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Stability margins for generalized fractional two-dimensional state space models

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In this paper, a new class of bidimensional fractional linear systems is considered. The stability radius of the disturbed system is described according to the $\mathcal{H}_\infty$ norm. Sufficient conditions to ensure the stability margins of the closed-loop system are offered in terms of linear matrix inequalities. The concept of $\mathcal{D}$ stability region for these systems is also considered. Examples are provided to verify the applicability of our main result.

Key words: fractional 2D systems, stability radius, stability region, linear matrix inequalities

1. Introduction

The fractional calculus theory is a powerful tool for representing many problems in different areas, such as mechanics, physics, chemistry, biology, economics, signal processing, and control theory; e.g. [3–5, 10–12, 15, 16, 19, 22]. The main reason for the success of fractional calculus theory is that these new fractional-order models are more accurate than integer-order models. On the other hand, significant attention has been focused on two-dimensional (2D) systems

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over the last few decades due to their involvement in many practical challenges. Comprehensive details can be found in ([2,7–9,14,23]). The importance of these systems lies in their ability to simplify and accurately illustrate complex behaviors and interactions, providing a clearer picture of real-world processes. [13,20]. These two-dimensional systems are characterized by the spread of information in two independent variables in two separate directions, such as time and distance or length and width etc.

Recently, the study of systems stability in the field of control theory has garnered significant interest from both researchers and practitioners [2,21,24].

Furthermore, recent research underscores that even minor disturbances in the controller coefficients can render the closed-loop system unstable or fragile relative to uncertainties that cannot be ignored.

Stability margins play a central role as an indicator, measuring disturbances or uncertainties before a system loses its stability. In [1], the authors use the stability radius to measure the instable distance of linear invariant systems. There are certain stability and stabilization results of fractional uncertain systems of order $(0 < \alpha < 1)$ [18]. Sufficient conditions for robust asymptotic stability of fractional closed loop systems with $(0 < \alpha < 1)$ and $(1 < \alpha < 2)$ through the linear matrix inequalities (LMIs) approach have been developed in [17].

In this paper, we look at the extension of the work in ([1]) to characterize the stability margins and $\mathcal{D}$-stability region conditions of fractional generalized two dimensional state space systems for both continous and discrete time cases expressed in a set of strict linear matrix inequalities. Numerical examples are given to illustrate the proposed methods.

Notation: $I($ or $0)$ is the identity (resp. zero) matrix with appropriate dimension. $X \succ 0$ $(X \prec 0)$ indicates that the matrix X is positive (negative) definite. $X^\top$ denote the matrix transposed with respect to the matrix X. $\mathbb{C}$ and $\mathbb{R}$ are the complex and the real spaces. The symbol $\otimes$ denotes the Kronecker product of two matrices, $eig(A)$ represente the eigenvalues of the matrix A.

2. Model description

Consider the fractional linear system described by the following form:

$$\begin{aligned} \lambda_1^\alpha \lambda_2^\alpha E x(t_1,t_2) &= A_0 x(t_1,t_2) + \lambda_2^\alpha A_1 x(t_1,t_2) + Bu(t_1,t_2), \\ y(t_1,t_2) &= Cx(t_1,t_2) + Du(t_1,t_2), \end{aligned} \tag{1}$$

where, α is the commensurate order $(0 < \alpha < 1)$, $E,\ A_0,\ A_1 \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$ are matrices of constants. $x(t_1,t_2) \in \mathbb{R}^n$, $u(t_1,t_2) \in \mathbb{R}^m$ and $y(t_1,t_2) \in \mathbb{R}^p$ are the state space, the input and the output vectors respectively,

λ_1 and λ_2 are the differential operators s_1, s_2 in the Laplace transform when (1) is continuous-time and for the delay operators z_1, z_2 (in the transformed domain) when (1) is discrete-time.

We establish the partial derivative for fractional two-dimensional continuous-time systems.

$$\begin{aligned}\lambda_1^\alpha \lambda_2^\alpha x(t_1,t_2) &= \frac{\partial^\alpha}{\partial t_1^\alpha}\frac{\partial^\alpha}{\partial t_2^\alpha} x(t_1,t_2)\\ &= \frac{1}{(\Gamma(n-\alpha))^2}\int_0^{t_1}\int_0^{t_2}\frac{x_{t_1}^{(n)}(\tau)}{(t_1-\tau)^{\alpha+1-n}}\frac{x_{t_2}^{(n)}(s)}{(t_2-s)^{\alpha+1-n}}\mathrm{d}s\,\mathrm{d}\tau,\end{aligned} \tag{2}$$

Γ is the Euler Gamma function defined by the formula

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\mathrm{d}t, \quad \mathcal{R}e(x) > 0. \tag{3}$$

