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Phase space analysis of semiconductor laser dynamics

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Abstract

The dynamics of semiconductor lasers are modelled in the time domain using a pair of differential equations known as rate equations. The analysis, based on temporal solutions of these equations, yields practical results utilised in various applications. Alternatively, an analysis employing the phase space method, a well-established analytical tool in applied mathematics, provides a more comprehensive perspective on semiconductor laser dynamics. The main purpose of this paper is to provide a detailed and intuitive introduction to phase space analysis in the context of semiconductor laser dynamics. The goal is to offer an easily comprehensible description of the mentioned method, placing emphasis on the graphical representation and physical interpretation of the results. The method effectiveness is shown through its application to selected practical problems. Furthermore, semiconductor laser dynamics can be treated as an illustrative example, showcasing the applicability of the method, which can be readily extended to other types of lasers or even more advanced dynamic systems.

1. Introduction

Semiconductor lasers are coherent light sources with unique features, including direct electrical pumping, high efficiency, and compact size. Their dynamic properties, characterised by very short switching times and a straightforward pulse generation, make them a perfect choice for various applications, such as telecommunication [1], medicine [2], or optical remote sensing [3]. Nevertheless, a more in-depth theoretical analysis reveals the inherently complex behaviour of this dynamic system [4]. A comprehensive understanding of the dynamic intricacies of semiconductor lasers unlocks pathways to optimise their performance, enhance reliability, and pave the way for innovative technological breakthroughs.

Semiconductor lasers dynamics are modelled with a system of differential equations, called rate equations, describing mutual interaction between carriers in an active medium and photons inside a laser cavity. They include all light and matter interaction mechanisms influencing the system output. Rate equations can be employed to derive various properties of lasers, such as DC characteristics or frequency response, however, for more advanced analysis, such as step response, equations have to be solved in time domain. This task can be complicated due to the nonlinear nature of the dynamic system and numerical methods are commonly used for accurate solutions.

Another approach to analyse the behaviour of a dynamic system is through a phase space analysis [5]. This method uses multidimensional space to demonstrate all possible states in which the dynamic object can exist, treating time as a parameter. This powerful tool offers a different perspective on laser dynamics, leading to a better understanding of such a nonlinear system. With support of the numerical methods it is possible to generate the phase space with a satisfactory resolution within a short computation time.

Phase space analysis itself is not a recent concept; it was developed by mathematicians in the 19th century in response to the demands of physics of that time [6]. As the method gained popularity, it found widespread application in solving dynamic problems across various fields, including laser technology [7]. Even early publications on semiconductor lasers dynamics described the use of phase space to illustrate the interaction between the matter and light [8]. Today, phase space analysis stands as a fundamental tool for describing semiconductor lasers, especially their chaotic behaviour under external perturbation.

The purpose of this paper is to present a thorough and straightforward introduction to phase plane analysis within the context of semiconductor laser dynamics. It was written to support researchers taking their initial steps into the topic. Emphasis will be placed on the graphical interpretation of results to enhance understanding of the problem and foster a proper intuition. Educators are encouraged to use this paper as a resource for their own teaching materials.

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The motivation behind this paper stems from the recognition of an existing gap in the literature concerning the clear introduction to the phase space analysis of semiconductor laser dynamics. While the method itself is widely used in advanced topics, a thorough and accessible presentation of the methodology appears to be lacking in current publications. This paper aims to fill that void, offering a concise and accessible overview to empower researchers, teachers and students in understanding and applying phase space analysis effectively.

While the authors' main focus is on semiconductor lasers, it is important to note that the phase plane analysis has universal applicability. The insights and description provided can be readily adopted for analysing various nonlinear dynamic systems, extending the utility of phase space analysis beyond the presented context of semiconductor lasers.

