously.

The obtained optimization theorems (Theorems 3 and 4) demand the assumption dealing with the non-empty interior of the set Q_2 representing the inequality constraints.

Therefore, we approximate the set Q_2 by the regular admissible cone (if $\text{int}Q_2 = \emptyset$, then this cone does not exist).

It is worth mentioning that the obtained results can be reinforced by omitting the assumption concerning the non-empty interior of the set Q_2 and utilizing the fact that the equality constraints in the form of the hyperbolic equations are "decoupling". The optimal control problem reduces to seeking $v^0 \in Q_2'$ and minimizing the performance index $I(v)$. Then we approximate the set Q_2' representing the inequality constraints by the regular tangent cone and for the performance index $I(v)$ we construct the regular improvement cone.

Making use of the Dubovicki-Milutin method the similar conditions of the optimality may be derived for a hyperbolic system with the Neumann boundary condition involving a time-varying lag.

The proposed methodology based on the Dubovicki-Milutin scheme can be presented as a specific case study concerning hyperbolic problems described by partial differential equations of the hyperbolic type including time lags appeared in the integral form both in the state equations and in the Neumann boundary conditions.

Another direction of research will be numerical examples concerning the determination of optimal control with constraints for multiple time delay hyperbolic systems.

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