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Research paper

Form-finding of optimal cable nets under self-weight based on the Force Density Method

Izabela Wójcik-Grząba[1]

Abstract: Form-finding of cable nets is the main topic of this paper. This initial stage of design path is grounded on the enhanced version of the Force Density Method. Apart from the basic form-finding it includes optimal shaping and adding self-weight of a cable structure. Minimal sum of cable lengths in the structure is treated here as a favourable initial configuration for reaching geometry and force distribution under prestress and self-weight. Regarding tensile forces obtained this way, cable sections can be proposed as the first approximation in further design process not included in this analysis. The basics of classic version of the Force Density Method are introduced in the paper. The nonlinear version of this method is used to solve an optimization problem of minimum weight cable net. The essentials of the procedures for achieving optimal shape and adding self-weight are also included and constitute the Extended Force Density Method proposed by the author. Defining proper input data for the self-weight analysis is crucial to find a new shape possibly close to the optimal one and is also discussed. A few examples of optimal or partially optimal cable nets are presented. It is shown that adding self-weight and elastic material properties can preserve the optimal shape with high accuracy. This allows to switch from the purely geometric problem of form-finding to the initial form of a structure with assumed sections and material. All calculations are performed with the use of the self-developed program UC-Form which is also briefly presented.

Keywords: form-finding, cable net, catenary, force density

[1]PhD., Eng., Warsaw University of Technology, Faculty of Civil Engineering, Al. Armii Ludowej 16, 00-637 Warsaw, Poland, e-mail: izabela.grzaba@pw.edu.pl, ORCID: 0000-0003-3769-8441

1. Introduction

Completing Paraboleum (or Dorton Arena) in Raleigh in 1952 was a breakthrough in the history of cable roof structures. This particular object designed by Maciej Nowicki and many others erected afterwards have been greatly appreciated by the architects and structural engineers around the world. Their specific features differ them from the traditional "rigid" structures and require special approach. Full utilization of cable cross sections carrying only tension is achieved, which is highly desirable in the age of sustainable architecture. Apart from being lightweight, economical and efficient, cable structures are also visually attractive. Steel cables are characterised by high tensile strength which enables spanning large areas without inner supporting colliding with the interior design. That is why they are mainly used in stadiums, sports, exhibition and concert halls.

The main disadvantages of cable structures result from negligibly small bending and shear stiffness of cables. In order to achieve transverse stiffness which enables transferring normal loads to the supports such structure needs to be prestressed and gain geometrical stiffness. Before that a cable net is geometrically instable. What is more, adding new loads results in significant changes in the shape and force distribution which means that cable structures are geometrically nonlinear [1]. These drawbacks have been examined since the sixties of 20th century and resulted in many methods of form-finding. The most appreciated are Dynamic Relaxation Method, Transient Stiffness Method, Force Density Method and their modifications [2]. The Force Density Method appears to be simple, elastic and universal tool for form-finding of not only cable but also tensegrity and membrane structures. Proposed by Schek in 1974 in [3] is constantly developed, refined and used to solve new problems [4–6].

Nowadays many researchers focus on the optimisation methods, as we tend to sustainable design. Force Density Method can be used as the main or auxiliary tool in optimisation process of tensile or compressive structures as it is shown in [7–11]. Some similarities with minimal surfaces are also examined in the case of cable and membrane structures to find minimal-length or minimal-area solutions [6, 12–14].

Form-finding is the first step of design process of tensile structures and is conducted to determine the initial geometry and prestress in the cables. This configuration is then loaded with self-weight and all the live loads and submitted for traditional static and/or dynamic design. Proper prestress of a cable net should provide required spatial stiffness and adding self-weight shouldn't have significant influence on the configuration, as it was shown in [15]. But self-weight is crucial when the structure consists of slack cables and particularly while analysing erection process or failure of chosen elements. Also self-weight of quite large cable sections can influence tensile force distribution which is pointed in [16]. Moreover using elastic, catenary element for form-finding gives information about the initial (unstrained) element lengths needed in subsequent design stages as it is stated in [17]. Finally, the design procedure presented in Eurocode 3 [18] advises to treat gravity loads and prestress as one load case. Hence it would be beneficial to get initial configuration under pretension and self-weight and better estimate target sections and unstrained lengths. In a few papers there was some attempts to manage this problem. For example authors of [16] and [19] loaded all the cable elements as if they were taut. They submitted point loads equal to half of elements self-weight in the nodes.