2. Rate equations

The fundamental model governing the dynamics of single-mode semiconductor lasers is mathematically described by a set of rate equations. These equations constitute a nonlinear system of differential equations that describes the mutual interaction between carriers and photons within an active region:

$$\frac{dN(t)}{dt} = \frac{I(t)}{eV} - \frac{N(t)}{\tau_N} - v_g G(N,S)S(t), \tag{1a}$$

$$\frac{dS(t)}{dt} = \Gamma v_g G(N,S)S(t) - \frac{S(t)}{\tau_P} + \frac{\beta_{sp} N(t)}{\tau_r}, \tag{1b}$$

where $N(t)$ and $S(t)$ denote the carrier and photon densities, respectively. The term $I(t)$ represents the current flowing through the junction, e is the elementary charge, V is the volume of the active region, τ_N is the carrier lifetime, v_g is the group velocity, $G(N,S)$ is the gain function, Γ is the confinement factor, τ_P is the photon lifetime, β_{sp} is the spontaneous emission factor, and τ_r is the radiative component of the carrier lifetime.

It is noteworthy that the literature often includes the third equation describing the electromagnetic wave phase evolution [9]. However, this equation is linearly dependent on the carrier density and does not significantly impact the laser intensity dynamics. For the sake of simplicity, the authors will neglect this mentioned equation.

Each term of the equations above has a physical interpretation originating from the quantum electronics. In (1a), the first expression describes the pumping rate directly related to the electric current flowing through the junction. The second term encompasses all mechanisms responsible for carrier loss that do not cause the light amplifications, such as spontaneous emission and Auger recombination. The last expression describes the carrier loss due to the stimulated emission. This expression, along with additional confinement factor, recurs in (1b) and it represents the change in photon number resulting from optical gain. The subsequent term describes photons escaping from the laser cavity due to transmission and dissipative losses. The final term describes a minor contribution of spontaneous emission in the lasing mode. While typically this term is neglected during a laser generation, its presence is essential to initiate the laser process.

An additional comment is necessary regarding the gain function, denoted in its general form as $G(N,S)$. In its most common form, it takes the shape of a linear function independent of the actual photon density:

$$G(N) = g_0(N - N_{tr}). \tag{2}$$

Here, g_0 represents the gain coefficient, and N_{tr} is the carrier density at which the active region becomes optically transparent. While this straightforward model is suitable for basic simulations and educational purposes, more advanced models are available in the literature, incorporating logarithmic relationships [10] or piecewise-defined functions [11]. Additionally, to account for gain saturation effects, another function dependent on photon density is used:

$$G(N,S) = \frac{g_0(N - N_{tr})}{1 + \varepsilon_{sat} S}. \tag{3}$$

In (3), ε_{sat} denotes the gain saturation coefficient. Although the saturation effect will be neglected for a fundamental understanding of semiconductor laser behaviour, its impact on laser dynamics will be demonstrated subsequently.

The brief overview above serves as a concise reminder of semiconductor laser rate equations, playing an integral role in ensuring the completeness of the phase plane method description. For a more in-depth understanding, readers are encouraged to consult various academic books and dedicated literature on the subject.

3. Phase plane construction

From the perspective of the article subject, it is more intriguing to examine the presented model from the control theory point of view. Rate equations are a system of two ordinary differential equations with two independent state variables, the carrier and photon densities. In (1a) and (1b), the variables are clearly separated from their derivatives, what means that for a given constant current, the change of the state variables depends only on the values of these variables.

It is possible to imagine a two-dimensional plane, where each X and Y axis represents one of the state variables: carrier N and photon S densities, respectively. In this conceptual framework, each point on the plane, denoted as $P(N_P, S_P)$, represents a different state of the dynamic system. According to the rate equations, the derivatives of the state variables depend only on the current state of the system, i.e., the actual values of N and S. Therefore, a vector $\vec{v} = \left[\frac{dN}{dt}, \frac{dS}{dt}\right]$ can be assigned at each point of the plane (Fig. 1), creating a vector field. These vectors describe the rate of state change and they point to the state of a laser in the next moment of time. Alternatively, the vector can be represented by its length $||\vec{v}||$ (a magnitude) and direction θ (an angle between the vector and the X axis). As time progresses, the state will continuously change, tracing a path on the phase plane. In mathematical terms, this path is called *a trajectory* or *an orbit*, and the velocity vector $\vec{v}$ is always tangent to this path.

