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Research paper

Refined energy method for the elastic flexural-torsional buckling of steel H-section beam-columns Part I: Formulation and solution

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Abstract: Closed form solutions for the flexural-torsional buckling of elastic beam-columns may only be obtained for simple end boundary conditions, and the case of uniform bending and compression. Moment gradient cases need approximate analytical or numerical methods to be used. Investigations presented in this paper deal with the analytical energy method applied for any asymmetric transverse loading case that produces a moment gradient. Part I of this paper is devoted entirely to the theoretical investigations into the energy based out-of-plane stability formulation and its general solution. For the convenience of calculations, the load and the resulting moment diagram are presented as a superposition of two components, namely the symmetric and antisymmetric ones. The basic form of a non-classical energy equation is developed. It appears to be a function dependent upon the products of the prebuckling displacements (know from the prebuckling analysis) and the postbuckling deformation state components (unknowns enabling the formulation of the stability eigenproblem according to the linear buckling analysis). Firstly, the buckling state solution is sought by presenting the basic form of the non-classical energy equation in several variants being dependent upon the approximation of the major axis stress resultant M_y and the buckling minor axis stress resultant M_z. The following are considered: the classical energy equation leading to the linear eigenproblem analysis (LEA), its variant leading to the quadratic eigenproblem analysis (QEA) and the other non-classical energy equation forms leading to nonlinear eigenproblem analyses (NEA). The novel forms are those for which the stability equation becomes dependent only upon the twist rotation and its derivatives. Such a refinement is allowed for by using the second order out-of-plane bending differential equation through which the minor axis curvature shape is directly related to the twist rotation shape. Secondly, the effect of coupling of the in-plane and out-of-plane buckling forms is taken into consideration by introducing approximate second order bending relationships. The accuracy of the classical energy method of solving FTB problems is expected to be improved for both H- and I-section beam-columns. The outcomes of research presented in this part are utilized in Part II.

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1. Introduction

The energy approach for the evaluation of elastic FTB problems of beam-columns was studied by many authors by using the classical energy method (CEM). The classical energy equation is referred to such an equation that neglects the effect of prebuckling displacements and results in the LBA problem being formulated as a linear eigenproblem (as presented by Roik [20], Trahair [23]). Giżejowski et al. [8] presented the general solution of FTB problems using the linear eigenproblem analysis (LEA) and an arbitrary loading pattern.

A way for the refinement of the classical energy approach in the case of lateral-torsional buckling (LTB) of beams was shown by Timishenko and Gere [22], therefore referred to TEM (Timoshenko's Energy Method). The accuracy of classical approach of solving LTB problems is improved by making use of the minor axis in-plane first order bending differential equilibrium equation. This method, in combination with the classical energy equation was used by many authors to solve different LTB problems of bisymmetric and monosymmetric double-tee section beams and beam-columns, e.g. [3–5, 15, 16, 19]. Barszcz et al. [1] derived recently the energy equation of elastic thin-walled members and used it for solving LTB problems of beams in a non-classical form of TEM in which the effect of prebuckling displacements is taken into consideration. As far as FTB problems are concerned, it may be proved that the direct use of TEM refinement for beam-columns in the same way as that used for the classical form of energy approach for beams, and based on the second order minor axis in-plane bending differential equilibrium equation, fails to yield the solution consistent with that obtained for beams [2, 6] (when setting the axial force to zero value in the resultant energy equation). It is therefore important to derive a more accurate energy equation in which the nonlinear prebuckling stress resultant terms are retained.

A more accurate closed form solution of FTB of fork-supported beam-columns of double-tee bisymmetric sections takes the form given by Trahair et al. [24]:

$$\left(\frac{\sqrt{k_1 k_2}\, M_y}{M_{\mathrm{cr},0}}\right)^2 = \left(1 - \frac{N}{N_y}\right)\left(1 - \frac{N}{N_z}\right)\left(1 - \frac{N}{N_T}\right) \tag{1.1}$$

where: N – prebuckling axial stress resultant being constant along the member length, M_y – prebuckling major axis bending stress resultant being constant along the member length, $k_1 = 1 - \dfrac{EI_z}{EI_y}$ and $k_2 = 1 - \dfrac{1}{2}\dfrac{GI_T}{EI_y}\left(1 + \dfrac{EI_w}{GI_T}\dfrac{\pi^2}{L^2}\right)$ – factors accounting for the second order prebuckling effects, E, G – Young modulus and Kirchhoff modulus of steel, I_y, I_z, I_w, I_T – major and minor axis moment of inertia, warping constant and Saint Venant torsion constant, L – member length, $M_{cr,0} = i_0\sqrt{N_z N_T}$ – critical moment in the case of uniform bending, i_0 – polar radius of gyration, N_y, N_z, N_T – major axis, minor axis and torsional bifurcation forces in pure compression.

Many authors tried to omit the direct formulation of a more accurate energy equation. Solutions for non-uniform bending was available by using the Galerkin approximate method in solving FTB differential equilibrium problems. The starting point for such an approach is to establish the set of second order differential equilibrium equations. Mohri et al. [14] solved analytically the FTB problems of beam-columns subjected to simple and symmetric loading patterns using the Galerkin method. More general case of the beam-column loading pattern of narrow flange I-sections, namely being a combination of unequal end moments and the uniformly distributed load over the entire length, was considered by Bijak [5] for solving analytically the FTB problem. The Bubnov–Galerkin method was used. Cuk and Trahair [7] studied the narrow flange I-section beam-column FTB under unequal end moments, and recently it was also done by Gizejowski et al. [11]. The summary of solutions for the elastic LTB and FTB formulation based on the classical energy equation is presented in Trahair et al. [24], Gizejowski and Uziak [12] and Gizejowski et al. [10].

The classical energy approach to solve elastic flexural-torsional problems of beam-columns subjected to load patterns dependent upon a single load parameter was used in [8]. Such an approach may represent the loads different on both half-lengths of the member. Investigations presented herein are a continuation of previously conducted research. In the energy method presented in this paper, the non-classical energy equation valid for H-section beam-columns, like hot-rolled steel HEB sections, affected by the problem addressed in the article is formulated first. The equation is valid also for narrow flange I-shape sections, like hot-rolled IPE sections. In order to be consistent with the energy solution developed earlier for beams [1], the higher order terms are retained in the flexural components of the strain equations. As a result, the developed non-classical energy equation term, associated with the prebuckling moment M_y, becomes a sum of two products, the first being the product of twist rotation ϕ_x and curvature v_0'' of the out-of-plane displacement state, while the second being the product of minor axis moment M_z at buckling and twist rotation ϕ_x. Since the single term of the second order derivative of the minor axis displacement and the twist rotation in CEM is replaced by two components, the proposed approach is referred hereafter to the refined energy method (REM). The integral terms of general CEM and REM solutions are distinguished hereafter by adopting the subscripts cem and rem, accordingly.

2. Elastic flexural-torsional buckling formulation

2.1. Refined energy equation

The starting point for strain components evaluation of the nonlinear buckling problem of thin-walled beam-columns is the formulation presented by Gizejowski et al. [13]. The displacement field relationship developed there enabled to express coupling between the torsion deformation state and those of bending and axial states, therefore extending the formulation of Pi and Bradford in [17, 18] which neglected the effect of axial deformations on the torsion state. In the present study, the first step is to formulate the rotation matrix in the member deflected configuration, assuming small rotations in that configuration. This allows to write an approximate, general matrix relationship for the displacement field in the deflected configuration as:

$$\begin{bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{bmatrix} = \boldsymbol{R} \begin{bmatrix} \mathrm{d}x \\ y \\ z \end{bmatrix} - \begin{bmatrix} \mathrm{d}x + \omega\kappa_x(x) \\ y \\ z \end{bmatrix} \tag{2.1}$$

where: $u(x,y,z)$, $v(x,y,z)$, $w(x,y,z)$ – displacements u, v, w of a member point identified by the coordinates x, y, z in the deflected configuration, ω – sectional warping coordinate, $\kappa_x(x) = (\phi_x)' - x$ – coordinate dependent twist along the axis indicated by the subscript symbol; $f(\ldots)$ – variable f being a function of selected arguments listed in the round bracket that indicate the coordinates of adopted Cartesian system (in the following, the arguments are dropped for the convenience of notation), $(f)' = \dfrac{\mathrm{d}f}{\mathrm{d}x}$.

The rotation matrix $\boldsymbol{R}$ defined for the deflected configuration in reference to the fixed Cartesian coordinate system in the initial configuration takes the form:

$$\boldsymbol{R} = \begin{bmatrix} \sqrt{1+2e} & -\phi_z & \phi_y \\ \phi_z & 1 - \frac{1}{2}(\phi_x)^2 & -\phi_x \\ -\phi_y & \phi_x & 1 - \frac{1}{2}(\phi_x)^2 \end{bmatrix} \tag{2.2}$$

where: ϕ_x, ϕ_y, ϕ_z – angles of rotation in the deflected configuration with reference to the initial configuration, $\sqrt{1+2e}\,\mathrm{d}x \approx (1+e)\,\mathrm{d}x$ – fiber length measured along the x-axis in the deflected configuration, $e = u_0' + \frac{1}{2}\phi^2$ – normal strain measure of the section fiber including the bowing effect, $\phi = \sqrt{\left(\dfrac{\partial v}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial x}\right)^2}$, u_0 – displacement along the member axis.