It is well known that the model (1) can be reduced to Roesser model in the following form

$$\begin{aligned}\begin{bmatrix}\lambda_1^\alpha E & 0\\ 0 & \lambda_2^\alpha I_n\end{bmatrix} x(t_1,t_2) &= \begin{bmatrix}A_1 & I_n\\ A_0 & 0\end{bmatrix} x(t_1,t_2) + \begin{bmatrix}0\\ B\end{bmatrix} u(t_1,t_2),\\ y(t_1,t_2) &= \begin{bmatrix}C & 0\end{bmatrix} x(t_1,t_2) + Du(t_1,t_2),\end{aligned} \tag{4}$$

when $x(t_1,t_2)$ is defined as

$$x(t_1,t_2) = \begin{bmatrix}x^h(t_1,t_2)\\ x^v(t_1,t_2)\end{bmatrix},$$

where, $x^h(t_1,t_2)$ and $x^v(t_1,t_2)$ represent the horizontal and the vertical states in $\mathbb{R}^n$ for any $t_1, t_2 \geqslant 0$.

3. Stability margins for continuous and discrete fractional two-dimensional systems

Inspired by [1], consider the system given by (4) and assume that the above realization is minimal and strictly stable, which means that the model (4) has all its eigenvalues in the open set $\Gamma_1 \times \Gamma_2$ of the complex plane and satisfing the condition

$$\left|\arg\left(\mathrm{eig}\left(\begin{bmatrix}E & 0\\ 0 & I_n\end{bmatrix}^{-1}\begin{bmatrix}A_1 & I_n\\ A_0 & 0\end{bmatrix}\right)\right)\right| > \alpha\frac{\pi}{2}. \tag{5}$$

Now, if we close the loop with $u = \Delta y$, we obtain

$$\begin{aligned} \begin{bmatrix} \lambda_1^\alpha E & 0 \\ 0 & \lambda_2^\alpha I_n \end{bmatrix} x(t_1,t_2) &= \begin{bmatrix} A_1 & I_n \\ A_0 & 0 \end{bmatrix} x(t_1,t_2) + \begin{bmatrix} 0 \\ B \end{bmatrix} \Delta y(t_1,t_2), \\ y(t_1,t_2) &= \begin{bmatrix} C & 0 \end{bmatrix} x(t_1,t_2) + D\Delta y(t_1,t_2), \end{aligned} \tag{6}$$

After elimination of $y(t_1,t_2)$, we get

$$\begin{bmatrix} \lambda_1^\alpha E & 0 \\ 0 & \lambda_2^\alpha I_n \end{bmatrix} x(t_1,t_2) = \begin{bmatrix} A_1 & I_n \\ A(\Delta) & 0 \end{bmatrix} x(t_1,t_2), \tag{7}$$

where,

$$A(\Delta) = A_0 + B\,(I_m - \Delta D)^{-1}\,\Delta C,$$

or, $(I_m - \Delta D)^{-1}\Delta = \Delta(I_p - D\Delta)^{-1}$ which is easily verified by the relation $\Delta(I_p - D\Delta) = (I_m - \Delta D)\Delta$.

Now, we want to know conditions to guarantee that the closed loop system (7) is also strictly stable.

We therefore define the corresponding stability radius of the perturbed system (7) as the smallest perturbation Δ destabilizing the system,

$$\begin{aligned} &r_C(E, A_0, A_1, B, C, D) \\ &:= \inf_{\Delta} \left\{ \|\Delta\|_2 : \begin{bmatrix} \lambda_1^\alpha E - A_1 & -I_n \\ -A(\Delta) & \lambda_2^\alpha I_n \end{bmatrix} \text{ has unstable eigenvalues} \right\}. \end{aligned} \tag{8}$$

The stability will be lost only when one of the eigenvalues crosses the boundary $\partial\Gamma_1 \times \partial\Gamma_2$ of the stability region $\Gamma_1 \times \Gamma_2$. An equivalent formulation of this stability radius is thus given by

$$\begin{aligned} &r_C(E, A_0, A_1, B, C, D) \\ &:= \inf_{(\lambda_1^\alpha,\lambda_2^\alpha)\in\partial\Gamma_1\times\partial\Gamma_2} \left\{ \inf_{\Delta} \left\{ \|\Delta\|_2 : \det \begin{bmatrix} \lambda_1^\alpha E - A_1 & -I_n \\ -A(\Delta) & \lambda_2^\alpha I_n \end{bmatrix} = 0 \right\} \right\}. \end{aligned} \tag{9}$$