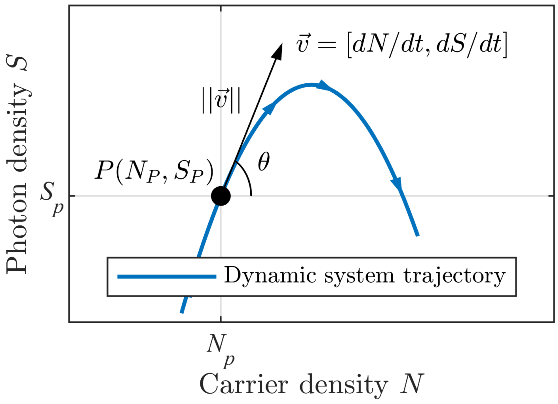

Fig. 1. A graphical interpretation of the trajectory of a dynamic system on the phase plane.

Rate equations are non-autonomous differential equations due to the presence of the excitation term $\frac{I(t)}{eV}$. Consequently, the vector field on the phase plane varies with different currents and to construct a two-dimensional phase plane a constant current must be assumed. For arbitrary current values, the vector field can be calculated using numerical methods by solving the rate equations for each point on a grid iteratively. This approach is fast and straightforward, with the main limitation being the resolution of the solution. To enhance the understanding of the phase plane method, the subsequent paragraphs will present numerically obtained plots using 1550 nm distributed-feedback (DFB) laser diode parameters adopted from literature [12] and gathered in Table 1. The current was arbitrarily set to $I = 5 \times I_{th}$ as a value frequently found in practical applications.

Table 1
Dynamic parameters of the semiconductor laser used in numerical calculations.

Parameter	**Value**
Active region volume V	30 μm^3
Group velocity v_g	$8.33 \cdot 10^7$ m/s
Gain coefficient g_0	$9.9 \cdot 10^{-16}$ cm^2
Carrier density at transparency N_{tr}	$1.23 \cdot 10^{18}$ cm^{-3}
Carrier lifetime τ_N	1.2 ns
Photon lifetime τ_P	1.7 ps
Radiative carrier lifetime τ_r	1.8 ns
Confinement coefficient Γ	0.2
Spontaneous emission factor β_{sp}	$3 \cdot 10^{-5}$
Gain saturation coefficient ε_{sat}	$2.77 \cdot 10^{-17}$ cm^3

The initial step in plotting a vector field on the phase plane is calculating its vertical and horizontal components separately by evaluating derivatives from (1a) and (1b) for each point. The results in the form of combined contour and quiver plots are presented in Fig. 2. Additionally, red curves mark points where the derivatives are equal to zero. These characteristic lines, essential for understanding the system behaviour, are referred to as *nullclines*. In the presented case, they divide the vector fields into two distinct regions. For carrier density dynamics, being on the left