In the first paper the force density was redefined in order to take the exact values of reactions into account. The approximate, parabolic formulation of a cable element is employed in the second article. In both papers new equations regarding behaviour of a cable net are added to the original FDM system of equations. Article [17] lacks the details of the utilized methods but force density is defined on the basis of the force value in one end of the element.

The main aim of this article is to present the method of adding self-weight of straight (prestressed) and catenary (slack) cables at the stage of form-finding. Moreover the author shows that such configuration can be very close to the optimal one which is a new approach. Universal method of adding self-weight of taut and slack elastic cable elements to Force Density Method was proposed by the author in [20] and [21] and is briefly explained in Chapter 3 and utilized in calculation examples in this article. Moreover, for economical material use, the optimisation method is proposed. Minimising self-weight of a cable net results in a configuration under uniform prestress which is called here optimal. Cable lengths found in this analysis are then used to define initial lengths for the analysis under prestress and self-weight. As it is proved in the examples in Chapter 6, such configuration can be very close to the optimal one. It also fulfils the idea by Wanda Lewis who claims that minimal-length or minimal-surface tensile structures are the most rational solutions for the initial shape [1]. Such an approach was also presented by the author in [22].

2. The essentials of the Force Density Method

The calculation procedure presented in this paper is based upon the Force Density Method (FDM) proposed by Schek in [3]. According to his idea a cable net can be defined as a set of straight, weightless truss elements working only in tension, ended with nodes. The structure is supported in fixed nodes with coordinates $\overline{x}, \overline{y}, \overline{z}$. The remaining nodes are free and can be submitted for point forces $\boldsymbol{p}_x, \boldsymbol{p}_y, \boldsymbol{p}_z$. Figure 1 presents 3-elements example of a cable net along with the incidence matrix $[\boldsymbol{C}\overline{C}]$ defined in FDM to describe the elements connectivity.

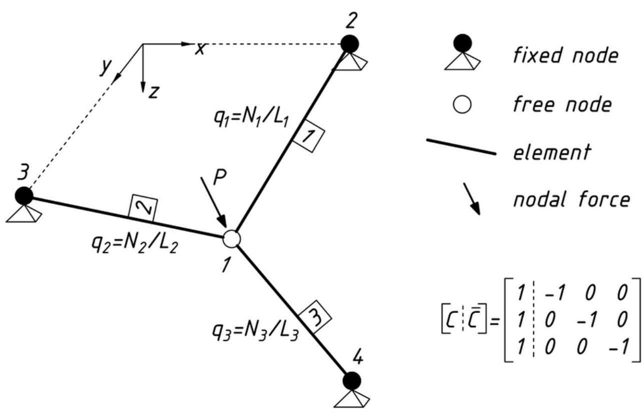

Fig. 1. 3-elements cable net and its definition

First columns of the matrix correspond to free nodes (i.e. $\boldsymbol{C}$), second columns correspond to fixed nodes (i.e. $\overline{C}$) and rows correspond to elements. The matrix consist only of the numbers 1, –1 and 0 which mean consecutively: starting and ending node of the element and no connection of the node with a particular element. Square, diagonal matrices and corresponding vertical vectors with elements from diagonals are marked with the same letter in capital and small version.