So,

$$\det \begin{pmatrix} \lambda_1^\alpha E - A_1 & -I_n \\ -A(\Delta) & \lambda_2^\alpha I_n \end{pmatrix} = 0, \tag{10}$$

is equivalent to testing

$$\det \left(\begin{bmatrix} \lambda_1^\alpha E - A_1 & -I_n \\ -A_0 & \lambda_2^\alpha I_n \end{bmatrix} - \begin{bmatrix} 0 \\ B \end{bmatrix} (I_m - \Delta D)^{-1} \begin{bmatrix} \Delta C & 0 \end{bmatrix} \right) = 0, \tag{11}$$

which can be rewritten as

$$\det\begin{pmatrix}\lambda_1^{\alpha}E - A_1 & -I_n & 0\\ -A_0 & \lambda_2^{\alpha}I_n & B\\ \Delta C & 0 & I_m - \Delta D\end{pmatrix} = 0. \tag{12}$$

where,

$$\begin{bmatrix}\lambda_1^{\alpha}E - A_1 & -I_n\\ -A_0 & \lambda_2^{\alpha}I_n\end{bmatrix} - \begin{bmatrix}0\\ B\end{bmatrix}(I_m - \Delta D)^{-1}\begin{bmatrix}\Delta C & 0\end{bmatrix},$$

is the Schur complement of

$$\begin{bmatrix}\lambda_1^{\alpha}E - A_1 & -I_n & 0\\ -A_0 & \lambda_2^{\alpha}I_n & B\\ \Delta C & 0 & I_m - \Delta D\end{bmatrix}.$$

Thus, the condition (12) can be written as

$$\det\left(\begin{bmatrix}\lambda_1^{\alpha}E - A_1 & -I_n & 0\\ -A_0 & \lambda_2^{\alpha}I_n & B\\ 0 & 0 & I_m\end{bmatrix} + \begin{bmatrix}0\\ 0\\ I_m\end{bmatrix}\Delta\begin{bmatrix}C & 0 & -D\end{bmatrix}\right) = 0. \tag{13}$$

The matrix $\begin{bmatrix}\lambda_1^{\alpha}E - A_1 & -I_n\\ -A_0 & \lambda_2^{\alpha}I_n\end{bmatrix}$ is invertible. So, testing (13) is equivalent to testing

$$\det\left(I_{2n+m} + \begin{bmatrix}-\begin{pmatrix}\lambda_1^{\alpha}E - A_1 & -I_n\\ -A_0 & \lambda_2^{\alpha}I_n\end{pmatrix}^{-1}\begin{pmatrix}0\\ B\end{pmatrix}\\ I_m\end{bmatrix}\Delta\begin{bmatrix}C & 0 & -D\end{bmatrix}\right) = 0,$$

since, $\det(I + RS) = 0$ implies that $\det(I + SR) = 0$ for all conformable matrices R and S, this finally gives

$$\det\left(I_m - \Delta G(\lambda_1^{\alpha}, \lambda_2^{\alpha})\right) = 0,$$

where,

$$G(\lambda_1^{\alpha}, \lambda_2^{\alpha}) = \begin{bmatrix}C & 0 & -D\end{bmatrix}\begin{bmatrix}-\begin{pmatrix}\lambda_1^{\alpha}E - A_1 & -I_n\\ -A_0 & \lambda_2^{\alpha}I_n\end{pmatrix}^{-1}\begin{pmatrix}0\\ B\end{pmatrix}\\ I_m\end{bmatrix}.$$

We can reformulate the stability radius as follows

$$\begin{aligned} r_C(E, A_0, A_1, B, C, D) \\ := \inf_{(\lambda_1^{\alpha}, \lambda_2^{\alpha})\in\partial\Gamma_1\times\partial\Gamma_2}\left\{\inf_{\Delta}\left\{\|\Delta\|_2 :\ \det\left(I_m - \Delta G(\lambda_1^{\alpha}, \lambda_2^{\alpha})\right) = 0\right\}\right\},\end{aligned}$$

which is equal to,

$$r_C(E, A_0, A_1, B, C, D) := \left[\sup_{(\lambda_1^\alpha, \lambda_2^\alpha)\in\partial\Gamma_1\times\partial\Gamma_2} \left\| G(\lambda_1^\alpha, \lambda_2^\alpha) \right\|_2 \right]^{-1}$$
$$= \|G(.,.)\|_\infty^{-1}. \tag{14}$$

For continuous-time fractional 2D systems

$$\partial\Gamma_1 = |\omega_1|^\alpha \left[\cos\left(\alpha\left(\frac{\pi}{2} + \arg(\omega_1)\right)\right) + j\sin\left(\alpha\left(\frac{\pi}{2} + \arg(\omega_1)\right)\right)\right],$$
$$\partial\Gamma_2 = |\omega_2|^\alpha \left[\cos\left(\alpha\left(\frac{\pi}{2} + \arg(\omega_2)\right)\right) + j\sin\left(\alpha\left(\frac{\pi}{2} + \arg(\omega_2)\right)\right)\right],$$

where, $\omega_1,\ \omega_2 \in \mathbb{R} - \{0\}$. Therefore, the stability radius obtained by Eq. (14) can be written as

