WARSAW UNIVERSITY OF TECHNOLOGY Index 351733
FACULTY OF CIVIL ENGINEERING
COMMITTEE FOR CIVIL AND WATER ENGINEERING
POLISH ACADEMY OF SCIENCES ISSN 1230-2945

DOI: 10.24425/ace.2023.147651

ARCHIVES OF CIVIL ENGINEERING

Vol. LXIX ISSUE 4 2023

pp. 125 – 140

Research paper

Robust and reliability-based design optimization of steel beams

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Abstract: In line with the principles of modern design a building structure should not only be safe but also optimized. In deterministic optimization, the uncertainties of the structures are not explicitly taken into account. Traditionally, uncertainties of the structural system (i.e. material parameters, loads, dimensions of the cross-sections) are considered by means of partial safety factors specified in design codes. Worth noticing, that optimal structures are sensitive to randomness design parameters and deterministic optimal solutions may lead to reduced reliability levels. It therefore seems natural to extend the formulation of deterministic optimization with the random scatter of parameter values. Such a formulation is offered by robust optimization and reliability-based design optimization. The applicability of RBDO is strongly dependent on the availability of the joint probability density function. A formulation of non-deterministic optimization that better adapts to the design realities is robust optimization. Unlike RBDO optimization, this formulation does not require estimation of failure probabilities. In the paper using the examples of steel beams, the authors compare the strengths and weaknesses of both formulations.

Keywords: first order reliability method, reliability index, reliability-based design optimization, robust optimization

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1. Introduction

In deterministic optimization, the random nature of design parameters is taken into account using partial safety coefficients. These factors are determined by the relevant design codes. In accordance with the standard provisions, the coefficients are calibrated so that their application covers the widest possible scope of design tasks. This approach leads to overly conservative solutions. In paper [1], the problem of optimal design of a galvanized girder in accordance with Eurocode 3 was considered. If ensuring a sufficiently high level of safety is one of the most important requirements for the designed structure, a reliability-based design optimization (RBDO) [2–6] is worth considering. In the RBDO framework, design constraints are formulated using failure probabilities. The failure probabilities are understood as the probability of exceeding certain allowable ultimate or serviceability states defined by appropriate limit state functions. The possibility of using reliability optimization in practice is strongly conditioned by the availability of the joint probability distribution of random parameters. The reliability of the estimated failure probabilities depends on the precise stochastic model. The formulation of nondeterministic optimization that better fits the design reality is robust optimization [7–10]. Contrary to reliability based optimization, this formulation does not require joint probability distribution of random parameters and precise probability distribution of variables. The random nature of the structure response is taken into account by the objective function and constraint definitions, which include mean values and variances. In the paper, we will show a comparative analysis of the results obtained from reliability based and robust optimization approaches using the examples of a steel beams. This analysis will allow us to present the strengths and weaknesses of both methods for optimal structural design with parameter randomness taken into account.

2. First order reliability method – FORM

The FORM method is one of most effective approximate methods of the calculation of reliability measures. In a general case, when the probability distribution of vector $\boldsymbol{X}$ of base variables is not the Gaussian distribution, transformation is used to transform this vector to the Gaussian vector whose coordinates are independent standard normal variables $\boldsymbol{Z}$. The existence of this type of transformation and the manner of its construction was shown for the first time by Rosenblatt [10] for the case when coordinates of vector $\boldsymbol{X}$ have uniform distributions. Hohenbichler and Rackwitz [12] adapted this transformation to reliability calculations. The transformation of basic random variables to the Gaussian standard space must assure the equivalence of the formulation of the reliability problem. The probability of failure, defined in space $\boldsymbol{X}$, must be equal to the probability defined in space $\boldsymbol{Z}$. After transformation of the variables from the original space to the standard normal space is performed $\boldsymbol{Z} = T(\boldsymbol{X})$. we obtain new limit state function $G(Z_1, \ldots, Z_n)$. Estimating the failure probability with the FORM method [13, 14] requires the determination of a point that is closest to the origin of the $\boldsymbol{Z}$ coordinate system. We refer to this point as design point. The design point lies on $G(Z_1, \ldots, Z_n) = 0$ and has coordinates

$(Z_1^*, \ldots, Z_n^*)$. It is the most probable point of failure from among all points in this area. The Hasofer-Lind reliability index β [15] represents the distance of the design point from origin of standard normal space. Finding a design point is a task for non-linear programming with constraints. The calculations can be performed using the iterative method. If we can define distributions of the random variables we improve algorithm of calculating β using Rackwitz-Fiessler procedure [16]. Probability distributions other than normal are replaced with standard normal distributions. The cumulative distribution function and the probability density function have the same value for the true and equivalent distributions at the design point. The FORM method transforms the integration problem into an optimization problem that can be solved by any optimization algorithm with inequality constraints. The accuracy of results obtained with the use of the Hasofer-Lind index is sufficient for practical needs. The index gained a considerable popularity as a reliability measure, particularly in conjunction with transformation methods which use full information about random variable distributions [17–22].