The objective of a form-finding problem is to find coordinates of free nodes $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ which satisfy equilibrium equations of the structure. Nodal equilibrium equations can be defined as shown below:

$$\begin{cases} \boldsymbol{C}^T \boldsymbol{X}_\Delta \boldsymbol{L}^{-1} \boldsymbol{n} = \boldsymbol{p}_x \\ \boldsymbol{C}^T \boldsymbol{Y}_\Delta \boldsymbol{L}^{-1} \boldsymbol{n} = \boldsymbol{p}_y \\ \boldsymbol{C}^T \boldsymbol{Z}_\Delta \boldsymbol{L}^{-1} \boldsymbol{n} = \boldsymbol{p}_z \end{cases} \tag{2.1}$$

where matrices $\boldsymbol{X}_\Delta$, $\boldsymbol{Y}_\Delta$, $\boldsymbol{Z}_\Delta$ contains projections of element lengths in x, y and z direction. These equations are nonlinear due to unknown coordinates $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$. Defining a new quantity called force density as ratio between element force and length $\boldsymbol{q} = \boldsymbol{L}^{-1}\boldsymbol{n}$ changes this system of equations into a linear form. The solution with the use of auxiliary matrices $\boldsymbol{D} = \boldsymbol{C}^T \boldsymbol{Q} \boldsymbol{C}$, $\overline{\boldsymbol{D}} = \boldsymbol{C}^T \boldsymbol{Q} \overline{\boldsymbol{C}}$ is shown below:

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{D}^{-1} \left(\boldsymbol{p}_x - \overline{\boldsymbol{D}\boldsymbol{x}}\right) \\ \boldsymbol{y} &= \boldsymbol{D}^{-1} \left(\boldsymbol{p}_y - \overline{\boldsymbol{D}\boldsymbol{y}}\right) \\ \boldsymbol{z} &= \boldsymbol{D}^{-1} \left(\boldsymbol{p}_z - \overline{\boldsymbol{D}\boldsymbol{z}}\right) \end{aligned} \tag{2.2}$$

In order to find the configuration and tensile force distribution of the previously defined prestressed cable net the values of force density in elements have to be proposed. The proportions between force density values determine the geometry and their absolute values yield the prestress level.

3. The Extended Force Density Method with self-weight

Prestress and self-weight of the cable structure are present in each load combination so it is convenient to treat them as a starting point for subsequent analyses under other live loads. Extended Force Density Method (EFDM) proposed by the author in [21] enables form-finding of the cable nets under these two loads. What is more in this approach a cable net can be fully or partially prestressed or even fully taut. For this purpose the exact, catenary element was introduced, as opposed to the commonly used parabolic element, which has some accuracy limitations. In this method a cable under self-weight which shape is catenary is replaced by statically equivalent straight element as it is shown in Fig. 2. The forces acting along the catenary element are separated into constant N_{sub} and variable components corresponding to prestress and self-weight. The latter is submitted to substitutive element as reaction forces R_A and R_B in nodes.

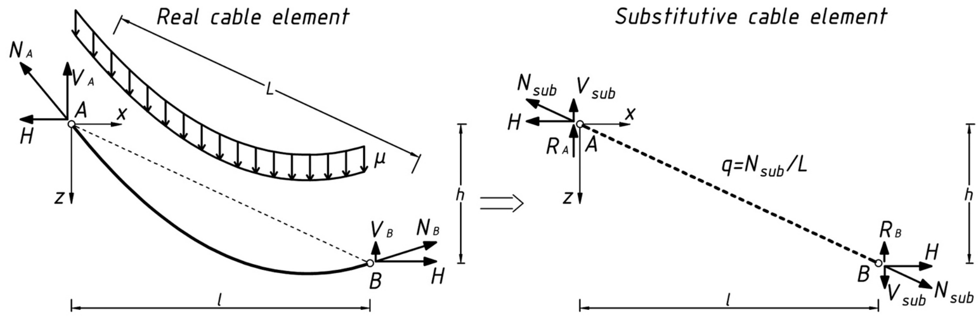

Fig. 2. Substitutive cable element in EFDM

In order to introduce this modification to the classis version of the FDM two stages are needed. First the external, vertical loads vector $\boldsymbol{p}_z$ have to be summed with the vector $\boldsymbol{p}_r$ of nodal reactions from self-weight. This vector stores the sums of reactions R_A and R_B in each node.