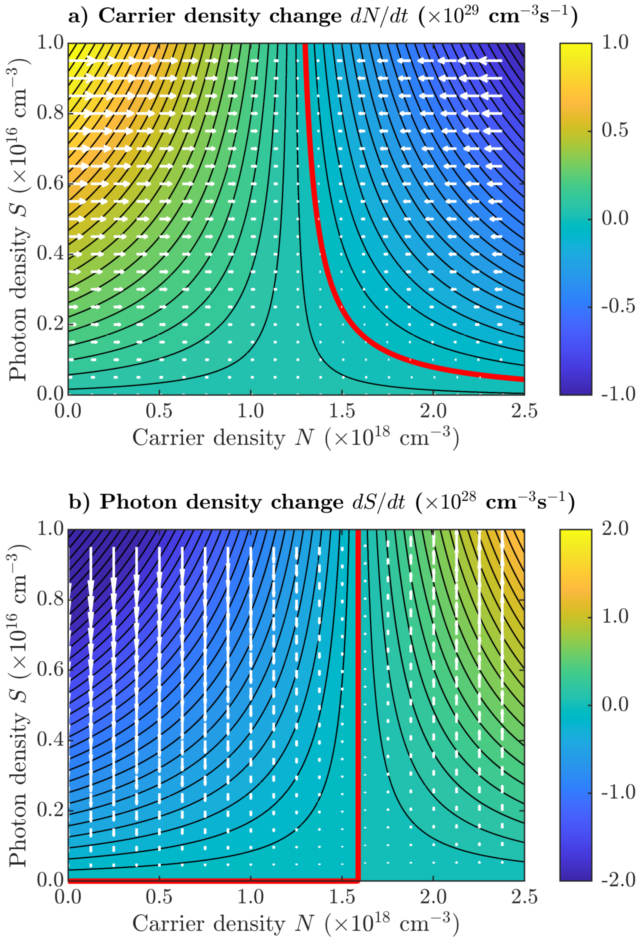

Fig. 2. Horizontal and vertical components of the vector field in the phase plane. Black lines are isoclines and red lines represents nullclines.

side of the red curve (N-nullcline) signifies an increase in carrier density (with positive derivative values), causing the point representing the system state to move to the right. Conversely, on the right side of the nullcline, the opposite behaviour occurs. For photon density dynamics, the red curve (S-nullcline) is located slightly over the horizontal axis and at specific point it becomes vertical. The derivative is positive on the right side of the plane, distinguishing it from the carriers. In this situation, as time passes, the point representing the actual state will move upward.

By combining the components, the complete vector field on the phase plane is obtained. Figure 3 shows magnitudes and direction of each vector on the plane. It can be observed that the majority of vectors are almost horizontal. The dominance of this single component arises from the fact that the vertical components are one order of magnitude smaller than the horizontal ones. This pattern changes only in the close neighbourhood of the N-nullcline. At the point where both nullclines intersect, the magnitude of a state change vector $\vec{v}$ is equal to zero. This characteristic point is referred to as *an attractor*, and over time, the dynamic system will evolve towards it.

Both nullclines divide the phase plane into four distinct regions. In the first region, lying along the horizontal axis [dark blue region in Fig. 3(b)], vectors are directed towards first quadrant of the coordinate system. The vectors in