The vector of rotation angles in the deflected configuration may be related to those in the initial configuration through the cosines direction matrix $\boldsymbol{T_{R\theta}}$ as follows:

$$\begin{bmatrix} \phi_x \\ \phi_z \\ \phi_y \end{bmatrix} = \begin{bmatrix} \theta_x \\ 0 \\ 0 \end{bmatrix} + \boldsymbol{T_{R\theta}} \begin{bmatrix} \theta_z \\ \theta_y \end{bmatrix} \tag{2.3}$$

where: $\boldsymbol{T_{R\theta}}$ – direction cosines matrix

$$\boldsymbol{T_{R\theta}} = \begin{bmatrix} -\frac{1}{2}\theta_y & \frac{1}{2}\theta_z \\ \cos\phi_x & -\sin\phi_x \\ \sin\phi_x & \cos\phi_x \end{bmatrix} \tag{2.4}$$

where: θ_x – angle of twist rotation, θ_y, θ_z – angles of flexural rotations about the member axis along y and z, respectively, $\theta_y = \arctan\left(\dfrac{-w_0'}{1+u_0'}\right)$, $\theta_z = \arctan\left(\dfrac{v_0'}{1+u_0'}\right)$, v_0, w_0 – displacements of the member axis along y and z coordinates, respectively.

After decomposing the square rotation matrix $\boldsymbol{R}$ into two components, namely into the vector corresponding to the rigid rotation of $\mathrm{d}x$ and the rectangular rotation matrix component $\boldsymbol{R}_{yz}$ corresponding to the section coordinates (y, z), the displacement field may be expressed as follows

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u_0' + \frac{1}{2}\phi^2 - \omega\kappa_x' \\ \theta_z \boxed{\cos\phi_x} - \theta_y \boxed{\sin\phi_x} \\ -\left(\theta_y \boxed{\cos\phi_x} + \theta_z \boxed{\sin\phi_x}\right) \end{bmatrix} \mathrm{d}x + \boldsymbol{R}_{yz} \begin{bmatrix} y \\ z \end{bmatrix} \tag{2.5}$$

The submatrix $\boldsymbol{R}_{yz}$ of the rotation matrix $\boldsymbol{R}$ takes the following form

$$\boldsymbol{R}_{yz} = \begin{bmatrix} -\left(\theta_z \boxed{\cos\phi_x} - \theta_y \boxed{\sin\phi_x}\right) & \left(\theta_y \boxed{\cos\phi_x} + \theta_z \boxed{\sin\phi_x}\right) \\ -\frac{1}{2}\left[\theta_x + \frac{1}{2}\left(\theta_y \boxed{\theta_z} - \boxed{\theta_y} \theta_z\right)\right]^2 & -\left[\theta_x + \frac{1}{2}\left(\theta_y \boxed{\theta_z} - \boxed{\theta_y} \theta_z\right)\right] \\ \left[\theta_x + \frac{1}{2}\left(\theta_y \boxed{\theta_z} - \boxed{\theta_y} \theta_z\right)\right] & -\frac{1}{2}\left[\theta_x + \frac{1}{2}\left(\theta_y \boxed{\theta_z} - \boxed{\theta_y} \theta_z\right)\right]^2 \end{bmatrix} \tag{2.6}$$

where all the framed quantities in this final displacement field relationship are the terms belonging to the direction cosines matrix $\boldsymbol{T}_{R\theta}$.

Barszcz et al. [1] presented the formulation based on the rotation matrix developed by Pi and Bradford in [17, 18] with reference to LBA buckling problems of thin-walled members. Comparison of the displacement field relationship developed in [1] and that presented herein yields the conclusion that both formulations are identical as far as LBA problems of thin-walled members are concerned. The difference between the above formulation, presented also in [13], and that developed by Pi and Bradford in [17, 18] may arise only when the nonlinear buckling problems (NBA) of thin-walled members are dealt with.

The solutions presented hereafter are related to LBA problems of thin-walled beam-columns in which the expression for the total potential energy formulation is the sum of the strain energy U being a sum of U_L (based on the linear terms of Green strain components, both normal and shear) and U_{NL} (based on the nonlinear term of the Green normal strain component), and the negative work of applied loads V:

$$\Pi = U_L + U_{NL} - V \tag{2.7}$$

In-plane transverse loads and/or end moments generate the in-plane bending moment M_y and the axial force N, compressive or tensile. Maintaining all the important terms of strain energy for a general case of the FTB of thin-walled beam-columns of the I-bisymmetric open cross-section, then neglecting the terms of higher order then 2 in relation to the postbuckling out-of-plane deformation state components and carrying out the required calculations for the out-of-plane LBA, the energy components of the total potential energy Π of H- and I-section beam-columns, under the loads of different values in two half-lengths but of symmetrical placement with reference to the mid-section, may be presented in compression and bending as follows:

(2.8)
$$U_L = \frac{1}{2}\int_0^L \Big[EI_z\,(v_0'')^2 - (1 - I_z/I_y)\,EI_y w_0''\phi_x\,(2v_0'' - w_0''\phi_x) + EI_w\,(\phi_x'')^2 + GI_T\,(\phi_x')^2\Big]\,\mathrm{d}x$$

(2.9)
$$U_{NL} = -\frac{1}{2}EAu_0'\int_0^L \Big[(v_0')^2 + i_0^2\,(\phi_x')^2\Big]\,\mathrm{d}x$$

(2.10)
$$V = \frac{1}{2}\Bigg\{\sum_i \int_{x_{q1,i}}^{x_{q2,i}} q_{z,i}z_{q,i}\left[\phi_{x,i}(x)\right]^2 \mathrm{d}x + \sum_j Q_{z,j}z_{Q,j}\left[\phi_{x,j}\left(x_{Q,j}\right)\right]^2\phi_{x,j}^2 + \sum_i \int_{L-x_{q2,i}}^{L-x_{q1,i}} \psi_q q_{z,i}z_{q,i}\left[\phi_{x,i}(x)\right]^2 \mathrm{d}x + \sum_j \psi_Q Q_{z,j}z_{Q,j}\left[\phi_{x,j}\left(L - x_{Q,j}\right)\right]^2\Bigg\}$$

where: A – cross-sectional area; remaining symbols according to Fig. 1, where $q_{z,i}$, $Q_{z,j}$ – load components in the first half-length of the member while $\psi_q q_{z,i}$, $\psi_Q Q_{z,j}$ – corresponding load components for the second half-length; the shear centre displacements correspond to the right hand rule coordinate system axes x, y and z (axis y is perpendicular to the figure plane).

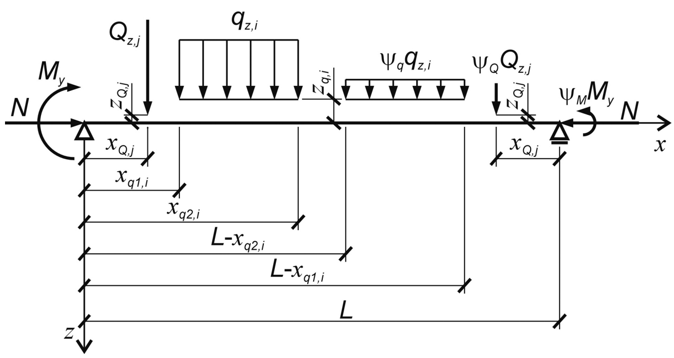


Fig. 1. Coordinate system adopted and general loading pattern in the undeflected configuration

Let us consider the definitions of the axial force $N = EAu_0'$ and the major axis bending moment $M_y = -EI_y w_0''$, the curvature of which is identified by the second derivative of the prebuckling displacement component w_0 and defined as follows:

(2.11)
$$w_0'' = -\frac{M_y}{EI_y}$$

Moreover, let us introduce the direct definition of the out-of-plane moment at the member buckled position:

$$M_z = EI_z w_0''\phi_x \;\rightarrow\; w_0''\phi_x = \frac{M_z}{EI_z} \tag{2.12}$$

Substituting Eqs. (2.11) and (2.12) to the energy equation (2.7), with contributions given by Eqs. (2.8)÷(2.10), results in the non-classical energy equation representing the stationary condition of the total potential energy at buckling:

$$\begin{aligned}
&\frac{1}{2}\int_0^L \Big\{ EI_z\delta\left[(v_0'')^2\right] + EI_w\delta\left[(\phi_x'')^2\right] + GI_T\delta\left[(\phi_x')^2\right] - N\delta\left[(v_0')^2 + i_0^2(\phi_x')^2\right] \\
&\quad + k_1 M_y\delta\left(2v_0''\phi_x - \frac{M_z}{EI_z}\phi_x\right)\Big\}\,\mathrm{d}x + \frac{1}{2}\Big\{\sum_i \int_{x_{q1,i}}^{x_{q2,i}} q_{z,i}z_{q,i}\delta\left[\phi_{x,i}(x)\right]^2 \mathrm{d}x \\
&\quad + \sum_j Q_{z,j}\, z_{Q,j}\delta\left[\phi_{x,j}\left(x_{Q,j}\right)\right]^2 \phi_{x,j}^2 + \sum_i \int_{L-x_{q2,i}}^{L-x_{q1,i}} \psi_q q_{z,i}z_{q,i}\delta\left[\phi_{x,i}(x)\right]^2 \mathrm{d}x \\
&\quad + \sum_j \psi_Q Q_{z,j}z_{Q,j}\delta\left[\phi_{x,j}\left(L - x_{Q,j}\right)\right]^2\Big\} = 0
\end{aligned} \tag{2.13}$$

In the following, beam-columns with simple member natural boundary conditions are dealt with. The shape functions used hereafter, approximating the out-of-plane displacement and twist rotation are of the well known format that has also been used in the previous studies [1,5,8,17,23]. The energy equation may therefore be expressed in terms of unknown buckled shape constants in relation to the post-buckling displacements:

$$v_0 = v_{0s} + v_{0a} \tag{2.14}$$

in which $v_{0s} = a_1 \sin(\pi\xi)$, $v_{0a} = a_2 \sin(2\pi\xi)$
and

$$\phi_x = a_3 \sin(\pi\xi) \tag{2.15}$$

where: ξ – dimensionless coordinate equal to x/L, a_1, a_2 and a_3 – unknown buckled shape constants.