3. Reliability-based design optimization – RBDO

The basic formulation of the reliability-based design optimization consists in minimizing the objective function under probabilistic constraints. The typical reliability-based design optimization formulation is written as:

$$\text{find:} \quad \boldsymbol{d},\ \boldsymbol{\mu}_x \tag{3.1}$$

$$\text{minimize:} \quad f(\boldsymbol{d}, \boldsymbol{\mu}_X, \boldsymbol{\mu}_P) \tag{3.2}$$

with constraints:

$$p[g_i(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{P}) \le 0] - \Phi(-\beta_i^t) \le 0 \qquad i = 1, \ldots, k_g \tag{3.3}$$

$$d_j^l \le d_j \le d_j^u \qquad j = 1, \ldots, n_d \tag{3.4}$$

$$\mu_{x_r}^l \le \mu_{x_r} \le \mu_{x_r}^u \qquad r = 1, \ldots, n_x \tag{3.5}$$

where: $\boldsymbol{X}$ and $\boldsymbol{P}$ are vectors of random variables with expected values respectively $\boldsymbol{\mu}_X$ and $\boldsymbol{\mu}_P$, $p_f^i = p[g_i(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{P}) \le 0]$ is the failure probability corresponding to the i-th limit function $g_i(\cdot)$, $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution, and β_i^t, $i = 1, \ldots, k_g$, are the minimum reliability indices established by the designer. The expressions $\Phi(-\beta_i^t)$ are therefore the maximum permissible values of the failure probability.

Variables $\boldsymbol{X}$ can be defined as design random variables because their expected values change in the optimization process, leading (in the case of constant values of other parameters describing the distribution) to shift of the probability density function. Unlike $\boldsymbol{X}$ variables, the probability distribution of the vector $\boldsymbol{P}$ does not change during optimization and therefore, in the context of the reliability optimization problem, these variables are called random parameters.

The probabilistic constraint is the key constraint in reliability-based design optimization. The main formulations are classified into three categories: the two-level approach, the mono-level approach, the decoupled approach. The first category considers the probabilistic constraints inside the optimization loop. This approach leads to nested optimization problem. The inner loop deals with reliability assessment and the outer loop deals with cost optimization [23]. Second category aims at solving the problem in a single loop procedure. The probabilistic constraints are replaced by the optimality conditions [24–26]. The third category consists in separating the reliability analysis from the optimization procedure. The RBDO problem is transformed to a sequence of deterministic optimization. The deterministic constraints are linked to the reliability analysis and performed after or before the deterministic design [27, 28].

In this paper the calculations of reliability-based design optimization are made using Costrel module of Strurel software [29]. The failure probabilities are computed by means of FORM. In Costrel use is made of the fact, that the so-called Kuhn–Tucker conditions must be meet for each reliability optimization. It is mono-level RBDO approaches.

4. Robust optimization

Robust design optimization belongs to non-deterministic optimization formulations. This approach includes the effect of structural parameter randomness on the response scatter, therefore, it usually increases structural reliability. The constraints may be deterministic or may be expressed by the first two statistical moments. More attention is paid to the adequate performance of structures subjected to small parameter variations. Unlike other types of optimization (e.g., reliability-based design optimization), imprecise specification of the types of probability distributions is not significant. The values of the first statistical moments of the response depend primarily on the first moments of the random variables. In the absence of adequate data, a uniform or normal distribution of the variables is often assumed. The typical robust optimization formulation is written as:

$$\text{find: } \boldsymbol{d}, \boldsymbol{\mu}_x \tag{4.1}$$

$$\text{minimize: } \{E[f(\boldsymbol{d},\boldsymbol{X},\boldsymbol{P})], \sigma[f(\boldsymbol{d},\boldsymbol{X},\boldsymbol{P})]\} \tag{4.2}$$

with constraints:

$$E\left[g_i(\boldsymbol{d},\boldsymbol{X},\boldsymbol{P})\right] - \widetilde{\beta}_i \sigma\left[g_i(\boldsymbol{d},\boldsymbol{X},\boldsymbol{P})\right] \geq 0, \qquad i = 1,\ldots,k_g \tag{4.3}$$

$$\sigma\left[c_k(\boldsymbol{d},\boldsymbol{X},\boldsymbol{P})\right] \leq \sigma_k^u, \qquad k = 1,\ldots,k_c \tag{4.4}$$

$$d_j^l \leq d_j \leq d_j^u, \qquad j = 1,\ldots,n_d \tag{4.5}$$

$$\mu_{xr}^l \leq \mu_{xr} \leq \mu_{xr}^u, \qquad r = 1,\ldots,n_x \tag{4.6}$$

where: $\boldsymbol{d}$ – deterministic design variables, $\boldsymbol{X}, \boldsymbol{P}$ – vectors of random variables with expected values of μ_x, μ_p, f – objective function, g_i – functions of constraints, c_k – functions, the standard deviations of which must not exceed the allowable values σ_k^u, $\widetilde{\beta}_i > 0$ – coefficients corresponding to the constraints $g_i \geq 0$ which represent the safety margin with which these constraints must be met.