$$\begin{cases} \boldsymbol{Dx} = \boldsymbol{p}_x - \overline{\boldsymbol{Dx}} \\ \boldsymbol{Dy} = \boldsymbol{p}_y - \overline{\boldsymbol{Dy}} \\ \boldsymbol{Dz} = (\boldsymbol{p}_z + \boldsymbol{p}_r) - \overline{\boldsymbol{Dz}} \end{cases} \tag{3.1}$$

Second step involves iterative procedure in which new force density values for the structure under self-weight are obtained. For this purpose catenary element equilibrium equation (3.2) is used to build a new system of nonlinear equations (3.3) for unknown force densities (details can be found in [22]).

$$g_w(q) = \frac{HL_0}{EA} + \frac{H}{\mu}\left[\operatorname{arsinh}\left(\frac{V_A}{H}\right) + \operatorname{arsinh}\left(\frac{\mu L_0 - V_A}{H}\right)\right] - l = 0 \tag{3.2}$$

In Eq. (3.2) L_0 is the initial cable length, EA is a longitudinal stiffness, μ is self-weight per meter and other symbols are presented in Fig. 2. Tensile force components V_A and H are also functions of the force density q. New system of nonlinear equations (3.3) applies for all the elements in the structure and can be solved with iterative Newton procedure.

$$\boldsymbol{g}_w(\boldsymbol{q}, \boldsymbol{x}(\boldsymbol{q}), \boldsymbol{y}(\boldsymbol{q}), \boldsymbol{z}(\boldsymbol{q})) = \boldsymbol{0} \tag{3.3}$$

4. Optimization – minimum-length problem

Schek in [3] presented three major problems of cable net minimisation with the aid of the FDM. Dzierżanowski and Wójcik-Grząba in [11] focused on one of them which is the sum of cable lengths minimisation. This problem appears to be obvious from the economy and sustainability point of view. It can be shown that starting from self-weight minimisation of a cable net we can get to the state of uniform prestress state.

Starting from the minimum-weight setup of the structure and assuming homogenous materials we get to minimum-volume problem. After putting limit tensile stress σ_T in all cables and constant cross-section A_i along the i-th element ($i = 1, \dots, s$) the overall volume V of a cable net can be written as:

$$V = \frac{1}{\sigma_T} \sum_{i=1}^{s} N_i L_i, \quad N_i = A_i \sigma_T \tag{4.1}$$

Now making the supposition of uniform prestress in the whole cable net to tensile force $N = A_0 \sigma_T$, where A_0 is a cross-section area sufficient for the assumed level of tension, the solution of minimisation problem becomes as shown in Eq. (4.2):

$$\hat{V} = A_0 \min \left\{ \sum_{i=1}^{s} L_i (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \,|\, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in R^m \right\} \tag{4.2}$$

Let's introduce a new, auxiliary problem of minimisation of the functional: $\wp(\boldsymbol{q}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \boldsymbol{l}^T \boldsymbol{Q} \boldsymbol{l}$. After defining force density matrix for uniformly prestressed cable net as: $\hat{q} = A_0 \sigma_T \boldsymbol{L}^{-1} \mathbf{1}$ and putting it in the above formula for $\wp(\boldsymbol{q}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$, we get a new form of the functional:

$$\frac{1}{\sigma_T} \wp(\hat{q}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = A_0 \boldsymbol{l}^T \boldsymbol{L}^{-1} \boldsymbol{l} = A_0 \sum_{i=1}^{s} L_i (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \tag{4.3}$$

Minimization of the functional gives the same formula as in Eq. (4.2), so the supposition of uniform prestress state is equivalent to minimum-volume structure.