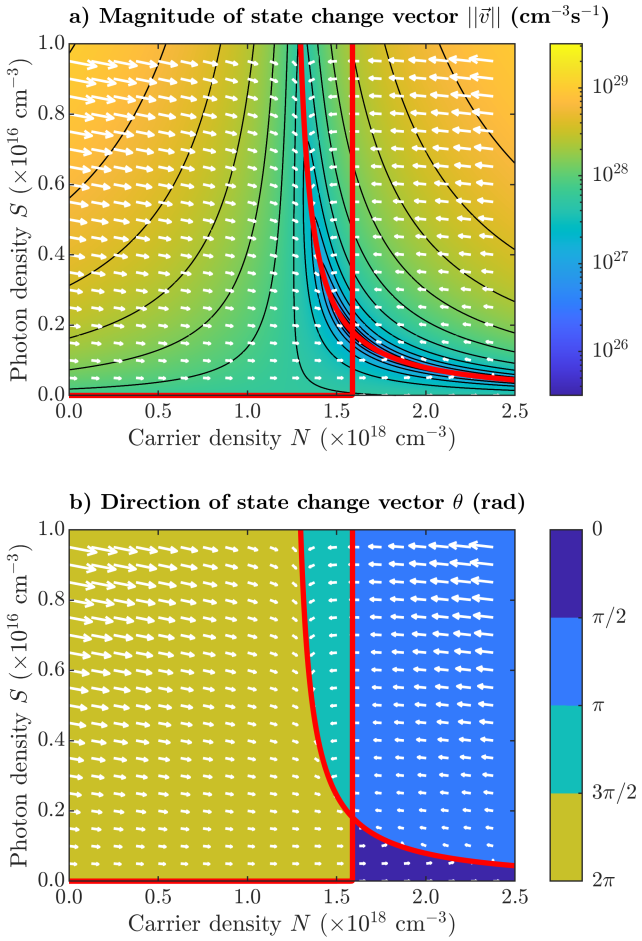

Fig. 3. Magnitude and direction of the vectors in the phase plane. Black lines are isoclines and red lines represent nullclines.

the rightmost region direct towards the second quadrant. A narrow region in the middle of the plane corresponds to the negative values of both rate equations and vectors are oriented to the third quadrant. Lastly, the olive green region in Fig. 3(b) points to the fourth quadrant of the coordinate system. This vector field arrangement implicates the dynamic system behaviour, as it cyclically transitions from one region to another. Vectors for points lying on the nullclines will be purely horizontal or vertical, causing the state variable to reach a local extreme every time the system crosses the red line. These properties collectively contribute to the emergence of oscillations before the system reaches a steady state.

Rate equations are non-autonomous differential equations due to the presence of the excitation term. It appears only in the equation for carrier density dynamics, what means that vertical components of the vectors on the phase plane remain unchanged for different currents. In (1a), the excitation term acts as a constant component and the shape of the vector field for horizontal component does not depend on the actual current value. As the current increases, the vector magnitude will change equally for the whole plane and the whole function will move towards higher values. Therefore, any isocline on the contour plot in Fig. 2(a) is a nullcline for a specific current. This remark is helpful to imagine the vector field for various currents and has a huge implication in practical applications of the phase plane analysis.

4. Physical interpretation

All previous considerations have neglected the physical context of the dynamic system. The mentioned properties of the vector field on the phase plane are a result of a structure of the assumed mathematical model. However, it is crucial to note that the outcomes obtained hold a direct correlation with physical phenomena occurring in the semiconductor laser.

Analysing the vector fields of each component separetly provides information on the state variables dynamics. The N-nullcline locates the points where driving current balances the carrier loss due to a stimulated and spontaneous emission. The impact of the light emission phenomena grows for higher carrier concentrations, i.e., on the right side of the Fig. 3(a), leading to the higher energy level depopulation.

The shape of the N-nullcline will change with the current, as noted in the remark. Since the current can only take positive values, the N-nullcline associated with no net current represents the isocline that intersects the coordinate system origin. For higher excitation values, the nullcline will shift to the right side of the plane, as shown in Fig. 4(a). Generally, it has a hyperbolic shape; however, for a specific excitation, the nullcline will become a vertical line. This occurs when the current is high enough to compensate the spontaneous emission and maintains the carrier density at medium transparency N_{tr}. Under these conditions, the resultant gain of the medium is zero, photons do not interact with carriers, and the carrier density remains constant, regardless of the actual photon density. It is noteworthy that this vertical line also serves as an asymptote for all isoclines on the plane.

The S-nullcline, presented in Fig. 3(b), has a shape of two segments. The first segment lies just above the horizontal axis and arises due to the presence of the spontaneous emission term in (1b). Although its impact on laser dynamics is minor, it plays a crucial role in initiating laser operation. At a certain carrier density value (the threshold), the nullcline becomes vertical, dividing the phase plane into two regions. On the left side of the plane, the intensity of stimulated emission is not sufficient to overcome the optical losses in the laser, resulting in a decrease in the number of photons. Conversely, on the right side, the medium optical gain exceeds total losses, leading to an increase of photons.