A very popular method of determining the points of the Pareto set is the multi-criteria optimization task scalarization method, in which a linear combination of criteria is used as the objective function. By changing the coefficients (weights) for individual components, Pareto set points are obtained. The values of the weights can be changed in a systematic way, so as to determine the set of non-dominated solutions as precisely as possible. Their values are also affected by the designer's preferences for minimizing the mean value and variance. So the task can be modified to the following scalar optimization task:

(4.7) find values of variables: $\boldsymbol{d}, \boldsymbol{\mu_x}$

(4.8) that minimize: $\widetilde{f} = \dfrac{1-\gamma}{\mu^*} E\left[f(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{P})\right] + \dfrac{\gamma}{\sigma^*} \sigma\left[f(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{P})\right]$

with the constraints (4.3)–(4.6)

The robust optimization algorithm has seven steps:

1. Specify the feasible region according to congruent with (4.7) and (4.8) and select the weighting factor γ.
2. Generate N realizations of the vector of design variables uniformly spaced over the current feasible region, in accordance with the optimal Latin-hypercube design.
3. Determine statistical moments of the objective and constraint functions for each of the N realizations of vector $\{\boldsymbol{d}, \boldsymbol{\mu_x}\}$.
4. Construct the response surface using methods such as kriging directly for individual statistical moments: $\hat{\mu}_f, \hat{\sigma}_f, \hat{\mu}_{g_i}, \hat{\sigma}_{g_i}, \hat{\sigma}_{c_k}$.
5. Solve the deterministic optimization problem

(4.9) find values of variables: $\boldsymbol{d}, \boldsymbol{\mu_x}$

(4.10) Minimizing: $\widetilde{f}^{\mathrm{DRS}} = \dfrac{1-\gamma}{\mu^*} \hat{\mu}_f(\boldsymbol{d}, \boldsymbol{\mu}_x) + \dfrac{\gamma}{\sigma^*} \hat{\sigma}_f(\boldsymbol{d}, \boldsymbol{\mu}_x)$

Subject to constraints:

(4.11) $\hat{\mu}_{g_i}(\boldsymbol{d}, \boldsymbol{\mu}_x) - \widetilde{\beta}_i \hat{\sigma}_{g_i}(\boldsymbol{d}, \boldsymbol{\mu}_x) \geq 0, \qquad i = 1, \ldots, k_g$

(4.12) $\hat{\sigma}_{c_k}(\boldsymbol{d}, \boldsymbol{\mu}_x) \leq \sigma_k^u, \qquad k = 1, \ldots, k_c$

(4.13) $\hat{d}_j^l \leq \hat{d}_j \leq \hat{d}_j^u, \qquad j = 1, \ldots, n_d$

(4.14) $\hat{\mu}_{x_r}^l \leq \mu_{x_r} \leq \hat{\mu}_{x_r}^u, \qquad r = 1, \ldots, n_x$

6. Check the condition for convergence; if it is satisfied, terminate the algorithm.
7. Shift the feasible region over the optimal point determined at step 5. Reduce the feasible region and return.

The weighting factor $\gamma \in [0, 1]$ in formula (4.10) defines the importance of each criterion. Assuming that $\gamma = 0$, an optimization problem transforms into the mean value minimization problem, whereas for $\gamma = 1$ it becomes a problem of minimizing the variance of the objective function. Values μ^* and σ^* are normalizing constants. Normalization coefficients are determined based on the extreme values of appropriate moments obtained in step 3. Quantities: $\hat{d}_j^l$, $\hat{d}_j^u$, $\hat{\mu}_{x_r}^l$, $\hat{\mu}_{x_r}^u$ are the current boundaries of the feasible region.

The key element of the algorithm for the realization of a robust optimization problem is an effective method of estimating mean values and standard deviations of the objective and constraint functions. In the paper techniques of approximating implicit functions of design variables using metamodels, i.e., response surface designs, were used. In Numpress Explore [30] was used the kriging algorithm [31] with optimal Latin hypercubes [32].

5. Numerical results and discussion

5.1. Example 1. Statically determinate structure

We consider a bent steel cantilever $L = 150$ cm long, rectangular cross-section with dimensions $D = 8.0$ cm and $d = 6.0$ cm, Young's modulus $E = 21,000$ kN/cm^2, Poisson's ratio $\nu = 0.3$. The cantilever was loaded with a force $Q = 0.05$ kN/cm as shown in Figure 1. The initial mass of the cantilever is $f_{M1} = 32.97$ kg.

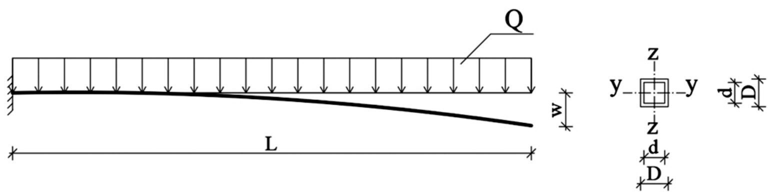

Fig. 1. Cantilever geometry and load

For such a modelled structure, the displacement of the cantilever end was determined according to the relationship: $w = \dfrac{QL^4}{8EI} = 0.521$ cm, where $I = \dfrac{D^4 - d^4}{12}$, bending moment $My = \dfrac{QL^2}{2} = 562.5$ kN·cm. The calculations shown below were performed with NumpressExplore [30] and Costrel [29].

5.1.1. Reliability analysis

In the first stage, reliability analysis was carried out using the FORM method. The value of the reliability index and the probability of structure failure were checked for two cases marked as SLS and ULS.