Eq. (2.13) is called hereafter as a refined energy equation that provides a basis for the eigenproblem formulation of FTB problems. Depending on the assumption used, the major axis moment M_y might be approximated using M_y^{II} or M_y^{I}, depending on whether the obtained solution accounts for the in-plane second order effect on the out-of-plane bifurcation state, or not. The minor axis moment M_z might be approximated by using M_z^{II} or M_z^{I}, or even $M_z = 0$, depending on whether all the second order minor axis effects are accounted for in the out-of-plane bifurcation state, or not.

2.2. Summary of solutions based on linear or quadratic eigenproblem formulations

The classical formulation is based on ignoring the prebuckling displacements on the buckling state ($k_1 = 1$), while the major axis moment M_y is approximated by M_y^{I} and the minor axis moment M_z is approximated by either its zero value (leading to the linear eigenproblem analysis, LEA) or taking it as M_z^{I} (leading to the quadratic linear analysis, QEA) as shown hereafter. Therefore, in both LEA and QEA formulations, the effect of axial compression on the minor axis bending state is neglected.

2.2.1. Solution based on LEA

Taking $k_1 = 1$, and approximating the energy equation by taking $M_z = 0$ and $M_y = M_y^{\mathrm{I}}$ in Eq. (2.13), results in the classical energy equation that has been widely used in the development of bifurcation solutions of variety of simple and more complex loading conditions, e.g. Trahair [23]. Such an energy equation is dependent linearly upon the in-plane moment, the axial force and the in-plane load acting away from the shear centre:

$$\begin{aligned}(2.16)\quad & \frac{1}{2}\int_0^L \left\{EI_z\delta\left[\left(v_0''\right)^2\right] + EI_w\delta\left[\left(\phi_x''\right)^2\right] + GI_T\delta\left[\left(\phi_x'\right)^2\right] + 2M_y^{\mathrm{I}}\delta\left(v_0''\phi_x\right)\right\}\mathrm{d}x \\ & - \frac{1}{2}N\int_0^L \delta\left[\left(v_0'\right)^2 + i_0^2\left(\phi_x'\right)^2\right]\mathrm{d}x \\ & + \frac{1}{2}\left\{\sum_i \int_{x_{q1,i}}^{x_{q2,i}} q_{z,i}z_{q,i}\delta\left[\phi_{x,i}(x)\right]^2\mathrm{d}x + \sum_j Q_{z,j}z_{Q,j}\delta\left[\phi_{x,j}\left(x_{Q,j}\right)\right]^2\phi_{x,j}^2\right. \\ & \left. + \sum_i \int_{L-x_{q2,i}}^{L-x_{q1,i}} \psi_q q_{z,i}z_{q,i}\delta\left[\phi_{x,i}(x)\right]^2\mathrm{d}x + \sum_j \psi_Q Q_{z,j}z_{Q,j}\delta\left[\phi_{x,j}\left(L - x_{Q,j}\right)\right]^2\right\} = 0\end{aligned}$$

The above equation is the basic relationship used in CEM, see Gizejowski et al [8]. The direct use of such an equation leads to the linear eigenproblem analysis (LEA). Using displacement field approximation given by Eqs. (2.14) and (2.15), the LEA closed form solution results from the eigenproblem involving the stiffness stability matrix $\boldsymbol{K}$ of size 3×3. The solution yields the following stability criterion [8]:

$$(2.17)\qquad K(3,3) - \left\{\frac{[K(1,3)]^2}{K(1,1)} + \frac{[K(2,3)]^2}{K(2,2)}\right\} = 0$$

in which $K(m,n)$ are the terms of the stiffness matrix $\boldsymbol{K}$, multiplied by $\frac{2L}{\pi^2}$ and associated with the vector of unknown buckled shape constants The terms of Eq. (2.17) are as follows:

$$(2.18)\qquad K(3,3) = i_0^2 N_T\left(1 - \frac{N}{N_T}\right)\zeta$$

$$\frac{[K(1,3)]^2}{K(1,1)} = \frac{\left(M_{y,s,\max} 2I_s\right)^2}{N_z\left(1-\dfrac{N}{N_z}\right)} \tag{2.19}$$

$$\frac{[K(2,3)]^2}{K(2,2)} = \frac{\left(4M_{y,a,\max} 2I_a\right)^2}{4N_{za}\left(1-\dfrac{N}{N_{za}}\right)} \tag{2.20}$$

where: $N_T = \left(\pi^2 EI_w/L^2 + GI_T\right)/i_0^2$, $N_z = \pi^2 EI_z/L^2$, N_{za} – second lowest bifurcation load equal to $4N_z$,

$$\zeta = 1 + \frac{C_{bF} z_F}{i_0^2 N_T\left(1-\dfrac{N}{N_T}\right)},$$

$$C_{bF} z_F = \begin{cases} \dfrac{2L^2}{\pi^2}\displaystyle\sum_i\left[\int\limits_{\xi_{q1,i}}^{\xi_{q2,i}} q_{z,i} z_{q,i}\sin^2(\pi\xi)\,\mathrm{d}\xi + \int\limits_{1-\xi_{q2,i}}^{1-\xi_{q1,i}} \psi_{q,i} q_{z,i} z_{q,i}\sin^2(\pi\xi)\,\mathrm{d}\xi\right] & \text{for distributed loads} \\ \dfrac{2L}{\pi^2}\displaystyle\sum_j\left\{Q_{z,j} z_{Q,j}\sin^2(\pi\xi_j) + \psi_{Q,j} Q_{z,j} z_{Q,j}\sin^2\left[\pi\left(1-\xi_j\right)\right]\right\} & \text{for concentrated loads} \end{cases}$$

$M_{y,s,\max}$, $M_{y,a,\max}$ – maximum absolute values of symmetric and antisymmetric moment components, scaling the elementary action field moments $M_{y,s}(\xi)$ and $M_{y,a}(\xi)$, respectively,

$$I_s = \int\limits_0^1 \frac{M_{y,s}(\xi)}{M_{y,s,\max}}\sin^2(\pi\xi)\,\mathrm{d}\xi \text{ – symmetric moment integral,}$$

$$I_a = \int\limits_0^1 \frac{M_{y,a}(\xi)}{M_{y,a,\max}}\sin(\pi\xi)\sin(2\pi\xi)\,\mathrm{d}\xi \text{ – antisymmetric moment integral.}$$

The term $\dfrac{[K(1,3)]^2}{K(1,1)}$ in Eq. (2.17) represents a decrease in the potential energy equal to the work done by the first order field moment component $M_{y,s}(\xi)$ as the element deflects w_0 due to the combined effects of the lateral deflection v_0 and twist rotation ϕ_x. Similarly, the term $\dfrac{[K(2,3)]^2}{K(2,2)}$ represents the work done by the first order field moment component $M_{y,a}(\xi)$.

Finally, the LEA solution takes the form:

$$\left(\frac{M_{y,\max}}{C_{bc} M_{cr,0}}\right)^2 \frac{1}{F_2(N)} = 1 \tag{2.21}$$

where: $M_{y,\max}$ – maximum moment under the considered loading system, C_{bc} – factor converting an arbitrary moment gradient case into an equivalent uniform moment case,

$F_2(N) = \left(1 - \dfrac{N}{N_z}\right)\left(1 - \dfrac{N}{N_T}\right)$ – coefficient representing the effect of out-of-plane buckling under compressive force on the LTB buckling moment.

The equivalent uniform moment factor C_{bc} is varying with the minor axis critical force utilization ratio N/N_z as given below:

$$\frac{1}{C_{bc}} = \frac{1}{\sqrt{\zeta}}\left[\left(\frac{M_{y,s,\max}}{M_{y,\max}}\frac{1}{C_{bs,\text{cem}}}\right)^2 + \frac{1-\dfrac{N}{N_z}}{1-\dfrac{N}{N_{za}}}\left(\frac{M_{y,a,\max}}{M_{y,\max}}\frac{1}{C_{ba,\text{cem}}}\right)^2\right]^{0.5} \tag{2.22}$$

where: $\dfrac{1}{C_{bs,\text{cem}}} = 2I_s$ and $\dfrac{1}{C_{ba,\text{cem}}} = 2I_a$ – conversion factors for the symmetric and antisymmetric moment diagram components.

Eq. (2.17) with its C_{bc} factor evaluated from (2.22) is of the same form as presented by Giżejowski et al. [8].