$$r_C(E, A_0, A_1, B, C, D)$$
$$:= \left[\sup_{(\omega_1,\omega_2)\in\mathbb{R}^2-\{0\}} \left\| G\left(\begin{array}{c} |\omega_1|^\alpha \begin{bmatrix} \cos\left(\alpha\left(\frac{\pi}{2} + \arg(\omega_1)\right)\right) \\ +j\sin\left(\alpha\left(\frac{\pi}{2} + \arg(\omega_1)\right)\right) \end{bmatrix}, \\ |\omega_2|^\alpha \begin{bmatrix} \cos\left(\alpha\left(\frac{\pi}{2} + \arg(\omega_2)\right)\right) \\ +j\sin\left(\alpha\left(\frac{\pi}{2} + \arg(\omega_2)\right)\right) \end{bmatrix} \end{array} \right) \right\|_2 \right]^{-1},$$

and for the discrete-time systems

$$\partial\Gamma_1 = e^{|\omega_1|^\alpha \left[\cos\left(\alpha\left(\frac{\pi}{2}+\arg(\omega_1)\right)\right)+j\sin\left(\alpha\left(\frac{\pi}{2}+\arg(\omega_1)\right)\right)\right]},$$
$$\partial\Gamma_2 = e^{|\omega_2|^\alpha \left[\cos\left(\alpha\left(\frac{\pi}{2}+\arg(\omega_2)\right)\right)+j\sin\left(\alpha\left(\frac{\pi}{2}+\arg(\omega_2)\right)\right)\right]},$$

where, $\omega_1,\ \omega_2 \in \mathbb{R} - \{0\}$.

Then,

$$r_C(E, A_0, A_1, B, C, D)$$
$$:= \left[\sup_{(\omega_1,\omega_2)\in\mathbb{R}^2-\{0\}} \left\| G\left(\begin{array}{c} e^{|\omega_1|^\alpha \begin{bmatrix} \cos\left(\alpha\left(\frac{\pi}{2} + \arg(\omega_1)\right)\right) \\ +j\sin\left(\alpha\left(\frac{\pi}{2} + \arg(\omega_1)\right)\right) \end{bmatrix}}, \\ e^{|\omega_2|^\alpha \begin{bmatrix} \cos\left(\alpha\left(\frac{\pi}{2} + \arg(\omega_2)\right)\right) \\ +j\sin\left(\alpha\left(\frac{\pi}{2} + \arg(\omega_2)\right)\right) \end{bmatrix}} \end{array} \right) \right\|_2 \right]^{-1}.$$

For the continuous and the discrete time cases we have,

$$\arg(\omega_i) = \begin{cases} 2k\pi & \text{if } \omega_i \in \mathbb{R}^+ - \{0\}, \\ \pi + 2k\pi & \text{if } \omega_i \in \mathbb{R}^- - \{0\}, \end{cases}$$

$k \in \mathbb{Z},\ i = 1, 2.$

For $E = I_n$ these connections is standard, based on [1] we recall them in the following theorem given for arbitrary E.

Theorem 1. *Assume that the open loop sytem (4) is strictly stable. Then the closed loop system (7) is strictly stable if and only if* $\Delta \in \mathbb{C}^{m\times p}$ *verifies*

$$\|\Delta\|_2 < \mu_\star^{-1}, \tag{15}$$

where,

$$\mu_\star := \|G(.,.)\|_\infty := \sup_{(\lambda_1^\alpha,\lambda_2^\alpha)\in\partial\Gamma_1\times\partial\Gamma_2} \left\|G(\lambda_1^\alpha,\lambda_2^\alpha)\right\|_2 . \tag{16}$$

$\partial\Gamma_i = j^\alpha\mathbb{R}$ *in the continuous-time case and* $\partial\Gamma_i = e^{j^\alpha\mathbb{R}}$*in the discrete-time case, for all* i, $i = 1, 2$.

We note that if we impose the condition that Δ is real, (15) and (16) only become sufficient conditions of stability. However, the theorem asserts that stability is ensured for any Δ (whether real or complex) which satisfies (15) and (16). The key issue for computing $\mu_\star$ is to construct computable conditions for an upper bound μ of $\mu_\star$. Such $\mu > \mu_\star$ must satisfy

$$\mu^2 I_m - G_\star(\lambda_1^\alpha,\lambda_2^\alpha)\, G(\lambda_1^\alpha,\lambda_2^\alpha) \succ 0,\ \ \forall(\lambda_1^\alpha,\lambda_2^\alpha) \in \partial\Gamma_1\times\partial\Gamma_2 , \tag{17}$$

where, $G_\star(\lambda_1^\alpha,\lambda_2^\alpha)$ is equal to

$$G_\star\left(\lambda_1^\alpha,\lambda_2^\alpha\right) := G\left(\begin{matrix} |\omega_1|^\alpha \begin{bmatrix} \cos\left(\alpha\left(\dfrac{\pi}{2}+\arg(\omega_1)\right)\right) \\ -j\sin\left(\alpha\left(\dfrac{\pi}{2}+\arg(\omega_1)\right)\right)\end{bmatrix}, \\ |\omega_2|^\alpha \begin{bmatrix} \cos\left(\alpha\left(\dfrac{\pi}{2}+\arg(\omega_2)\right)\right) \\ -j\sin\left(\alpha\left(\dfrac{\pi}{2}+\arg(\omega_2)\right)\right)\end{bmatrix}\end{matrix}\right)^\top$$