For the SLS case, the form of the limit function was formulated as a condition of not exceeding the permissible vertical displacement of the node $w_d = 1$ cm:

$$f_{\text{SLS}}(\boldsymbol{x}) = w_d - \frac{Q \cdot L^4}{8 \cdot E \cdot I} = 1 - \frac{12 \cdot Q \cdot 150^4}{8 \cdot E \cdot (D^4 - d^4)} \tag{5.1}$$

For the ULS case, the form of the limit function was formulated as a condition of not exceeding the permissible bending moment $M_d = f_y \cdot W_y = f_y \cdot \dfrac{D^4 - d^4}{6 \cdot D}$, where $f_y = 23.5$ kN/cm^2:

$$f_{\mathrm{ULS}}(\boldsymbol{x}) = M_d - \frac{Q \cdot L^2}{2} = 23.5 \cdot \frac{D^4 - d^4}{6 \cdot D} - \frac{Q \cdot 150^2}{2} \tag{5.2}$$

Conducting the reliability analysis, random variables D, d, Q and E shown in Table 1 were defined. Random variables are described by normal probability distribution and are not correlated.

Table 1. Description of random variables

Random variable	Mean value	Standard deviation	Coefficient of variation
D	8 [cm]	0.16 [cm]	2 [%]
d	6 [cm]	0.12 [cm]	2 [%]
E	21 000 [kN/cm^2]	630 [kN/cm^2]	3 [%]
Q	0.05 [kN/cm]	0.001 [kN/cm]	2 [%]

The value of the reliability index was respectively for SLS and ULS: $\beta^{\mathrm{SLS}} = 3.038$ and $\beta^{\mathrm{ULS}} = 5.879$, while the probability of failure $p_f^{\mathrm{SLS}} = 1.190e{-}3$ and $p_f^{\mathrm{ULS}} = 2.068e{-}9$. In the PN-EN-1990 standard, for the serviceability limit state, the value of the reliability index for the RC2 structure class is $\beta^{\mathrm{SLS}} = 1.5$ while for the ultimate limit state, it is $\beta^{\mathrm{ULS}} = 3.8$. Therefore, the designed structure meets the standard conditions.

5.1.2. Deterministic optimization

Design variables were defined as the dimensions of the cross-section D and d, while the objective function will be the mass of the structure:

$$f_C = \min\left(\left(D^2 - d^2\right) \cdot L \cdot \rho\right) \tag{5.3}$$

where: $\rho = 0.00785 \left[\dfrac{\mathrm{kg}}{\mathrm{cm}^3}\right]$ – assumed steel density, L [cm] – cantilever length.

In this case, simple constraints are described in Table 2. They constitute the upper and lower limits of the searched values of D and d.

Table 2. Description of simple constraints

Random variable	Lower limit	Upper limit
D	7.5 [cm]	8.5 [cm]
d	5.5 [cm]	6.5 [cm]

Both the serviceability limit state and the ultimate limit state were verified, in which the unequal constraints, were formulated as the conditions (5.1), (5.2).

The obtained dimensions of the cross-section are: for SLS, $D = 7.742$ cm and $d = 6.5$ cm, for ULS, $D = 7.5$ cm and $d = 6.5$ cm. The value of the objective function for these cases was, respectively, for SLS and ULS: 20.828 kg, 16.485 kg. The probability of failure and the reliability index were also verified, which in this case were respectively for SLS and ULS: $\beta_D^{\text{SLS}} = 3.198e{-}6$ and $\beta_D^{\text{ULS}} = 1.125$, while the probability of failure $p_{fD}^{\text{SLS}} = 0.5$ and $p_{fD}^{\text{ULS}} = 0.130$.

5.1.3. Robust optimization

To perform robust optimization in NumpressExplore [33–35], you have to define random variables, design variables, objective functions, and constraint functions. For this type of optimization, it is necessary to use an appropriate type of design variables named in the program "random design variable". The design variable is linked to the random variable in such a way that the value of the design variable is also the mean value of the random variable. Additionally, it is possible to set a constant value of the coefficient of variation for the standard deviation, which means that the standard deviation of a random variable changes and is always equal to the product of the mean value and the coefficient of variation. In this case, the value of the coefficient of variation was set at 2%.

For this case, the tasks of robust optimization take the form:

– For the serviceability limit state:

(5.4) Find the values of the variables: μ_D, μ_d

$$\text{(5.5)}\quad \text{Minimizing: } f_C = \frac{1-\gamma}{\eta^*} E\left[\left(\left(D^2 - d^2\right) \cdot L \cdot \rho\right)\right] + \frac{\gamma}{\sigma^*}\sigma\left[\left(\left(D^2 - d^2\right) \cdot L \cdot \rho\right)\right]$$

– With constraints:

$$\text{(5.6)}\quad E\left[1 - \frac{12 \cdot Q \cdot 150^4}{8 \cdot E \cdot (D^4 - d^4)}\right] - \widetilde{\beta}_i \cdot \sigma\left[1 - \frac{12 \cdot Q \cdot 150^4}{8 \cdot E \cdot \left(D^4 - d^4\right)}\right] > 0$$

$$\text{(5.7)}\quad 7.5 \le \mu_D \le 8.5$$

$$\text{(5.8)}\quad 5.5 \le \mu_d \le 6.5$$

– For the ultimate limit state:

(5.9) Find the values of the variables: μ_D, μ_d

$$\text{(5.10)}\quad \text{Minimizing: } f_C = \frac{1-\gamma}{\eta^*} E\left[\left(\left(D^2 - d^2\right) \cdot L \cdot \rho\right)\right] + \frac{\gamma}{\sigma^*}\sigma\left[\left(\left(D^2 - d^2\right) \cdot L \cdot \rho\right)\right]$$

– With constraints:

$$\text{(5.11)}\quad E\left[23.5 \cdot \frac{D^4 - d^4}{6 \cdot D} \frac{Q \cdot 150^2}{2}\right] - \widetilde{\beta}_i \cdot \sigma\left[23.5 \cdot \frac{D^4 - d^4}{6 \cdot D} - \frac{Q \cdot 150^2}{2}\right] > 0$$

$$\text{(5.12)}\quad 7.5 \le \mu_D \le 8.5$$

$$\text{(5.13)}\quad 5.5 \le \mu_d \le 6.5$$

where: $\gamma \in [0, 1]$ – determines the importance of each of the criteria, $\rho = 0.00785 \left[\frac{\text{kg}}{\text{cm}^3}\right]$ – assumed steel density, L [cm] – the length of cantilever, η^*, σ^* – normalizing constants.

Solving the problem begins with determining the initial allowable area based on the previously defined constraints. The parameter was adopted for the calculations $\gamma = 0.5$, the parameters $\widetilde{\beta_i^{\text{SLS}}} = 2.0$ and $\widetilde{\beta_i^{\text{ULS}}} = 3.8$. The response surfaces in this case are built using the kriging method, while the experiments are generated according to the plan of optimal Latin cubes. The solution to the presented optimization problem, i.e. the minimization of the f_c function, was performed using the Nelder Mead simplex algorithm. After the carried out robust optimization, the finally obtained values of the width and height of the cross-section were, respectively, for the serviceability limit state $D = 8.056$ cm and $d = 6.500$ cm and for the ultimate limit state $D = 7.5$ cm and $d = 5.83$ cm. The weight of the structure was: for SLS $f_c = 26.669$ kg and for ULS $f_c = 26.224$ kg. A slight increase in the cross-section height and an increase in the weight of the structure result in a significant change in the value of the reliability index and the probability of failure, which in this case are, respectively, for SLS and ULS: $\beta_R^{\text{SLS}} = 1.721$ and $\beta_R^{\text{ULS}} = 4.079$, while the probability of failure $p_{fR}^{\text{SLS}} = 4.264e{-}02$ and $p_{fR}^{\text{ULS}} = 2.259e{-}05$.

5.1.4. Reliability based design optimization

The conditions (5.1), (5.2) were taken into account in the reliability optimization task.

The objective function is the mass of the structure:

$$f_C = \min\left((D^2 - d^2) \cdot L \cdot \rho\right) \tag{5.14}$$

where: $\rho = 0.00785 \left[\frac{\text{kg}}{\text{cm}^3}\right]$ – assumed steel density, L [cm] – the length of cantilever.

In the case of reliability optimization [33], it is necessary to assume a limit reliability index (failure probability). In the case of SLS, the limitation was set at the level of $\beta = 1.5$, while for ULS $\beta = 3.8$. After the performed reliability optimizations, the obtained values of the width and height of the cross-section were, respectively, for the serviceability limit state $D = 8.038$ cm and $d = 6.5$ cm ($\beta = 1.628$) and for the ultimate limit state $D = 7.5$ cm and $d = 5.891$ cm ($\beta = 3.808$). The weight of the structure was: for SLS $f_c = 26.328$ kg and for ULS $f_c = 25.371$ kg.

Observing the results obtained in the robust and reliability-based optimization, we can see their convergence for case C and A (Table 3). Both solutions give the level of failure probability at a level acceptable in the standards.

Additionally the authors decided to check the failure probability level for the solution $D = 8.0$ cm, $d = 6.5$ cm for both the ultimate limit state and the serviceability limit state. The probability of failure and the reliability index were verified, which in this case were respectively for SLS and ULS: $\beta_N^{\text{SLS}} = 1.423$ and $\beta_N^{\text{ULS}} = 3.697$, while the probability of failure $p_{f_N}^{\text{SLS}} = 0.077$ and $p_{f_N}^{\text{ULS}} = 1e{-}04$. From an engineering point of view, this solution is optimal.

Table 3. Reliability indexes for robust and RBDO analysis for both limit functions (5.1), (5.2)

	Robust		RBDO	
	A	B	C	D
D	8.056 [cm]	7.5 [cm]	8.038 [cm]	7.5 [cm]
d	6.5 [cm]	5.83 [cm]	6.5 [cm]	6.5 [cm]
SLS function (5.1)	$\beta = 1.723$	$\beta = 0.736$	$\beta = 1.626$	$\beta = 0.551$
ULS function (5.2)	$\beta = 3.971$	$\beta = 4.080$	$\beta = 3.882$	$\beta = 3.810$

5.2. Example 2. Statically indeterminate structure

We consider a bent steel beam $L = 300$ cm long, rectangular cross-section with dimensions $D = 8.0$ cm and $d = 6.0$ cm, Young's modulus $E = 21,000$ kN/cm^2, Poisson's ratio $v = 0.3$. The beam was loaded with a force $Q = 0.05$ kN/cm as shown in Figure 2. The initial mass of the beam is $f_{M1} = 65.94$ kg.