2.2.2. Solution based on QEA

The solution may be obtained in the form of QEA when k_1 =1 in Eq. (2.13), the minor axis moment $M_z = M_z^{\text{I}} = -M_y^{\text{I}}\phi$. The major axis curvature is evaluated from the second order differential equilibrium equation:

$$v_0'' = -\frac{M_y^{\text{I}}\phi + Nv_0}{EI_z} \tag{2.23}$$

As a result, the energy equation (2.13) is dependent nonlinearly upon the in-plane stress resultants, while it remains dependent linearly upon the in-plane load components acting away from the shear centre:

$$\begin{aligned} &\frac{1}{2}\int_0^L \Bigg\{EI_z\delta\left[(v_0'')^2\right] + EI_w\delta\left[(\phi_x'')^2\right] + GI_T\delta\left[(\phi_x')^2\right] \\ &- M_y^{\text{I}}\delta\left(\frac{M_y^{\text{I}}\phi_x^2}{EI_z} + N\frac{2v_0\phi_x}{EI_z}\right)\Bigg\}\,\mathrm{d}x - \frac{1}{2}N\int_0^L \left\{\delta\left[(v_0')^2 + i_0^2(\phi_x')^2\right]\right\}\mathrm{d}x \\ &+ \frac{1}{2}\Bigg\{\sum_i \int_{x_{q1,i}}^{x_{q2,i}} q_{z,i}z_{q,i}\delta\left[\phi_{x,i}(x)\right]^2\mathrm{d}x + \sum_j Q_{z,j}z_{Q,j}\delta\left[\phi_{x,j}(x_{Q,j})\right]^2\phi_{x,j}^2 \\ &+ \sum_i \int_{L-x_{q2,i}}^{L-x_{q1,i}} \psi_q q_{z,i}z_{q,i}\delta\left[\phi_{x,i}(x)\right]^2\mathrm{d}x + \sum_j \psi_Q Q_{z,j}z_{Q,j}\delta\left[\phi_{x,j}(L-x_{Q,j})\right]^2\Bigg\} = 0 \end{aligned} \tag{2.24}$$

The energy equation (2.24) becomes dependent quadratically upon the in-plane stress resultants, therefore its use leads to the quadratic eigenproblem analysis (QEA). The solution of energy equation (2.24) yields the stability criterion of the same format as Eq. (2.17) but with different structure of the stiffness matrix terms $K(m,n)$ as far as the terms $K(3,3)$ and off-diagonal terms for $m \neq n$ are concerned. As a result, the following holds:

$$K(3,3) = i_0^2 N_T \left(1 - \frac{N}{N_T}\right)\zeta - \frac{(M_{y,\max})^2}{N_z}\left[\left(\frac{M_{y,s,\max}}{M_{y,\max}}\right)^2 2J_s + \left(\frac{M_{y,a,\max}}{M_{y,\max}}\right)^2 2J_a\right] \tag{2.25}$$

$$\frac{[K(1,3)]^2}{K(1,1)} = \left(\frac{N}{N_z}\right)^2 \frac{(M_{y,s,\max} 2I_s)^2}{N_z\left(1 - \dfrac{N}{N_z}\right)} \tag{2.26}$$

$$\frac{[K(2,3)]^2}{K(2,2)} = \left(\frac{N}{N_z}\right)^2 \frac{(M_{y,a,\max} 2I_a)^2}{4N_{za}\left(1 - \dfrac{N}{N_{za}}\right)} \tag{2.27}$$

where: $J_s = \int\limits_0^1 \left[\frac{M_{y,s}(\xi)}{M_{y,s,\max}}\right]^2 \sin^2(\pi\xi)\,\mathrm{d}\xi$, $J_a = \int\limits_0^1 \left[\frac{M_{y,a}(\xi)}{M_{y,a,\max}}\right]^2 \sin^2(\pi\xi)\,\mathrm{d}\xi$ and the other unexplained symbols are the same as in Eq. (2.17).

Finally, the solution of Eq. (2.24) takes the form given by Eq. (2.21) in which:

$$\frac{1}{C_{bc}} = \frac{1}{\sqrt{\zeta}}\left\{\left(\frac{M_{y,s,\max}}{M_{y,\max}}\right)^2\left[\left(1 - \frac{N}{N_z}\right)\left(\frac{1}{C_{bs,\mathrm{rem}}}\right)^2 + \left(\frac{N}{N_z}\frac{1}{C_{bs,\mathrm{cem}}}\right)^2\right] + \left(\frac{M_{y,a,\max}}{M_{y,\max}}\right)^2\left[\left(1 - \frac{N}{N_z}\right)\left(\frac{1}{C_{ba,\mathrm{rem}}}\right)^2 + \frac{1 - \dfrac{N}{N_z}}{1 - \dfrac{N}{N_{za}}}\left(\frac{N}{N_{za}}\frac{1}{C_{ba,\mathrm{cem}}}\right)^2\right]\right\}^{0.5} \tag{2.28}$$

where: $\frac{1}{C_{bs,\mathrm{rem}}} = \sqrt{2J_s}$ and $\frac{1}{C_{ba,\mathrm{rem}}} = \sqrt{2J_a}$ – conversion factors for the symmetric and antisymmetric moment diagram components.

The solution based on Eq. (2.24) with its C_{bc} factor evaluated from (2.28) seems to be similar to that given by Bijak [5]. Differences between the Bijak's solution and that of present study arise from the power of N/N_z before the conversion factor $1/C_{bs,\mathrm{cem}}$ squared and the power of N/N_{za} before $1/C_{ba,\mathrm{cem}}$ squared:

$$\frac{1}{C_{bc}} = \frac{1}{\sqrt{\zeta}}\left\{\left(\frac{M_{y,s,\max}}{M_{y,\max}}\right)^2\left[\left(1 - \frac{N}{N_z}\right)\left(\frac{1}{C_{bs,\mathrm{rem}}}\right)^2 + \frac{N}{N_z}\left(\frac{1}{C_{bs,\mathrm{cem}}}\right)^2\right] + \left(\frac{M_{y,a,\max}}{M_{y,\max}}\right)^2\left[\left(1 - \frac{N}{N_z}\right)\left(\frac{1}{C_{ba,\mathrm{rem}}}\right)^2 + \frac{1 - \dfrac{N}{N_z}}{1 - \dfrac{N}{N_{za}}}\frac{N}{N_{za}}\left(\frac{1}{C_{ba,\mathrm{cem}}}\right)^2\right]\right\}^{0.5} \tag{2.29}$$

Eq. (2.29) contains the terms that are dependent linearly upon N/N_z and N/N_{za} while those terms in Eq. (2.28) are being squared. For $N = 0$, both equations convert to that yielding from the classical TEM.

In the following subsection, novel solutions are being looked at by using the second order moments of $M_y = M_y^{\mathrm{II}}$ and $M_z = M_z^{\mathrm{II}}$, therefore considering all the other effects being neglected in the energy equations presented in this section.

2.3. Summary of novel solutions based on the higher order eigenproblem formulations

When there are used $M_z = M_z^{\mathrm{II}}$ and the minor axis curvature is determined from the differential second order equilibrium equation, the energy equation becomes dependent nonlinearly upon the in-plane stress resultants, therefore its use leads to the nonlinear eigenproblem analysis (NEA).

The nonlinear minor axis moment term in Eq. (2.13) takes the form:

$$\frac{M_z}{EI_z} = \frac{M_z^{\mathrm{II}}}{EI_z} = -\frac{M_y^{\mathrm{I}}\phi + Nv_0}{EI_z} \tag{2.30}$$

while the term associated with M_y in the same equation (2.13) expressed in the postbuckling displacement field components (v_0, ϕ) yields:

$$2v_0''\phi = -2\frac{M_z^{\mathrm{II}}\phi}{EI_z} = -2\frac{M_y^{\mathrm{I}}\phi + Nv_0}{EI_z}\phi \tag{2.31}$$

As a result, the non-classical refined second-order energy equation takes the form:

$$\begin{aligned}
&\frac{1}{2}\int_0^L \Bigg\{EI_z\delta\left[\left(v_0''\right)^2\right] + EI_w\delta\left[\left(\phi_x''\right)^2\right] + GI_T\delta\left[\left(\phi_x'\right)^2\right] \\
&\quad - k_1\delta\left[M_y^{\mathrm{II}}\left(\frac{M_z^{\mathrm{II}}\phi_x}{EI_z}\right)\right]\Bigg\}\mathrm{d}x - \frac{1}{2}N\int_0^L\left\{\delta\left[\left(v_0'\right)^2 + i_0^2\left(\phi_x'\right)^2\right]\right\}\mathrm{d}x \\
&\quad + \frac{1}{2}\Bigg\{\sum_i \int_{x_{q1,i}}^{x_{q2,i}} q_{z,i}z_{q,i}\delta\left[\phi_{x,i}(x)\right]^2\mathrm{d}x + \sum_j Q_{z,j}z_{Q,j}\delta\left[\phi_{x,j}\left(x_{Q,j}\right)\right]^2\phi_{x,j}^2 \\
&\quad + \sum_i \int_{L-x_{q2,i}}^{L-x_{q1,i}} \psi_q q_{z,i}z_{q,i}\delta\left[\phi_{x,i}(x)\right]^2\mathrm{d}x + \sum_j \psi_Q Q_{z,j}z_{Q,j}\delta\left[\phi_{x,j}\left(L-x_{Q,j}\right)\right]^2\Bigg\} = 0
\end{aligned} \tag{2.32}$$

The accuracy of FTB solutions based on Eq. (2.32) is dependent upon the approximations used in evaluation of the in-plane moment M_y^{II} and the out-of-plane moment M_z^{II} at buckling. Table 1a presents the summary of different options in approximating the moment M_y^{II}, identified by the Roman number I (where I–A used for the second order moment amplification approximation of $M_y^{\mathrm{II}} = M_{y,\mathrm{amp}}^{\mathrm{II}}$ and I–B for the second order moment $P-\delta$ approximation of $M_y^{\mathrm{II}} = M_{y,P-\delta}^{\mathrm{II}}$). Additionally, $N_y = \pi^2 EI_y/L^2$, N_{ya} – second lowest bifurcation load equal to $4N_y$.