in the continuous-time, and

$$G_\star\left(\lambda_1^\alpha,\lambda_2^\alpha\right)=G\begin{pmatrix} e^{|\omega_1|^\alpha\begin{bmatrix}\cos\left(\alpha\left(\frac{\pi}{2}+\arg\left(\omega_1\right)\right)\right)\\ -j\sin\left(\alpha\left(\frac{\pi}{2}+\arg\left(\omega_1\right)\right)\right)\end{bmatrix}}, \\ e^{|\omega_2|^\alpha\begin{bmatrix}\cos\left(\alpha\left(\frac{\pi}{2}+\arg\left(\omega_2\right)\right)\right)\\ -j\sin\left(\alpha\left(\frac{\pi}{2}+\arg\left(\omega_2\right)\right)\right)\end{bmatrix}}\end{pmatrix}^\top,$$

in the discrete-time.

It was shown in [1] that for continuous time systems, $\mu>\mu_\star\geqslant 0$, if and only if

$$\left[\begin{array}{cc|c} -E^\top Y_1A_1-A_1^\top Y_1E & -E^\top Y_1-A_0^\top Y_2 & 0\\ -Y_2A_0-Y_1E & 0 & -Y_2B\\ \hline 0 & -B^\top Y_2 & \mu^2 I_m\end{array}\right] -\begin{bmatrix}C^\top\\0\\D^\top\end{bmatrix}\begin{bmatrix}C & 0 & D\end{bmatrix}\succ 0, \tag{18}$$

where,

$$Y=\begin{bmatrix}Y_1 & 0\\0 & Y_2\end{bmatrix},\qquad Y=Y^\top,$$

and for the discrete time, $\mu>\mu_\star\geqslant 0$ if and only if

$$\left[\begin{array}{cc|c} E^\top Y_1E-A_1^\top Y_1A_1-A_0^\top Y_2A_0 & -A_1^\top Y_1 & -A_0^\top Y_2B\\ -Y_1A_1 & Y_2-Y_1 & 0\\ \hline -B^\top Y_2A_0 & 0 & \mu^2 I_m-B^\top Y_2B\end{array}\right] -\begin{bmatrix}C^\top\\0\\D^\top\end{bmatrix}\begin{bmatrix}C & 0 & D\end{bmatrix}\succ 0, \tag{19}$$

where,

$$Y=\begin{bmatrix}Y_1 & 0\\0 & Y_2\end{bmatrix},\qquad Y=Y^\top.$$

If $(E=I_n)$, then the state space system will be standard. So, other linear matrix inequalities were established in [1].

For the continuous-time case

$$\left[\begin{array}{cc|c|c} -X_1A_1 - A_1^\top X_1 & -X_1 - A_0^\top X_2 & 0 & C^\top \\ -X_2A_0 - X_1 & 0 & -X_2B & 0 \\ \hline 0 & -B^\top X_2 & \mu I_m & D^\top \\ \hline C & 0 & D\mu I_p & \end{array}\right] \succ 0, \tag{20}$$

where,

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}], \quad X = X^\top.$$

For the discrete-time

$$\left[\begin{array}{cc|c|c} X_1 - A_1^\top X_1A_1 - A_0^\top X_2A_0 & -A_1^\top X_1 & -A_0^\top X_2B & C^\top \\ -X_1A_1 & X_2 - X_1 & 0 & 0 \\ \hline -B^\top X_2A_0 & 0 & \mu I_m - B^\top X_2B & D^\top \\ \hline C & 0 & D & \mu I_p \end{array}\right] \succ 0, \tag{21}$$

where,

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}, \qquad X = X^\top.$$

If the matrix E as a full rank then, we substitute X by $\begin{bmatrix} E^\top Y_1E & 0 \\ 0 & Y_2 \end{bmatrix}$. Subsequently, new conditions are defined for the standard state-space realization

$$\left[\begin{array}{cc|c} E^{-1}A_1 & E^{-1} & 0 \\ A_0 & 0 & B \\ \hline C & 0 & D \end{array}\right],$$

for the continuous-time systems,

$$\left(\begin{array}{cc|c|c} -E^\top Y_1A_1 - A_1^\top Y_1E & -E^\top Y_1 - A_0^\top Y_2 & 0 & C^\top \\ -Y_2A_0 - Y_1E & 0 & -Y_2B & 0 \\ \hline 0 & -B^\top Y_2 & \mu I_m & D^\top \\ \hline C & 0 & D & \mu I_p \end{array}\right) \succ 0, \tag{22}$$

$$Y = \begin{bmatrix} Y_1 & 0 \\ 0 & Y_2 \end{bmatrix}, \qquad Y = Y^\top \tag{23}$$