During the laser operation, the phase plane is divided by nullclines into four regions. Using previously adopted notation, the first region corresponds to the rise of a carrier density resulting from the driving current. The photon density is too small to obtain an effective stimulated emission and to depopulate the carriers. For higher carrier densities, vectors become more vertical due to the presence of stimulated emission. Moving to the second area, the photon density is high, and stimulated emission is sufficiently strong to decrease the number of carriers in the active region. In the third region, the photon density remains high enough to continue carrier depopulation. However, the optical gain becomes lower than losses, leading to a simultaneous decrease in photons. As the photon density drops sufficiently to allow carriers to build up, the laser enters the fourth region. Here, the increase in carriers, resulting from the current flow, exceeds its decrease caused by stimulated emission.

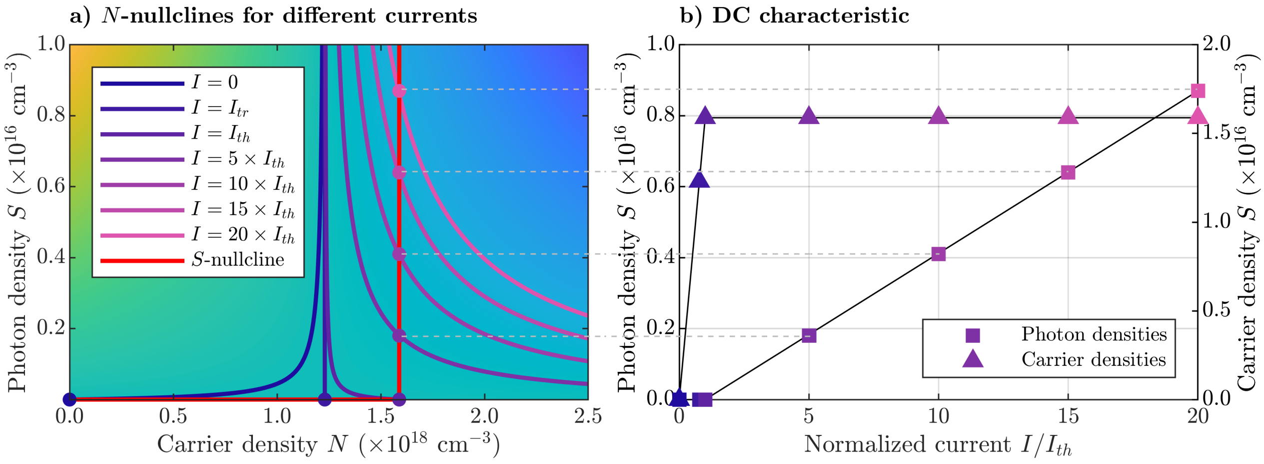

Fig. 4. Location of N-nullclines for different excitation currents (a) and obtaining the DC characteristic using phase space (b).

5. Practical applications

The presented approach, based on graphical interpretation of the phase plane, allows for an intuitive understanding of the properties of semiconductor laser dynamics. However, this method is not limited solely to rate equation analysis; it extends to numerous practical applications. Phase plane analysis is widely used, for instance, in describing chaotic generation [4] or optimising laser diode steering [13]. Here, the authors present much simpler examples of the use of phase plane analysis as alternatives to the analytical approach.

5.1. DC characteristics

Phase planes obtained for a given semiconductor laser provide direct information about its DC characteristics. In the classical approach, to establish quantitative relationships, rate equations are solved by setting them to zero. In the phase plane, this operation corresponds to localising the attractor for different currents. Since the position of the N-nullcline depends on the current, the intersection between nullclines shifts along the S-nullcline. The current value at which the N-nullcline intersects the bending of S-nullcline corresponds to the threshold current I_{th}. Beyond this point,the attractor moves upward on the plane, which corresponds to higher output powers, but the horizontal coordinate of the intersection remains unchanged, indicating carrier density or gain clamping. If the N-nullclines are obtained for uniformly spaced currents, the distance between consecutive attractors on a vertical section of the S-nullcline is fixed,which corresponds to the linear relationship between current and output power.

The concept of plotting a DC characteristics directly from the phase plane is illustrated in Fig. 4(b). The procedure involves identifying the intersection points of two nullclines for different excitation and subsequently plotting each X and Y coordinate as a function of current. This straightforward graphical method provides an immediate understanding of laser stationary behaviour, demonstrating the effectiveness of phase space analysis.