Table 1a. Approximations of the second order in-plane field moment M_y^{II}

Symbol		Field moment equation
I	A	$M_{y,\mathrm{amp}}^{\mathrm{II}} = \dfrac{M_{y,s}^{\mathrm{I}}}{1 - \dfrac{N}{N_y}} + \dfrac{M_{y,a}^{\mathrm{I}}}{1 - \dfrac{N}{N_{ya}}}$
	B	$M_{y,P-\delta}^{\mathrm{II}} = M_{y,s}^{\mathrm{I}} + \dfrac{N w_s^{\mathrm{I}}}{1 - \dfrac{N}{N_y}} + M_{y,a}^{\mathrm{I}} + \dfrac{N w_a^{\mathrm{I}}}{1 - \dfrac{N}{N_{ya}}}$

where:
$M_{y,s}^{\mathrm{I}} = M_{y,s,\mathrm{supp}}^{\mathrm{I}} + M_{y,s,\mathrm{span}}^{\mathrm{I}}$ – symmetric component of the field moment,
$M_{y,s,\mathrm{supp}}^{\mathrm{I}}$ – field moment produced by the symmetric component of support applied moments,
$M_{y,s,\mathrm{span}}^{\mathrm{I}}$ – field moment produced by the symmetric component of span loads,
$M_{y,a}^{\mathrm{I}} = M_{y,a,\mathrm{supp}}^{\mathrm{I}} + M_{y,a,\mathrm{span}}^{\mathrm{I}}$ – antysymmetric component of the field moment,
$M_{y,a,\mathrm{supp}}^{\mathrm{I}}$ – field moment produced by the antisymmetric component of support moments,
$M_{y,a,\mathrm{span}}^{\mathrm{I}}$ – field moment produced by the antisymmetric component of span loads,
$w_s^{\mathrm{I}} = \delta_{z,s}^{\mathrm{I}} \sin(\pi\xi)$, $\delta_{z,s}^{\mathrm{I}}$ – amplitude of the first order single curvature deflected profile for symmetric field in-plane moment component,
$w_a^{\mathrm{I}} = \delta_{z,a}^{\mathrm{I}} \sin(2\pi\xi)$, $\delta_{z,a}^{\mathrm{I}}$ – amplitude of the first order double curvature deflected profile for antisymmetric field in-plane moment component.

The accuracy of approximate relationships might be checked by comparison of their maximum moments with that yielding from the exact relationship in meaning of the second order bending theory. Investigations into the accuracy of the in-plane moment M_y approximations, for the case of unequal end moments were carried out by Gizejowski and Stachura [9] and verified by using an exact second order relationship [24]:

$$M_y^{\mathrm{II}} = M_{yM,\max} \left\{ \cos\left(\pi\sqrt{\frac{N}{N_y}}\xi\right) + \left[\psi_M \operatorname{cosec}\left(\pi\sqrt{\frac{N}{N_y}}\right) - \cot\left(\pi\sqrt{\frac{N}{N_y}}\right)\right] \sin\left(\pi\sqrt{\frac{N}{N_y}}\xi\right)\right\} \tag{2.33}$$

where: $\psi_M = M_{yM,\min}/M_{yM,\max}$ – moment gradient ratio ($M_{yM,\min}$ and $M_{yM,\max}$ applied support moments).

Based on the results of investigations presented in [9], one may conclude that options I–A and I–B for the approximation of second order in-plane moments are rather close to each other as far as the maximum moment is considered. Considering the whole range of the axial force utilization parameter N/N_y, the solution of FTB problems based on option I–A seems to be preferable for design applications. The options mentioned above are considered for the investigations presented hereafter.

2.4. General solution of non-classical second order energy equation

Since the nonlinear term of minor axis curvature may be evaluated by its decomposition, the buckling minor axis second moment may be approximated with use of equations given in Table 1b.

Table 1b. Approximations of the second order out-of-plane field moment M_z^{II}

Symbol		Field moment equation
II	A	$M_{z,\mathrm{amp}}^{\mathrm{II}} = \dfrac{M_{z,s}^{\mathrm{I}}}{1-\dfrac{N}{N_z}} + \dfrac{M_{z,a}^{\mathrm{I}}}{1-\dfrac{N}{N_{za}}}$
	B	$M_{z,P-\delta}^{\mathrm{II}} = M_{z,s}^{\mathrm{I}} + \dfrac{Nv_s^{\mathrm{I}}}{1-\dfrac{N}{N_z}} + M_{z,a}^{\mathrm{I}} + \dfrac{Nv_a^{\mathrm{I}}}{1-\dfrac{N}{N_{za}}}$

where:
$M_{z,\mathrm{amp}}^{\mathrm{II}}$, $M_{y,P-\delta}^{\mathrm{II}}$ – approximations of the second order out-of-plane moments based on the amplification rule and the $P-\delta$ rule, respectively,
$M_{z,s}^{\mathrm{I}} = M_{y,s}^{\mathrm{I}}\phi$, $M_{z,a}^{\mathrm{I}} = M_{y,a}^{\mathrm{I}}\phi$ – first order out-of-plane moments,
$v_s^{\mathrm{I}} = \delta_{y,s}^{\mathrm{I}}\sin(\pi\xi)$, $\delta_{y,s}^{\mathrm{I}} = \delta_{z,s}^{\mathrm{I}}\phi$ – amplitude of the first order single curvature deflected profile for symmetric field out-of-plane moment component,
$v_a^{\mathrm{I}} = \delta_{y,a}^{\mathrm{I}}\sin(2\pi\xi)$, $\delta_{y,a}^{\mathrm{I}} = \delta_{z,a}^{\mathrm{I}}\phi$ – amplitude of the first order double curvature deflected profile for antisymmetric field out-of-plane moment component.

Hence, the following approximations hold for option II–A and II–B from Table 1b:
– for Option II–A:

$$\frac{M_z^{\mathrm{II}}}{EI_z} = \frac{1}{EI_z}\left[\frac{M_{y,s}^{\mathrm{I}}}{1-\frac{N}{N_z}} + \frac{M_{y,a}^{\mathrm{I}}}{1-\frac{N}{N_{za}}}\right]\phi_x \tag{2.34}$$

– for Option II–B:

$$\frac{M_z^{\mathrm{II}}}{EI_z} = \frac{M_y^{\mathrm{II}}\phi_x}{EI_z} = \frac{1}{EI_z}\left[M_{y,s}^{\mathrm{I}} + \frac{N\delta_{z,s}^{\mathrm{I}}}{1-\dfrac{N}{N_z}}\sin(\pi\xi) + M_{y,a}^{\mathrm{I}} + \frac{N\delta_{z,a}^{\mathrm{I}}}{1-\dfrac{N}{N_{za}}}\sin(2\pi\xi)\right]\phi_x \tag{2.35}$$

The moment terms in the energy equation (2.32) are therefore dependent only upon one post-buckling displacement component, namely the twist rotation ϕ_x, instead of the product of minor axis curvature and twist rotation, like in the energy approaches presented in section 2.2. As a result, the refined second order energy equation may be formulated in two ways, presented in the following subsections.

2.4.1. Options 1 based on NEA

Option 1 in the versions 1a and 1b is based on Eq. (2.34), therefore Eq. (2.29) takes the form:

$$\begin{aligned}(2.36)\quad & \frac{1}{2}\int_0^L \Big\{ EI_z\delta\left[(v_0'')^2\right] + EI_w\delta\left[(\phi_x'')^2\right] + GI_T\delta\left[(\phi_x')^2\right] \\ & -k_1\delta\left[\frac{M_y^{\mathrm{II}}\phi_x}{EI_z}\left(\frac{M_{y,s}^{\mathrm{I}}\phi}{1-\dfrac{N}{N_z}} + \frac{M_{y,a}^{\mathrm{I}}\phi_x}{1-\dfrac{N}{N_{za}}}\right)\right]\Big\}\,\mathrm{d}x - \frac{1}{2}N\int_0^L \left\{\left[\delta\left(v_0'\right)^2 + i_0^2\delta\left(\phi_x'\right)^2\right]\right\}\mathrm{d}x \\ & + \frac{1}{2}\Bigg\{\sum_i \int_{x_{q1,i}}^{x_{q2,i}} q_{z,i}z_{q,i}\delta\left[\phi_{x,i}(x)\right]^2\mathrm{d}x + \sum_j Q_{z,j}z_{Q,j}\delta\left[\phi_{x,j}\left(x_{Q,j}\right)\right]^2\phi_{x,j}^2 \\ & + \sum_i \int_{L-x_{q2,i}}^{L-x_{q1,i}} \psi_q q_{z,i}z_{q,i}\delta\left[\phi_{x,i}(x)\right]^2\mathrm{d}x + \sum_j \psi_Q Q_{z,j}z_{Q,j}\delta\left[\phi_{x,j}\left(L-x_{Q,j}\right)\right]^2\Bigg\} = 0\end{aligned}$$

Buckling deformation state in Eq. (2.32) is decoupled since the terms associated with components (v_0, ϕ_x) are independent from each other. Three independent buckling modes are related to: a) the flexural-torsional mode governing the prediction of the beam-column buckling state, and two additional modes representing two lowest flexural buckling modes in pure compression. The matrix representation of the stability criterion corresponding to Eq. (2.32) means that the said terms are the diagonal terms $K(1,1)$ and $K(2,2)$ of $\boldsymbol{K}$ matrix of the size 3×3 while the corresponding off-diagonal terms $K(m,n)$ are of zero values (for $m \neq n$ and $m, n = 1,2,3$).

Since the second order minor axis moment terms account for the minor axis buckling modes under compression, the stability criterion for the flexural-torsional buckling based on the proposed refined energy method is that yielding from $K(3,3)$ equated to zero:

$$(2.37)\qquad \delta a_3\left[K(3,3)\right]a_3 = 0 \;\rightarrow\; K(3,3) = 0$$

The structure of $K(3,3)$ term depends upon the option used for the approximation of the in-plane moment M_y.