and for the discrete time

$$\left(\begin{array}{cc|c|c} E^\top Y_1 E - A_1^\top Y_1 A_1 - A_0^\top Y_2 A_0 & -A_1^\top Y_1 & -A_0^\top Y_2 B & C^\top \\ Y_1 A_1 & Y_2 - Y_1 & 0 & 0 \\ \hline -B^\top Y_2 A_0 & 0 & \mu I_m - B^\top Y_2 B & D^\top \\ \hline C & 0 & D & \mu I_p \end{array}\right) \succ 0, \tag{24}$$

$$Y = \begin{bmatrix} Y_1 & 0 \\ 0 & Y_2 \end{bmatrix}, \qquad Y = Y^\top.$$

4. $\mathcal{D}$-Stability region of fractional two-dimensional systems

We first define the LMI region which we will need later.

Definition 1. *A subset $\mathcal{D}$ in the complex plan associated to bidimensional systems is called an LMI region if there exist a symmetric matrices $R_{i0} \in \mathbb{R}^{d\times d}$ and $R_{i1} \in \mathbb{R}^{d\times d}$, $i = 1, 2$, such that*

$$\mathcal{D} = \left\{ (z_1, z_2) \in \mathbb{C}^2 : \sum_{i=1}^{2} R_{i0} + z_i R_{i1} + z_i^* R_{i1}^\top \prec 0 \right\}. \tag{25}$$

Definition 2. *The fractional 2-D sytem (4) is $\mathcal{D}$-stable if and only if all its poles are in the $\mathcal{D}$ region.*

In what follows, we present our second main result.

Theorem 2. *The sytem (4) with $rank(E) = n$ is $\mathcal{D}$-stable if there exist a symmetric positive definite matrix Y such that*

$$\begin{aligned} \mathcal{M}_{\mathcal{D}} := &\begin{pmatrix} R_{10} & 0 \\ 0 & R_{20} \end{pmatrix} \otimes \begin{pmatrix} E^\top Y_1 E & 0 \\ 0 & Y_2 \end{pmatrix} + \begin{pmatrix} R_{11} & 0 \\ 0 & R_{21} \end{pmatrix} \otimes \begin{pmatrix} E^\top Y_1 E & E^\top Y_1 \\ Y_2 A_0 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} R_{11}^\top & 0 \\ 0 & R_{21}^\top \end{pmatrix} \otimes \begin{pmatrix} A_1^\top Y_1 E & A_0^\top Y_2 \\ Y_1 E & 0 \end{pmatrix} \prec 0, \end{aligned} \tag{26}$$

$$Y = \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix}.$$

Proof. Its readily from applying the result obtained in [6] to the eigenvalue problem $\begin{pmatrix} E^{-1}A_1 & E^{-1} \\ A_0 & 0 \end{pmatrix}$ and replacing X by $\begin{pmatrix} E^\top Y_1 E & 0 \\ 0 & I_n \end{pmatrix}$.

Now, we will analyze the location of eigenvalues of the closed loop system (7) in the complex plan. We will suppose that the open loop system (4) is $\mathcal{D}$ stable and we will search sufficient conditions that the system (7) is $\mathcal{D}$ stable for given subset $\mathcal{D}$ included in $\mathbb{C}_-$.

Theorem 3. *The system* (7) *with uncertainty*

$$\Delta \in \mathbb{C}^{p\times m}, \ \|\Delta\|_2 < \mu^{-1}$$

is $\mathcal{D}$-stable if there exist a symmetric definite positive matrices $Y \in \mathbb{R}^{2n\times 2n}$ and $P \in \mathbb{R}^{r\times r}$ such that

$$\left(\begin{array}{cc|c|c} \multicolumn{2}{c|}{\multirow{2}{*}{$\mathcal{M}_{\mathcal{D}}(Y)$}} & 0 & Q_2^\top P \otimes C^\top \\ & & Q_1^\top \otimes Y_2 B & 0 \\ \hline 0 & Q_1 \otimes B^\top Y_2 & -\mu P \otimes I_n & P \otimes D^\top \\ \hline PQ_2 \otimes C & 0 & P \otimes D & -\mu P \otimes I_n \end{array}\right) \prec 0, \tag{27}$$

$$Y = \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix},$$

or, $R_1 = Q_1^\top Q_2$ is a factorization with Q_1 and Q_2 complete row rank r.

Proof. It comes easily from the application of the result of [6] to the standard state space realization

$$\left[\begin{array}{cc|c} E^{-1}A_1 & E^{-1} & 0 \\ A_0 & 0 & B \\ \hline C & 0 & D \end{array}\right],$$

then we substitute X by $\begin{bmatrix} E^\top Y_1 E & 0 \\ 0 & Y_2 \end{bmatrix}$. □

5. Examples

We end this work by giving some examples for the above results. We then present a numerical simulation to illustrate the theoretical results.