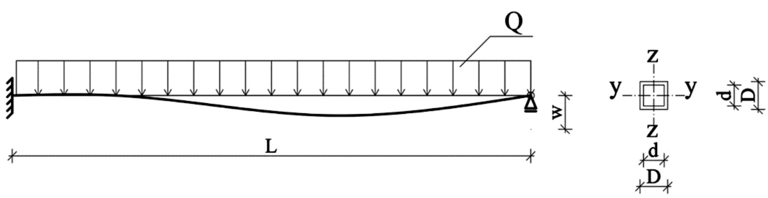

Fig. 2. Statically indeterminate structure geometry and load

For such a modelled structure, the maximum displacement was determined according to the relationship: $w = \dfrac{QL^4}{185EI} = 0.447$ cm, where $I = \dfrac{D^4 - d^4}{12}$, bending moment $My = \dfrac{QL^2}{8} = 562.5$ kN·cm. The calculations shown below were performed with NumpressExplore [30] and Costrel [29].

5.2.1. Reliability analysis

In the first stage, reliability analysis was carried out using the FORM method. The value of the reliability index and the probability of structure failure were checked for two cases marked as SLS and ULS. The random variables are normally distributed and are not correlated (Table 1).

For the SLS case, the form of the limit function was formulated as a condition of not exceeding the permissible vertical displacement of the node $w_d = 1$ cm:

$$f_{\text{SLS}}(\boldsymbol{x}) = w_d - \frac{Q \cdot L^4}{185 \cdot E \cdot I} = 1 - \frac{12 \cdot Q \cdot 300^4}{185 \cdot E \cdot (D^4 - d^4)} \tag{5.15}$$

For the ULS case, the form of the limit function was formulated as a condition of not exceeding the permissible bending moment $M_d = f_y \cdot W_y = f_y \cdot \dfrac{D^4 - d^4}{6 \cdot D}$, where $f_y = 23.5$ kN/cm^2:

$$f_{\text{ULS}}(\boldsymbol{x}) = M_d - \frac{Q \cdot L^2}{8} = 23.5 \cdot \frac{D^4 - d^4}{6 \cdot D} - \frac{Q \cdot 300^2}{8} \tag{5.16}$$

The value of the reliability index was respectively for SLS and ULS: $\beta^{\text{SLS}} = 5.006$ and $\beta^{\text{ULS}} = 5.879$, while the probability of failure $p_f^{\text{SLS}} = 2.777e{-}3$ and $p_f^{\text{ULS}} = 2.068e{-}9$.

5.2.2. Deterministic optimization

Design variables were defined as the dimensions of the cross-section D and d, while the objective function will be the mass of the structure (5.3)

In this case, simple constraints are described in Table 4. They constitute the upper and lower limits of the searched values of D and d.

Table 4. Description of simple constraints

Random Variable	Lower limit	Upper limit
D	7 [cm]	9 [cm]
d	5 [cm]	7 [cm]

Both the serviceability limit state and the ultimate limit state were verified, in which the unequal constraints, were formulated as the conditions (5.15), (5.16).

The obtained dimensions of the cross-section are: for SLS, $D = 7.705$ cm and $d = 6.905$ cm, for ULS, $D = 7.28$ cm and $d = 6.48$ cm. The value of the objective function for these cases was, respectively, for SLS and ULS: 27.53 kg, 25.92 kg. The probability of failure is very high, $p_{fD}^{\text{SLS}} = 0.5$ and $p_{fD}^{\text{ULS}} = 0.5$. These values are unacceptable.

5.2.3. Robust optimization

For this case, the tasks of robust optimization take the form:

– For the serviceability limit state:

(5.17) Find the values of the variables: μ_D, μ_d

(5.18) Minimizing: $f_C = \dfrac{1-\gamma}{\eta^*} E\left[\left(\left(D^2 - d^2\right) \cdot L \cdot \rho\right)\right] + \dfrac{\gamma}{\sigma^*} \sigma\left[\left(\left(D^2 - d^2\right) \cdot L \cdot \rho\right)\right]$

– With constraints:

$$E\left[1 - \frac{12 \cdot Q \cdot 300^4}{185 \cdot E \cdot \left(D^4 - d^4\right)}\right] - \widetilde{\beta}_i \cdot \sigma\left[1 - \frac{12 \cdot Q \cdot 300^4}{185 \cdot E \cdot \left(D^4 - d^4\right)}\right] > 0 \tag{5.19}$$

$$7 \le \mu_D \le 9 \tag{5.20}$$

$$5 \le \mu_d \le 7 \tag{5.21}$$

– For the ultimate limit state:

(5.22) Find the values of the variables: μ_D, μ_d

(5.23) Minimizing: $f_C = \frac{1-\gamma}{\eta^*} E\left[\left(\left(D^2 - d^2\right) \cdot L \cdot \rho\right)\right] + \frac{\gamma}{\sigma^*} \sigma\left[\left(\left(D^2 - d^2\right) \cdot L \cdot \rho\right)\right]$

– With constraints:

(5.24) $$E\left[23.5 \cdot \frac{D^4 - d^4}{6 \cdot D} \frac{Q \cdot 300^2}{8}\right] - \widetilde{\beta_i} \cdot \sigma\left[23.5 \cdot \frac{D^4 - d^4}{6 \cdot D} \frac{Q \cdot 300^2}{8}\right] > 0$$

(5.25) $$7 \leq \mu_D \leq 9$$

(5.26) $$5 \leq \mu_d \leq 7$$

where: $\gamma \in [0, 1]$ – determines the importance of each of the criteria, $\rho = 0.00785 \left[\frac{\text{kg}}{\text{cm}^3}\right]$ – assumed steel density, L [cm] – the length of beam, η^*, σ^* – normalizing constants.