Option 1a based on NEA

Adopting option I–A from Table 1a, that is, substituting the amplified first order in-plane moment $M_y^{\mathrm{II}} = M_{y,\mathrm{amp}}^{\mathrm{II}}$ to Eq. (2.36), the refined second order energy equation takes the form:

$$\frac{1}{2}\int_0^L \Bigg\{ EI_z\delta\left[\left(v_0''\right)^2\right] + EI_w\delta\left[\left(\phi_x''\right)^2\right] + GI_T\delta\left[\left(\phi_x'\right)^2\right] - \frac{k_1}{EI_z}\delta\left[\left(\frac{M_{y,s}^{\mathrm{I}}}{1-\frac{N}{N_y}} + \frac{M_{y,a}^{\mathrm{I}}}{1-\frac{N}{N_{ya}}}\right)\phi_x\left(\frac{M_{y,s}^{\mathrm{I}}\phi_x}{1-\frac{N}{N_z}} + \frac{M_{y,a}^{\mathrm{I}}\phi_x}{1-\frac{N}{N_{za}}}\right)\right]\Bigg\}\mathrm{d}x - \frac{1}{2}N\int_0^L\left[\delta\left(v_0'\right)^2 + i_0^2\delta\left(\phi_x'\right)^2\right]\mathrm{d}x + \frac{1}{2}\Bigg\{\sum_i\int_{x_{q1,i}}^{x_{q2,i}} q_{z,i}z_{q,i}\delta\left[\phi_{x,i}(x)\right]^2\mathrm{d}x + \sum_j Q_{z,j}z_{Q,j}\delta\left[\phi_{x,j}\left(x_{Q,j}\right)\right]^2\phi_{x,j}^2 + \sum_i\int_{L-x_{q2,i}}^{L-x_{q1,i}}\psi_q q_{z,i}z_{q,i}\delta\left[\phi_{x,i}(x)\right]^2\mathrm{d}x + \sum_j\psi_Q Q_{z,j}z_{Q,j}\delta\left[\phi_{x,j}\left(L-x_{Q,j}\right)\right]^2\Bigg\} = 0 \tag{2.38}$$

Carrying out the integrations of moment independent terms and multiplying by $\frac{2L}{\pi^2}$ leads to the following relationship:

$$K(3,3) = i_0^2 N_T\left(1-\frac{N}{N_T}\right)\zeta - \frac{k_1 M_{y,\max}^2}{N_z}\left[\frac{\left(\frac{M_{y,s,\max}}{M_{y,\max}}\right)^2 2J_s}{\left(1-\frac{N}{N_y}\right)\left(1-\frac{N}{N_z}\right)} + \frac{\left(\frac{M_{y,a,\max}}{M_{y,\max}}\right)^2 2J_a}{\left(1-\frac{N}{N_{ya}}\right)\left(1-\frac{N}{N_{za}}\right)}\right] \tag{2.39}$$

Hence, the general solution of the Option 1a becomes that of Eq. (2.21) in which $F_2(N)$ is replaced by $F_3(N)$ – coefficient representing the effect of compressive force with regard to both in-plane and out-of-plane buckling on the LTB buckling moment:

$$F_3(N) = \left(1-\frac{N}{N_y}\right)\left(1-\frac{N}{N_z}\right)\left(1-\frac{N}{N_T}\right) \tag{2.40}$$

The conversion factor C_{bc} is varying with the major axis critical force utilization ratio N/N_y, in addition to that of the minor axis critical force utilization ratio N/N_z:

$$\frac{1}{C_{bc}} = \sqrt{\frac{k_1}{\zeta}}\left[\left(\frac{M_{y,s,\max}}{M_{y,\max}}\frac{1}{C_{bs,\mathrm{rem}}}\right)^2 + \frac{\left(1-\frac{N}{N_y}\right)\left(1-\frac{N}{N_z}\right)}{\left(1-\frac{N}{N_{ya}}\right)\left(1-\frac{N}{N_{za}}\right)}\left(\frac{M_{y,a,\max}}{M_{y,\max}}\frac{1}{C_{ba,\mathrm{rem}}}\right)^2\right]^{0.5} \tag{2.41}$$

Option 1b based on NEA

Adopting option I–B from Table 1a, the refined second order energy equation takes the form:

$$\begin{aligned}(2.42)\quad & \frac{1}{2}\int_0^L \Bigg\{ EI_z\delta\left[(v_0'')^2\right] + EI_w\delta\left[(\phi_x'')^2\right] \\ & + GI_T\delta\left[(\phi_x')^2\right] - \frac{k_1}{EI_z}\delta\Bigg\{\Bigg[M_{y,s}^{\mathrm{I}} + M_{y,a}^{\mathrm{I}} + \frac{N\delta_{z,s}^{\mathrm{I}}}{1-\dfrac{N}{N_y}}\sin\left(\pi\frac{x}{L}\right) \\ & + \frac{N\delta_{z,a}^{\mathrm{I}}}{1-\dfrac{N}{N_{ya}}}\sin\left(2\pi\frac{x}{L}\right)\Bigg]\phi_x\left(\frac{M_{y,s}^{\mathrm{I}}\phi_x}{1-\dfrac{N}{N_z}} + \frac{M_{y,a}^{\mathrm{I}}\phi_x}{1-\dfrac{N}{N_{za}}}\right)\Bigg\}\Bigg\}\,\mathrm{d}x \\ & - \frac{1}{2}N\int_0^L \left\{\delta\left[(v_0')^2 + i_0^2(\phi_x')^2\right]\right\}\mathrm{d}x + \frac{1}{2}\Bigg\{\sum_i \int_{x_{q1,i}}^{x_{q2,i}} q_{z,i}z_{q,i}\delta\left[\phi_{x,i}(x)\right]^2\mathrm{d}x \\ & + \sum_j Q_{z,j}z_{Q,j}\delta\left[\phi_{x,j}(x_{Q,j})\right]^2\phi_{x,j}^2 + \sum_i \int_{L-x_{q2,i}}^{L-x_{q1,i}} \psi_q q_{z,i}z_{q,i}\delta\left[\phi_{x,i}(x)\right]^2\mathrm{d}x \\ & + \sum_j \psi_Q Q_{z,j}z_{Q,j}\delta\left[\phi_{x,j}(L-x_{Q,j})\right]^2\Bigg\} = 0\end{aligned}$$

Carrying out integrations of the moment independent terms in Eq. (2.42) and multiplying by $\dfrac{2L}{\pi^2}$ leads to the following relationship:

$$\begin{aligned}(2.43)\quad K(3,3) = {} & i_0^2 N_T\left(1-\frac{N}{N_T}\right)\zeta \\ & - \frac{k_1 M_{y,\max}^2}{N_z}\Bigg\{\left(\frac{M_{y,s,\max}}{M_{y,\max}}\right)^2\left[\frac{1}{1-\dfrac{N}{N_z}}\left(\frac{1}{C_{bs,\mathrm{rem}}}\right)^2 + \frac{N}{N_y}\frac{\pi^2 c_{\delta s}}{\left(1-\dfrac{N}{N_y}\right)\left(1-\dfrac{N}{N_z}\right)}2J_{s1}\right] \\ & + \left(\frac{M_{y,a,\max}}{M_{y,\max}}\right)^2\left[\frac{1}{1-\dfrac{N}{N_{za}}}\left(\frac{1}{C_{ba,\mathrm{rem}}}\right)^2 + \frac{N}{N_{ya}}\frac{\pi^2 c_{\delta a}}{\left(1-\dfrac{N}{N_{ya}}\right)\left(1-\dfrac{N}{N_{za}}\right)}2J_{a1}\right]\Bigg\}\end{aligned}$$

where: $J_{s1} = \displaystyle\int_0^1 \left[\frac{M_{y,s}(\xi)}{M_{y,s,\max}}\right]\sin^3(\pi\xi)\,\mathrm{d}\xi$, $J_{a1} = \displaystyle\int_0^1 \left[\frac{M_{y,a}(\xi)}{M_{y,a,\max}}\right]\sin^2(\pi\xi)\sin(2\pi\xi)\,\mathrm{d}\xi$,

$$c_{\delta s} = \frac{\delta_{z,s} E I_y}{M_{y,s,\max} L^2}, \quad c_{\delta a} = \frac{\delta_{z,a} E I_y}{M_{y,a,\max} (L/2)^2}.$$

As a result, the general solution becomes that of Eq. (2.21) in which $F_2(N)$ is replaced by $F_3(N)$ and the conversion factor C_{bc} is varying with the major axis critical force utilization ratio N/N_y, in addition to that of minor axis critical force utilization ratio N/N_z. Thus:

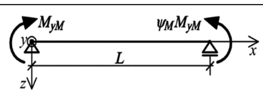
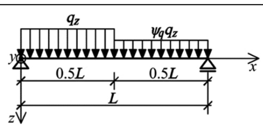
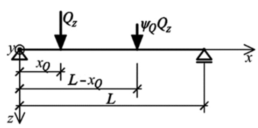
$$(2.44)\quad \frac{1}{C_{bc}} = \sqrt{\frac{k_1}{\zeta} \left\{ \left(\frac{M_{y,s,\max}}{M_{y,\max}}\right)^2 \left[\left(1 - \frac{N}{N_y}\right) \left(\frac{1}{C_{bs,\text{rem}}}\right)^2 + \frac{N}{N_y} \pi^2 c_{\delta s} \frac{1}{C_{bs,\text{rem}1}} \right] \right.}$$
$$\left. + \left(\frac{M_{y,a,\max}}{M_{y,\max}}\right)^2 \frac{1 - \frac{N}{N_z}}{1 - \frac{N}{N_{za}}} \left[\left(1 - \frac{N}{N_y}\right) \left(\frac{1}{C_{ba,\text{rem}}}\right)^2 + \frac{1 - \frac{N}{N_y}}{1 - \frac{N}{N_{ya}}} \frac{N}{N_{ya}} \pi^2 c_{\delta a} \frac{1}{C_{ba,\text{rem}1}} \right] \right\}^{0.5}$$

where: $\frac{1}{C_{bs,\text{rem}1}} = 2J_{s1}$ and $\frac{1}{C_{ba,\text{rem}1}} = 2J_{a1}$.