Example 1. Consider the fractional two-dimensional discrete-time system with the following matrices

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix}; \quad A_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0 \end{bmatrix};$$

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}; \quad D = \begin{bmatrix} 0.4 & 0 & 0 \end{bmatrix}.$$

In [25], Zou et al. showed that the system in question is stable. In our case, by comparing the results with our approach and by applying the LMI (21), we find a feasible solution given by

$$X_1 = \begin{bmatrix} 0.4504 & 0 \\ 0 & 1.0094 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0.5630 & 0 \\ 0 & 1.9613 \end{bmatrix}, \quad \mu = 2.9954,$$

which show that the closed-loop system is strictly stable.

Example 2. Consider the fractional continuous state-space system described by (4) with $\alpha = 0.4$ as follows,

$$\begin{aligned} \begin{bmatrix} 0.02\,\lambda_1^{0.4} & 0 \\ 0 & \lambda_2^{0.4} \end{bmatrix} x(t_1, t_2) &= \begin{bmatrix} -0.0055 & 1 \\ -0.114 & 0 \end{bmatrix} x(t_1, t_2) + \begin{bmatrix} 0 \\ 0.001 \end{bmatrix} u(t_1, t_2), \\ y(t_1, t_2) &= \begin{bmatrix} -1 & 0 \end{bmatrix} x(t_1, t_2) + 0.4\, u(t_1, t_2). \end{aligned} \tag{28}$$

It can be easily verified that the sytem (28) is stable, as its eigenvalues are -0.2750 and -0.1140, satisfying the condition (5). Additionally, by applying LMI (22), a feasible solution is found

$$Y_1 = 4.2173; \quad Y_2 = 0.7399; \quad \mu = 1.2138 \times 10^3.$$

This implies that the closed-loop system is strictly stable.

Example 3. Consider the fractional-order linear sytem (4) with $\alpha = 0.5$ as follows

$$\begin{aligned} \begin{bmatrix} \lambda_1^{0.5} x^h(t_1, t_2) \\ \lambda_2^{0.5} x^v(t_1, t_2) \end{bmatrix} &= \begin{bmatrix} -0.89 & 1 \\ -142.6 & 0 \end{bmatrix} \begin{bmatrix} x^h(t_1, t_2) \\ x^v(t_1, t_2) \end{bmatrix} + \begin{bmatrix} 0 \\ 178.25 \end{bmatrix} u(t_1, t_2), \\ y(t_1, t_2) &= \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} x^h(t_1, t_2) \\ x^v(t_1, t_2) \end{bmatrix}. \end{aligned} \tag{29}$$

We consider the LMI region (25) with

$$R_{10} = \begin{bmatrix} -2\cos\frac{\pi}{3} & 0 \\ 0 & -2\cos\frac{\pi}{3} \end{bmatrix}, \quad R_{11} = \begin{bmatrix} \cos\frac{\pi}{3} & 2\sin\frac{\pi}{3} \\ -2\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix}$$

$$R_{20} = \begin{bmatrix} -150 & 0 \\ 0 & -150 \end{bmatrix}, \qquad R_{21} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

which means that the region associated by the first direction represents a disk with center zero and radius 150 and the second region associated by the second direction is a conic sector with angle $\frac{\pi}{6}$.

By using our method on the open loop sytem (29) we find that the LMI given in Theorem 2 is feasible and a feasible solution is

$$Y_1 = 1.1709 \times 10^{-16}, \quad Y_2 = 7.2167 \times 10^{-19},$$

which shows that is strictly stable in the region $\mathcal{D}$.

By applying the result given in Theorem 3 of the closed loop system we find that the LMI is feasible and is feasible solution is as follows

$$\mu = 1.6440, \quad Y_1 = 5.2940 \times 10^{-23}, \quad Y_2 = 1.5510 \times 10^{-25},$$

we can take $P = I_{2\times2}$.

Hence, we can conclude that the closed loop system is also strictly $\mathcal{D}$-stable.

6. Concluding remarks

In this paper, a new fractional generalized 2D model is presented. Extended results on the stability margins conditions of the perturbed system are derived. A new $\mathcal{D}$ region in the complex plan associated to such system is introduced. $\mathcal{D}$ stability region of the considered problem using the Kronecker product and LMIs are studied. Some numerical examples are shown the effectiveness of our main results.