The analysis parameters are the same as in example 5.1. The finally obtained values of the width and height of the cross-section were, respectively, for the serviceability limit state $D = 7.487$ cm and $d = 6.120$ cm and for the ultimate limit state $D = 7$ cm and $d = 5.09$ cm. The weight of the structure was: for SLS $f_c = 43.805$ kg and for ULS $f_c = 54.381$ kg. A slight increase in the cross-section height and an increase in the weight of the structure result in a significant change in the value of the reliability index and the probability of failure, which in this case are, respectively, for SLS and ULS: $\beta_R^{\text{SLS}} = 1.811$ and $\beta_R^{\text{ULS}} = 4.504$, while the probability of failure $p_{fR}^{\text{SLS}} = 3.5e{-}02$ and $p_{fR}^{\text{ULS}} = 3.34e{-}06$.

5.2.4. Reliability based design optimization

The conditions (5.15), (5.16) were taken into account in the reliability optimization task. The objective function is the mass of the structure (5.3). In the case of SLS, the limitation was set at the level of $\beta = 1.5$, while for ULS $\beta = 3.8$. After the performed reliability optimizations, the obtained values of the width and height of the cross-section were, respectively, for the serviceability limit state $D = 8.074$ cm and $d = 7$ cm ($\beta = 1.555$) and for the ultimate limit state $D = 7.129$ cm and $d = 5.424$ cm ($\beta = 3.808$). The weight of the structure was: for SLS $f_c = 35.404$ kg and for ULS $f_c = 50.404$ kg.

Observing the results obtained in the robust and reliability-based optimization, we can see their convergence for case B and D (Table 5). Both solutions give the level of failure probability at a level acceptable in the standards.

Additionally the authors decided to check the failure probability level for the solution $D = 7.1$ cm, $d = 5.4$ cm for both the ultimate limit state and the serviceability limit state. The probability of failure and the reliability index were verified, which in this case were respectively for SLS and ULS: $\beta_N^{\text{SLS}} = 2.1$ and $\beta_N^{\text{ULS}} = 3.75$, while the probability of failure $p_{fN}^{\text{SLS}} = 0.018$ and $p_{fN}^{\text{ULS}} = 8.98e{-}05$. From an engineering point of view, this solution is optimal.

Table 5. Reliability indexes for robust and RBDO analysis for both limit functions (5.15), (5.16)

	Robust		RBDO	
	A	B	C	D
D	7.487 [cm]	7.5 [cm]	8.074 [cm]	7.129 [cm]
d	6.12 [cm]	5.83 [cm]	7.0 [cm]	5.424 [cm]
SLS function (5.15)	$\beta = 1.811$	$\beta = 2.480$	$\beta = 1.555$	$\beta = 2.196$
ULS function (5.16)	$\beta = 2.725$	$\beta = 4.504$	$\beta = 1.911$	$\beta = 3.808$

6. Conclusions

The analysis of the influence of the random nature of the parameters describing the modelled phenomenon is extremely important in the process of optimal design. Solutions that fullfill their function for nominal parameter values may be unacceptable after taking into account random imperfections. These imperfections may relate to the scattering of material parameters, dimensions, and external influences. In the analysed work, the variability of material parameters is described by Young's modulus E with a coefficient of variation equal to 3%. The distribution of cross-sectional dimensions is described by variables D, d with a coefficient of variation of 2%. The external action defines a uniformly distributed load Q with a coefficient of variation of 2%. The results of deterministic optimization, while maintaining the previously defined coefficients of variation, turned out to be completely useless. Failure probabilities calculated for the ultimate limit state and the serviceability limit state are very high. In order to find a solution that is insensitive to imperfections of model parameters or external influences that are difficult to control, we have two options. The first one is robust optimization. The second one is optimization based on the reliability of the so-called RBDO. If guaranteeing a high level of safety is the most important requirement for the designed structure, it is worth choosing RBDO. Within RBDO, design constraints are formulated using failure probabilities. The applicability of RBDO is strongly dependent on the availability of the joint probability density function. The reliability of the estimated failure probability values depends on the precise stochastic model. A formulation of non-deterministic optimization that better adapts to the design realities is robust optimization. Unlike RBDO optimization, this formulation does not require estimation of failure probabilities. The random nature of the structure's response is taken into account by defining the objective function and constraints, including mean values and variances. The computational complexity of this approach is related to the use of effective methods of estimating statistical moments.