The coefficients of first order amplitudes of in-plane prebuckling deflected profiles for the single curvature of the symmetric field moment component $c_{\delta s}$ and the double curvature of the antisymmetric field moment component $c_{\delta a}$ are given in Table 2.

The following notation is used in Table 2: $\delta_{z,s,\max}$ – amplitude of the first order single curvature deflected profile for symmetric field in-plane moment component, $\delta_{z,a,\max}$ – amplitude of the first order double curvature deflected profile for antisymmetric field in-plane moment component, $\delta_{z,a,x=L/4}$ – displacement at $x = L/4$ of the first order double curvature deflected profile for antisymmetric field in-plane moment component.

Table 2. The coefficients of first order amplitudes of in-plane prebuckling deflected profiles

Symbol and scheme of load case				$c_{\delta s}$ for symmetric deflected profile	$c_{\delta a}$ for antisymmetric deflected profile	
				$\delta_{z,s,\max}$	$\delta_{z,a,\max}$	$\delta_{z,a,x=L/4}$
M				0.125	0.0641	0.0625
q				0.104	0.104	0.104
Q		$x_0 = L/2$	$\psi_Q = 0$	0.0833	–	–
		$x_0 = 3L/8$	$\psi_Q \neq 0$	0.102	0.0774	0.0764
		$x_0 = L/3$		0.106	0.0806	0.0798
		$x_0 = L/4$		0.115	0.0833	0.0833
		$x_0 = L/6$		0.120	0.0806	0.0798
		$x_0 = L/8$		0.122	0.0776	0.0764

The coefficients $C_{bs,\text{cem}}$, $C_{ba,\text{cem}}$, $C_{bs,\text{rem}}$, $C_{ba,\text{rem}}$ as well as $C_{bs,\text{rem1}}$, $C_{ba,\text{rem1}}$ are dependent upon the in-plane moment distribution decomposed into its two components, symmetric and antisymmetric. Their values are listed in Table 3 and dealt with the same loading patterns as that considered in Table 2. The last column coefficients C_{bF} are used for the evaluation of the term ζ associated with the off-shear load application. This coefficient becomes equal to zero when the in-span load is applied at the cross section shear centre.

Table 3. The loading pattern dependent coefficients

Loading case symbol		Basic coefficients				Additional coefficients for option 2		C_{bF}
		Coefficients cem		Coefficients rem				
		$C_{bs,\text{cem}}$	$C_{ba,\text{cem}}$	$C_{bs,\text{rem}}$	$C_{ba,\text{rem}}$	$C_{bs,\text{rem1}}$	$C_{ba,\text{rem1}}$	
M		1	2.78	1	2.77	1.18	4.19	–
q		1.15	1.43	1.13	1.37	1.31	1.94	$q_z L^2/\pi^2$
Q	$x_0 = L/2$	1.42	–	1.37	–	1.59	–	$2Q_z L/\pi^2$
	$x_0 = 3L/8$	1.17	1.76	1.14	1.49	1,32	2.22	$3.414Q_z L/\pi^2$
	$x_0 = L/3$	1.12	1.74	1.10	1.56	1.27	2.25	$3Q_z L/\pi^2$
	$x_0 = L/4$	1.05	1.81	1.04	1.73	1.21	2.47	$2Q_z L/\pi^2$
	$x_0 = L/6$	1.01	2.01	1.01	1.98	1.18	2.89	$Q_z L/\pi^2$
	$x_0 = L/8$	1.01	2.15	1.01	2.14	1.18	3.18	$0.586Q_z L/\pi^2$

2.4.2. Option 2 based on NEA

Option 2 is based on Eq. (2.35), therefore Eq. (2.32) takes the form:

$$
\begin{aligned}
(2.45)\quad & \frac{1}{2}\int_0^L \Bigg\{ EI_z\delta\left[(v_0'')^2\right] + EI_w\delta\left[(\phi_x'')^2\right] + GI_T\delta\left[(\phi_x')^2\right] \\
& - k_1\delta\left[\frac{M_y^{\text{II}}\phi_x}{EI_z}\left(M_{y,s}^{\text{I}}\phi_x + \frac{N\delta_{z,s}^{\text{I}}\phi_x}{1-\dfrac{N}{N_z}}\sin\left(\pi\frac{x}{L}\right) + M_{y,a}^{\text{I}}\phi + \frac{N\delta_{z,a}^{\text{I}}\phi_x}{1-\dfrac{N}{N_{za}}}\sin\left(2\pi\frac{x}{L}\right)\right)\right]\Bigg\}\,dx \\
& - \frac{1}{2}N\int_0^L \left\{\left[\delta\left(v_0'\right)^2 + i_0^2\delta\left(\phi_x'\right)^2\right]\right\}dx + \frac{1}{2}\Bigg\{\sum_i \int_{x_{q1,i}}^{x_{q2,i}} q_{z,i}z_{q,i}\delta\left[\phi_{x,i}(x)\right]^2 dx \\
& + \sum_j Q_{z,j}z_{Q,j}\delta\left[\phi_{x,j}\left(x_{Q,j}\right)\right]^2\phi_{x,j}^2 + \sum_i \int_{L-x_{q2,i}}^{L-x_{q1,i}} \psi_q q_{z,i}z_{q,i}\delta\left[\phi_{x,i}(x)\right]^2 dx \\
& + \sum_j \psi_Q Q_{z,j}z_{Q,j}\delta\left[\phi_{x,j}\left(L-x_{Q,j}\right)\right]^2\Bigg\} = 0
\end{aligned}
$$

It is rational to suggest that the in-plane moment in the form of I–B from Table 1a is adopted in Eq. (2.45). This results in:

$$\begin{aligned}
(2.46)\quad & \frac{1}{2}\int_0^L \Bigg\{ EI_z\delta\left[(v_0'')^2\right] + EI_w\delta\left[(\phi_x'')^2\right] + GI_T\delta\left[(\phi_x')^2\right] \\
& - \frac{k_1}{EI_z}\delta\Bigg\{\Bigg[M_{y,s}^{\mathrm{I}} + M_{y,a}^{\mathrm{I}} + \frac{N\delta_{z,s}^{\mathrm{I}}}{1-\dfrac{N}{N_y}}\sin\left(\pi\frac{x}{L}\right) + \frac{N\delta_{z,a}^{\mathrm{I}}}{1-\dfrac{N}{N_{ya}}}\sin\left(2\pi\frac{x}{L}\right)\Bigg] \\
& \times\Bigg[M_{y,s}^{\mathrm{I}} + M_{y,a}^{\mathrm{I}} + \frac{N\delta_{z,s}^{\mathrm{I}}}{1-\dfrac{N}{N_z}}\sin\left(\pi\frac{x}{L}\right) + \frac{N\delta_{z,a}^{\mathrm{I}}}{1-\dfrac{N}{N_{za}}}\sin\left(2\pi\frac{x}{L}\right)\Bigg]\phi_x^2\Bigg\}\Bigg\}\,\mathrm{d}x \\
& - \frac{1}{2}\int_0^L\left[N\delta\left(v_0'\right)^2 + Ni_0^2\delta\left(\phi_x'\right)^2\right]\mathrm{d}x \\
& + \frac{1}{2}\Bigg\{\sum_i\int_{x_{q1,i}}^{x_{q2,i}} q_{z,i}z_{q,i}\delta\left[\phi_{x,i}(x)\right]^2\mathrm{d}x + \sum_j Q_{z,j}z_{Q,j}\delta\left[\phi_{x,j}\left(x_{Q,j}\right)\right]^2\phi_{x,j}^2 \\
& + \sum_i\int_{L-x_{q2,i}}^{L-x_{q1,i}}\psi_q q_{z,i}z_{q,i}\delta\left[\phi_{x,i}(x)\right]^2\mathrm{d}x + \sum_j\psi_Q Q_{z,j}z_{Q,j}\delta\left[\phi_{x,j}\left(L-x_{Q,j}\right)\right]^2\Bigg\} = 0
\end{aligned}$$

Carrying out the calculations as in the Options of 1a and 1b, the general solution becomes that of Eq. (2.21) with the conversion factor C_{bc} derived from the following:

$$\begin{aligned}
(2.47)\quad \frac{1}{C_{bc}} = \sqrt{\frac{k_1}{\zeta}}\Bigg\{ & \left(\frac{M_{y,s,\max}}{M_{y,\max}}\right)^2\Bigg[\left(1-\frac{N}{N_y}\right)\left(1-\frac{N}{N_z}\right)\left(\frac{1}{C_{bs,\mathrm{rem}}}\right)^2 \\
& + \left(2-\frac{N}{N_y}-\frac{N}{N_z}\right)\frac{N}{N_y}\frac{\pi^2 c_{\delta s}}{C_{bs,\mathrm{rem1}}} + \frac{3}{4}\left(\frac{N}{N_y}\right)^2\left(\pi^2 c_{\delta s}\right)^2\Bigg] \\
& + \left(\frac{M_{y,a,\max}}{M_{y,\max}}\right)^2\Bigg[\left(1-\frac{N}{N_y}\right)\left(1-\frac{N}{N_z}\right)\left(\frac{1}{C_{ba,\mathrm{rem}}}\right)^2 + \frac{\left(1-\dfrac{N}{N_y}\right)\left(1-\dfrac{N}{N_z}\right)}{\left(1-\dfrac{N}{N_{ya}}\right)\left(1-\dfrac{N}{N_{za}}\right)} \\
& \times\left(\left(2-\frac{N}{N_{ya}}-\frac{N}{N_{za}}\right)\frac{N}{N_{ya}}\frac{\pi^2 c_{\delta a}}{C_{ba,\mathrm{rem1}}} + \frac{1}{2}\left(\frac{N}{N_{ya}}\right)^2\left(\pi^2 c_{\delta a}\right)^2\right)\Bigg]\Bigg\}^{0.5}
\end{aligned}$$

3. Summary and conclusions

A novelty of present study yields from a generalization of the energy method by including the second order effects on the decrease of bending energy at the out-of-plane buckling state of beam-columns. As a result, the refined form of the energy equation has been obtained and used for the evaluation of the interaction between the axial force and the first order bending moment at the flexural-torsional buckling state. The solutions presented in the paper are valid for an arbitrary asymmetric major axis bending load case in which the in-plane moment terms represent the symmetric and antisymmetric load components. The integral moment terms being the product of sinus function of the mean twist rotation might be calculated either by the direct integration or by a numerical integration as shown in Serna et al. [21]. The obtained solutions may be directly compared with those existing in literature for single load cases when the load is symmetric and under a single load parameter.