References

[1] D. Bouagada and P. Van Dooren: Stability margins for generalized state space systems. *Applied Mathematics Letters*, **19**(5), (2006), 451–457. DOI: 10.1016/j.aml.2005.06.015

[2] K. Benyettou and D. Bouagada: Admissibility tests for multidimensional singular fractional continuous-time models. *Archives of Control Sciences*, **32**(3), (2022), 607–625. DOI: 10.24425/acs.2022.142851

[3] K. Bettou and A. Charef: Fractional order PIλDμA controller design based on Bode's ideal function. *Archives of Control Sciences*, **33**(2), (2023), 425–458. DOI: 10.24425/acs.2023.146428

[4] R. Hilfer, R. Metzler, A. Blumen and J. Klafter: Strange kinetics. *Chemical Physics*, **284**(1), (2002), 104–301. DOI: 10.1016/S0301-0104(02)00801-7

[5] T. Chiranjeevi, R. Devarapalli, N.R. Babu, K.B. Vakkapatla, R.G.S. Rao and F.P. Garcia Marquez: Fixed terminal time fractional optimal control problem for discrete time singular system. *Archives of Control Sciences*, **32**(3), (2022), 489–506. DOI: 10.24425/acs.2022.142846

[6] M. Chilali, P. Gahinet and P. Apkarian: Robust pole placement in LMI regions. *IEEE Transactions on Automatic Control*, **44**(12), (1999), 2257–2269. DOI: 10.1109/9.811208

[7] E. Fornasini and G. Marchesini: State-space realization theory of two-dimensional filters. *IEEE Transactions on Automatic Control*, **21**(4), (1976), 484–492. DOI: 10.1109/TAC.1976.1101305

[8] K. Galkowski: The Fornasini–Marchesini and the Roesser models: Algebraic methods for recasting. *IEEE Transactions on Automatic Control*, **41**(1), (1996), 107–112. DOI: 10.1109/9.481611

[9] D.D. Givone and R.P. Roesser: Multidimensional linear iterative circuits – general properties. *IEEE Transactions on Computers*, **21**(10), (1972), 1067–1073. DOI: 10.1109/T-C.1972.223453

[10] R. Hilfer: *Applications of Fractional Calculus in Physics*. World Scientific Publishing Company, Singapore, 2000, DOI: 10.1142/3779

[11] T. Kaczorek and K. Rogowski: *Fractional Linear Systems and Electrical Circuits*. Studies in Systems, Decision and Control, Springer International Publishing Switzerland, 2015.

[12] T. Kaczorek: *Selected Problems of Fractional Systems Theory*. Springer-Verlag, Berlin Heidelberg, 2011.

[13] T. Kaczorek: Positivity and stabilization of fractional 2D Roesser model by state-feedbacks. LMI approach. *Archives of Control Sciences*, **19**(2), (2009), 165–177.

[14] T. Kaczorek: *Two Dimensional Linear Systems*. Springer Verlag, Berlin, 1985. DOI: 10.1007/BFb0005617

[15] N.A. Khan, M.A. Qureshi, S. Akbar and A. Ara: Probing 3D chaotic Thomas' cyclically attractor with multimedia encryption and electronic circuitry. *Archives of Control Sciences*, **33**(1), (2023), 239–271. DOI: 10.24425/acs.2023145120

[16] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo: *Theory and Applications of Fractional Differential Equations*. Elsevier r.v., Amsterdam, The Netherlands, **204** 2006.

[17] Y-H. Lan and Y. Zhou: LMI-based robust control of fractional-order uncertain linear systems. *Computers and Mathematics with Applications*, **61**(3), (2011), 1460–1471. DOI: 10.1016/j.camwa.2011.03.028

[18] J-G. Lu, and Y-Q. Chen: Robust stability and stabilization of fractional order interval systems with the fractional order: The case $0 < \alpha < 1$. *IEEE Transactions on Automatic Control*, **55**(1), (2010), 152–158. DOI: 10.1109/TAC.2009.2033738

[19] F. Mainardi: *Fractals and Fractional Calculus in Continuum Mechanics*. Springer-Verlag Wien, New York, **378**, 1997.

[20] W. Marszalek: Two-dimensional state-space discrete models for hyperbolic partial differential equations. *Applied Mathematical Modelling*, **8**(1), (1984), 11–14. DOI: 10.1016/0307-904X(84)90170-7

[21] D. Peaucelle, D. Arzelier, O. Bachelier and J. Bernussou: A new robust $\mathcal{D}$-stability condition for real convex polytopic uncertainty. *Systems & Control Letters*, **40**(1), (2000), 21–30. DOI: 10.1016/S0167-6911(99)00119-X

[22] I. Podlubny: *Fractional Differntial Equations*. Mathematics in science and Engineering, **198**, 1998.

[23] R. Roesser: A discrete state-space model for linear image processing. *IEEE Transactions on Automatic Control*, **20**(1), (1975), 1–10. DOI: 10.1109/TAC.1975.1100844

[24] A. Soukkou, Y. Soukkou, S. Haddad, M. Benghaem and A. Rabhi: Review, design, stabilization and synchronization of fractional-order energy resources demand-supply hyperchaotic systems using fractional-order PD-based feedback control scheme. *Archives of Control Sciences*, **33**(3), (2023), 539–563. DOI: 10.24425/acs.2023.146957

[25] Y. Zou, H. Xu and W. Wang: Stability for two-dimensional singular discrete systems described by general model. *Multidimensional Systems and Signal Processing*, **19**, (2008), 219–229.