5.2. Step response

Step response of the semiconductor laser is a part of its large signal analysis and is crucial in applications requiring fast switching or pulse operation. In this case, the impact of nonlinear effects cannot be neglected and the analytical approach requires some simplifications. Using numerical methods the time evolution of carrier and photon densities can be determined, revealing the features of the step response, such as relaxation oscillations or turn-on delay. As in the phase plane analysis, the time is treated as a parameter, it focuses more on the energy exchange between carrier and photon densities.

When there is no current flowing through the junction, the laser remains in a stable idle state, depicted on the phase plane as a point at the origin of the coordinate system. The sudden change in current causes a reorganisation of the vector field, initiating movement of the point representing the laser actual state along the local state change vectors $\vec{v}$. The trajectory traced by the laser states will connect the idle state with the new stability point in the manner determined solely by the vector field structure. Hence, the trajectory shape unveils the dynamic features of the semiconductor laser step response.

As an example the step response for current change from $I = 0$ to $I = 5 \times I_{th}$ is considered. The complete trajectory depicted on the phase plane is presented in Fig. 5. After a rapid current change, the point representing the actual state moves from the idle state to the right side of the plane along the X axis. In this region, the vectors are nearly horizontal, resulting in an increase in carrier density with a negligible rise in photons. In the time domain, this phase corresponds to the turn-on delay of the laser. Once the carrier density reaches the threshold value, the local vectors become more inclined upwards, leading to the rise in photon density. At the intersection of the trajectory and the N-nullcline, the carrier density reaches its maximum and the current state point moves to the next region. Here, the photon density increases and it reaches its peak value at next intersection with nullcline. Since the vectors in the successive areas point towards the following region, the trajectory passes them subsequently, taking a shape of an asymmetric spiral. By definition, the

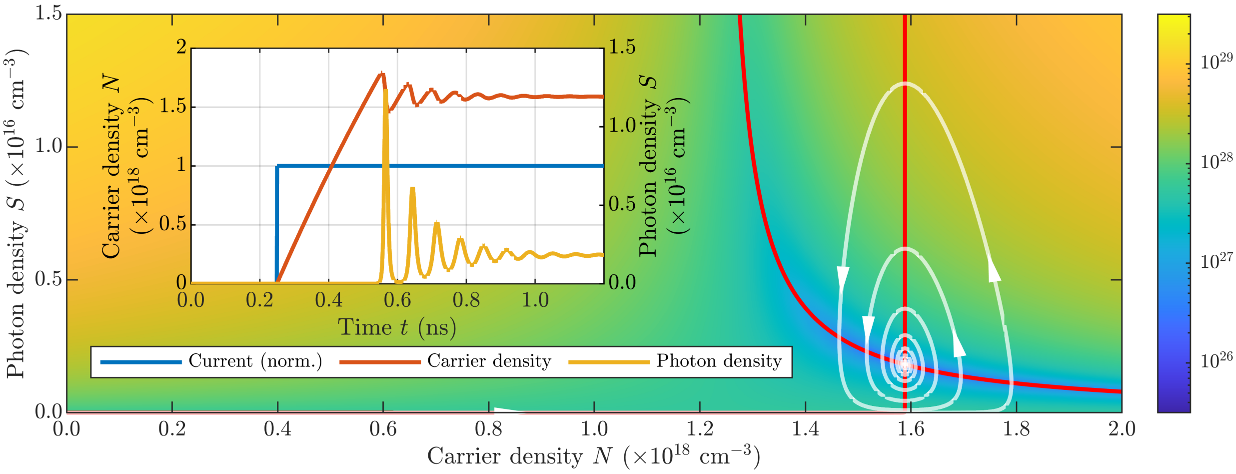

Fig. 5. Step response trajectory (obtained through time domain integration) on the phase plane. Inset: Step response in the time domain.

trajectory intersects nullclines always at a right angle at the local extreme of the corresponding state variable. The transition ends when the system reaches the stability point represented as the attractor point located at the intersection of both nullclines. The spiral shape of the trajectory reveals the resonant nature of laser dynamics, exposed by phase-shifted carrier and photon densities oscillations.