Reciprocals of the conversion factors $1/C_{bc}$ of the present study for the most robust option 1a solution are for simple load cases dependent upon the moment parameters $\dfrac{M_{y,s,\max}}{M_{y,\max}}$ and $\dfrac{M_{y,a,\max}}{M_{y,\max}}$ as well as the reciprocals of elementary conversion factors $1/C_{bs,\text{rem}}$ and $1/C_{bs,\text{rem}}$. The elementary conversion factors depend only upon the bending moment distributions along the beam-column length for the symmetric and antisymmetric components. Moreover, the conversion factors depend upon the lowest bifurcation flexural buckling force utilization ratios N/N_y and N/N_z, and the second lowest bifurcation flexural buckling force utilization ratios N/N_{ya} and N/N_{za}. This is a novelty of the present study since the solutions presented in earlier investigations and referred to asymmetric loading cases [8] were developed from the classical energy method in which the effects of prebuckling displacements and in-plane buckling were neglected. Moment distribution dependent coefficients $C_{bs,\text{cem}}$ and $C_{ba,\text{cem}}$ used in the classical energy method were based on LEA (linear eigenproblem analysis). The coefficients $C_{bs,\text{rem}}$ and $C_{ba,\text{rem}}$ used in the refined energy method are based on NEA Option 1a. In the other NEA option solutions, coefficients $C_{bs,\text{rem}}$ and $C_{ba,\text{rem}}$ are used together with $C_{bs,\text{cem}}$ and $C_{ba,\text{cem}}$. The comparison of elementary cem and rem equivalent uniform moment factors of C_{bs} and C_{ba} as well as the C_{bF} factor was presented in Table 3 of this paper. One may notice that coefficients $C_{bs,\text{cem}}$ and $C_{ba,\text{cem}}$ are generally of a higher value than those of $C_{bs,\text{rem}}$ and $C_{ba,\text{rem}}$ for all the considered loading cases shown in Table 3, especially for the loading cases dominated by an antisymmetric component. On the other hand, the energy term of the off-shear centre span loads is independent from the distribution of bending moment components, therefore factors C_{bF} of present study are the same as those from earlier studies, regardless whether the energy method is classical or refined to its non-classical form as presented in this study.

For simple boundary conditions considered in this study, the following notations may be used $N/N_y = (1 - k_1)\,(N/N_z)$, $N/N_{ya} = (1 - k_1)\,(N/N_{za})$, therefore solutions based on Eq. (2.32) may be expressed as a function of k_1 and N/N_z. The general Option 1a solution, according to Eq. (2.38), may be used for comparing the results with those obtained in earlier analytical studies, e.g. for load cases presented in [8], namely end moments (EMs), half-span unequal uniformly distributed loads (UDLs) and half-span unequal concentrated loads (CLs), the inequality of which in the half-spans is identified by the load factors ψ_M, ψ_q, ψ_Q, cf. Fig. 1. Comparing constants $C_{bs,\mathrm{rem}}$ and $C_{ba,\mathrm{rem}}$ of present study with those of $C_{bs,\mathrm{cem}}$ and $C_{ba,\mathrm{cem}}$ given in [8] and obtained from the classical energy method leads to the conclusion that for extreme symmetric or antisymmetric cases, the elastic flexural-torsional buckling limit curves $(M_y - N)_{cr}$ of present study, based on the Option 1a for the non-classical energy method in which $k_1 = 1$, are placed below those corresponding to the classical energy method. On the other hand, the solution of present non-classical energy method gives, for $k_1 = 0$, the values of $1/C_{bc} = 0$, therefore $C_{bc} = \infty$, i.e. the lateral-flexural mode seems not to be possible and only the buckling modes possible are those related to the axial compression.

The classical solution is based on the assumption that the effect of prebuckling displacements along $z - z$ axis, resulting from the bending action about $y - y$ axis and the second order effects in the form of amplification of first order moments or according to $P - \delta$ rule, may be neglected. Such an assumption is valid only for beam-columns laterally and torsionally unrestrained (ULT) between end points and having a narrow flange I-section (NFI). In order to account for the effect of prebuckling displacements and second order effects, one has to use the energy equation derived from the displacement field in which the circular trigonometric functions of twist rotation are maintained up to the final stage of the strain energy derivation as shown in [13] and used in this study. Such an approach is desired for beam-columns restrained laterally and torsionally (RLT) between end points and/or having wide flange I-sections (WFI). The classical FTB model seems therefore to be too conservative, especially in the situations referred to RLT and WFI. The conservatism is more and more visible when the second moment of inertia ratio I_z/I_y is much closer to unity than to its zero value. At the extreme situation of I_z/I_y close to unity, the critical moment becomes so high that the buckling state is related only to compression, regardless the level of prebuckling bending action.

Acknowledgements

This paper presents a continuation of research undertaken in [13] where the theoretical basis has been developed. The general formulation shown in the background paper is further developed towards its practical application in solving the flexural-torsional problems. Authors write the paper in the memory of late Professor Nicholas Snowden Trahair of the University of Sydney and with thank to his constant encouragement in research on the space instability of steel structures. The first author of this paper carried out the postdoctoral study under his leadership in 80s of the last century.

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Udoskonalona metoda energetyczna sprężystego wyboczenia giętno-skrętnego stalowych elementów ściskanych i zginanych o przekroju dwuteowym
Część I: Sformułowanie i rozwiązanie

Słowa kluczowe: stalowa belka-słup, dwuteownik bisymetryczny, zachowanie sprężyste, wyboczenie giętno-skrętne, klasyczna metoda energetyczna, udoskonalona metoda energetyczna, rozwiązania analityczne

Streszczenie:

Rozwiązania w postaci zamkniętej dla wyboczenia giętno-skrętnego (FTB) sprężystych belek-słupów można uzyskać tylko dla prostych warunków brzegowych oraz przypadku równomiernego zginania i ściskania. Przypadki zmiennego momentu zginającego wymagają zastosowania przybliżonych metod analitycznych lub numerycznych. Badania przedstawione w niniejszym artykule dotyczą analitycznej metody energetycznej, stosowanej dla dowolnych przypadków asymetrycznych obciążeń poprzecznych, wywołujących nierównomierny moment zginający. Część I prezentowanego artykułu jest w całości poświęcona badaniom teoretycznym nad energetyczną formułą utraty stateczności z płaszczyzny zginania i jej ogólnemu rozwiązaniu. Dla wygody obliczeń obciążenie i wykres momentów zginających przedstawiono jako superpozycję dwóch składowych: symetrycznej i antysymetrycznej. Opracowano podstawową postać nieklasycznego (udoskonalonego) równania energetycznego. Jest ono funkcjonałem zależnym od iloczynów odkształceń stanu przedwyboczeniowego, przemieszczeń osi pręta i ich pochodnych, odpowiednio – u_0 i w_0, oraz składowych stanu odkształcenia pokrytycznego, przemieszczenia z płaszczyzny zginania przedkrytycznego i kąta skręcenia, odpowiednio – v_0 i ϕ_x. Przemieszczenia przedwyboczeniowe u_0 osi pręta i w_0 w płaszczyźnie zginania są znane i mogą być powiązane z siłą osiową N i momentem zginającym względem osi głównej M_y otrzymanymi z analizy pierwszego rzędu (LA). Składowe stanu deformacji v_0 i ϕ_x z płaszczyzny płaskiego stanu zginania oraz ich pochodne są niewiadomymi umożliwiającymi sformułowanie problemu stateczności jako problemu wartości własnych (LBA). W artykule, po pierwsze, poszukiwane jest rozwiązanie stanu wyboczenia poprzez przedstawienie podstawowej postaci nieklasycznego równania energetycznego w kilku wariantach, zależnych od aproksymacji momentu M_z, a mianowicie klasycznego, prowadzącego do analizy liniowego problemu własnego (LEA) i kwadratowego problemu własnego (QEA) oraz innych form prowadzących do nieliniowych analiz problemów własnych (NEA). Nowe formy to te, dla których równanie stateczności zależy tylko od kąta skręcenia i jego pochodnych. Takie udoskonalenie jest możliwe, gdy do zginania z płaszczyzny zastosowane zostanie równanie różniczkowe drugiego rzędu, za pomocą którego krzywizna osi słabszej jest bezpośrednio powiązana z kątem skręcenia. Po drugie, uwzględniono efekt sprzężenia form wyboczenia w płaszczyźnie i z płaszczyzny zginania przedwyboczeniowego przez wprowadzenie przybliżonych zależności zginania drugiego rzędu. Dzięki uwzględnieniu tych efektów znacznie poprawiono dokładność klasycznej metody energetycznej rozwiązywania problemów FTB elementów ściskanych i zginanych w płaszczyźnie większej bezwładności przekroju, zarówno w wypadku przekroju dwuteowego H, jak i I. Wyniki tej części są wykorzystywane w Części II, dotyczącej porównania i weryfikacji rozwiązań uzyskanych w formie zamkniętej w Części I artykułu.

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