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Optymalizacja odpornościowa i niezawodnościowa stalowych belek

Słowa kluczowe: Analiza niezawodnościowa pierwszego rzędu, wskaźnik niezawodności, optymalizacja niezawodnościowa, optymalizacja odpornościowa

Streszczenie:

Analiza wpływu, jaki na modelowane zjawisko ma losowy charakter opisujących je parametrów jest niezwykle istotna w procesie optymalnego projektowania. Rozwiązania, które spełniają swoją funkcję dla nominalnych wartości parametrów mogą okazać się nie do zaakceptowania po uwzględnieniu losowych imperfekcji. Imperfekcje te mogą dotyczyć nieuniknionego rozrzutu parametrów materiałowych, wymiarów, oddziaływań zewnętrznych. W analizowanej pracy zmienność parametrów materiałowych opisuje moduł Younga E ze współczynnikiem zmienności równym 3%. Rozrzut wymiarów geometrycznych przekroju poprzecznego opisują zmienne D, d ze współczynnikiem zmienności 2%. Oddziaływanie zewnętrzne definiuje obciążenie równomiernie rozłożone Q o współczynniku zmienności równym 2%. Rezultaty optymalizacji deterministycznej, przy zachowaniu zdefiniowanych wcześniej współczynników zmienności, okazały się całkowicie nieprzydatnymi. Prawdopodobieństwa awarii obliczone dla stanu granicznego nośności i stanu granicznego użytkowania są bardzo wysokie. Dążąc do znalezienia rozwiązania niewrażliwego na trudne do kontrolowania

imperfekcje parametrów modelu lub oddziaływań zewnętrznych mamy do dyspozycji dwie opcje. Pierwsza z nich to optymalizacja typu robust. Druga to optymalizacja oparta na niezawodności tzw. RBDO. Jeżeli zagwarantowanie wysokiego poziomu bezpieczeństwa jest najważniejszym wymaganiem stawianym projektowanej konstrukcji warto wybrać RBDO. W ramach RBDO, ograniczenia projektowe formułowane są za pomocą prawdopodobieństw awarii. Możliwość zastosowania RBDO jest silnie uwarunkowana dostępnością łącznej funkcji gęstości prawdopodobieństwa. Od precyzyjnego modelu stochastycznego zależy wiarygodność szacowanych wartości prawdopodobieństwa awarii. Sformułowaniem optymalizacji niedeterministycznej, które lepiej dopasowuje się do realiów projektowych jest optymalizacja typu robust. Celem optymalizacji odpornościowej powinna być jednoczesna minimalizacja wartości średniej oraz odchylenia standardowego funkcji celu. W odróżnieniu od optymalizacji RBDO, sformułowanie to nie wymaga szacowania prawdopodobieństw awarii. Losowy charakter odpowiedzi konstrukcji uwzględniany jest poprzez definicje funkcji celu i ograniczeń, zawierających wartości średnie oraz wariancje. Złożoność obliczeniowa tego podejścia wiąże się z użyciem efektywnych metod szacowania momentów statystycznych. W pracy do obliczeń RBDO wykorzystano moduł Costrel środowiska obliczeniowego Strurel. W module Costrel obliczenia realizowane są zgodnie z ideą metod jednopoziomowych. Celem tych metod jest wyeliminowanie wewnętrznej pętli związanej z analizą niezawodności poprzez rozszerzenie zbioru zmiennych decyzyjnych oraz zastąpienie ograniczeń niezawodnościowych poprzez kryteria optymalności zadań poszukiwania punktów projektowych. Obliczenia związane z „robust" optymalizacją wykonano za pomocą oprogramowania Numpress Explore. Odpowiednia aproksymacja funkcji celu i ograniczeń ma kluczowe znaczenie dla efektywności oraz zbieżności przeprowadzanych analiz. W pracy wykorzystano metodę krigingu w wersji aproksymacyjnej wraz z planem eksperymentu opartym na koncepcji optymalnej łacińskiej hiperkostki. Obserwując wyniki uzyskane w „robust" optymalizacji i RBDO, możemy zobaczyć ich zbieżność dla przykładu 1 w przypadku C i A (Tabela 3) oraz dla przykładu 2 w przypadku B i D (tabela 5). Oba rozwiązania dają poziom prawdopodobieństwa awarii na poziomie akceptowalnym w normach. Dodatkowo autorzy zdecydowali się sprawdzić poziom prawdopodobieństwa awarii dla rozwiązania $D = 8{,}0$ cm, $d = 6{,}5$ cm (przypadek 1) oraz $D = 7{,}1$ cm, $d = 5{,}4$ cm (przypadek 2), zarówno dla stanu granicznego nośności jak i stanu granicznego użytkowania. Zweryfikowano również wskaźnik niezawodności oraz prawdopodobieństwo awarii dla nowo założonych wymiarów. Dla przypadku 1 wyniosły one odpowiednio dla SGU i SGN: $\beta_N^{\mathrm{SGU}} = 1{,}423$ i $\beta_N^{\mathrm{SGN}} = 3{,}697$, $p_{fN}^{\mathrm{SGU}} = 0{,}077$ i $p_{fN}^{\mathrm{SGN}} = 1e{-}04$, natomiast dla przypadku 2 dla SGU i SGN: $\beta_N^{\mathrm{SGU}} = 2{,}1$ i $\beta_N^{\mathrm{SGN}} = 3{,}75$, $p_{fN}^{\mathrm{SGU}} = 0{,}018$ i $p_{fN}^{\mathrm{SGN}} = 8{,}98e{-}05$. Z punktu widzenia inżynierskiego te rozwiązania są optymalne.

Received: 2023-03-24, Revised: 2023-05-04