5.3. *Impact of gain saturation*

In the paper, a linear gain model which neglects the gain saturation effect was used for simplicity. Nevertheless, the gain saturation effect plays significant role in laser dynamics, particularly as photon density reaches relatively high values. Through phase space analysis, the impact of gain saturation can be readily determined.

In Fig. 6, a phase plane with a vector field for rate equations including gain saturation is presented. Comparing to the image from Fig. 3, few major differences can be noticed. Firstly, the two-dimensional function of the vector magnitude is sheared, i.e., for higher photon densities the function is shifted to the right. It implicates the shape of both nullclines, especially the S-nullcline. Previously vertical segment is now inclined, which means that horizontal coordinate of the attractor will rise as current increases. Secondly, as the trajectory of a step response is a counter-clockwise spiral, it intersects nullclines earlier than in previous case. It means that the peak values of photon density are significantly smaller and the number of oscillations is reduced. In the time domain, it is manifested by much stronger damping of relaxation oscillations.

5.4. *Other examples*

The presented examples demonstrate the use of phase plane analysis to intuitively interpret fundamental topics in semiconductor laser dynamics. The list of demonstrations can be easily expanded to additional scenarios such as gain switching, Q-switching, or digital modulation. The analysis of the proposed cases may contribute to a better understanding of the phase plane concept.

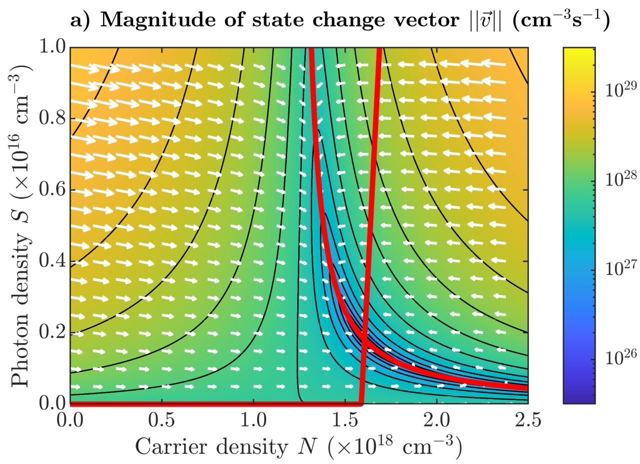

Fig. 6. Magnitude of the vectors in the phase plane for rate equations including gain saturation effect. Black lines are isoclines and red lines represents nullclines.

6. Conclusions

The paper demonstrates fundamentals of a phase space analysis in the context of semiconductor laser dynamics. Its primary objective is to offer an intuitive introduction to this qualitative method. The example results are obtained using simplified rate equations with a linear gain function, while the numerical results are derived based on typical parameters of a DFB laser diode operating at 1550 nm. The step-by-step process of constructing the phase plane and its physical interpretation is described. To demonstrate the method effectiveness, it has been used to solve basic dynamic problems, such as plotting DC characteristics or analysing step response of semiconductor laser. Additionally, the comparison of different mathematical models of gain function is presented.

The presented approach focuses in particular on the rate equations of semiconductor lasers for clarity. However, this concept can be applied to other laser dynamics models. By exploring various scenarios and comparing different laser models, readers can foster an intuition for phase plane analysis and its applications in laser research.

Authors' statement

The authors confirm contribution to the paper as follows: study conception and design, analysis, draft manuscript preparation: P.G.; conceptualisation, literature review, draft manuscript editing: M.M.; supervision, draft manuscript editing, language correction: J.Ś. All authors reviewed the results and approved the final version of the manuscript.

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