As it was shown, in order to find the optimal configuration, it is necessary to impose the state of uniform prestress in the whole cable net. It can be achieved by adding to the FDM new nonlinear system of equations for unknown values of force density as Schek proposed in [3]:

$$\boldsymbol{g}(\boldsymbol{q}^*) = \boldsymbol{n}^* - A_0 \sigma_T \mathbf{1} = \boldsymbol{Q}^* \boldsymbol{l}^* - A_0 \sigma_T \mathbf{1} = \mathbf{0} \tag{4.4}$$

This version is further called nlFDM and is used to achieve the optimal configuration. The vectors with asterisks contains only elements taking part in the optimisation process. In particular cases it is justified to optimise only the selected parts of the structure, as it is shown in Chapter 6. Solving the system of equations (4.4) gives the new force density vector defining geometry of the optimal structure. The solution can be found with the use of iterative Newton procedure. The case of partial optimisation yields underdetermined system of equations, which can be solved with the aid of least square approximation. The increment of the force density vector in each iteration can be defined as: $\boldsymbol{\Delta q} = \boldsymbol{G}^+ \boldsymbol{b}$, where: $\boldsymbol{G}^+ = \boldsymbol{G}^T \left(\boldsymbol{G} \boldsymbol{G}^T \right)^{-1}$, $\boldsymbol{G} = \left. \frac{\partial g}{\partial g} \right|_{\boldsymbol{q}_{(n-1)}}$, $\boldsymbol{b} = -\boldsymbol{g}\left(\boldsymbol{q}_{(n-1)} \right)$ and all the elements of matrix $\boldsymbol{G}$ are given in [21].

Optimal configuration can only be achieved, when the equilibrium equations in all nodes are satisfied under the uniform prestress. It means that all the forces with equal values converging in a given node have to get into a specific position of balance in three directions. Generally it is hard to verify if such configuration is possible before the calculations. When the convergence of the nlFDM solution is not achieved, it usually means that the optimal configuration is not possible for given input data.

As it was shown in this and previous chapters, Force Density Method can be extended by adding new system of equations which provide different constraints on cable nets. Apart from the two shown above some authors proposed also different constraints like nodes distance [3], node coordinates [23], reaction component values [24], elastic properties of prestressed element [3, 23]. Also some modifications were made in order to analyse cable structures with compression members [25] and bending elements [26]. It means that Force Density Method is still very elastic and versatile tool for cable structures analyses.

5. Calculation procedure

The aim of this paper is to find configuration of a cable net under self-weight and prestress which is possibly close to the optimal one. Methods nlFDM and EFDM presented in previous chapters are employed in proper sequence. Some details concerning verification, accuracy, convergence and recommended values of initial parameters of the method are presented in [21] and [22]. Generally, fast convergence is achieved for both nlFDM and EFDM, when starting configuration is sufficiently close to the solution.

In general, configuration of a cable net under self-weight differs from the optimal one, loaded only with the uniform prestress. In order to find a good agreement between them it is crucial to assume proper initial lengths of elastic cable elements. For this purpose initial (unstrained) element lengths L_0 for EFDM analysis are calculated on the basis of the lengths from the optimal configuration L_{opt}. Elastic elongation is subtracted from the optimal lengths L_{opt} as it is shown below:

$$L_0 = L_{\text{opt}} - \frac{N L_{\text{opt}}}{EA} \tag{5.1}$$

As it can be seen in the Eq. (5.1), the assumed cross-section area and Young modulus of material are necessary to perform the analysis of the structure under self-weight with the use of EFDM.

Different options of calculations presented in this paper are employed consecutively in the procedure listed below:

(0. find initial configuration – FDM);
1. assume prestress forces in all or in the part of elements;
2. find configuration in uniform prestress state – optimisation with nlFDM;
3. collect optimal element lengths L_{opt};
4. calculate unstrained element lengths L_0 (Eq. (5.1));
5. assume elements cross-sections satisfying ultimate limit state for cable elements;
6. find final configuration under self-weight – EFDM.

Step 0 is necessary when optimal solution is not converging. Usually FDM analysis with force density values from 0.1 to 1 kN/m in each cable is sufficient to get the auxiliary configuration for the optimal one. As a result of the calculation procedure the real geometry and force distribution in the cable net under self-weight and prestress is obtained. This solution is also close to the optimal one and can be transferred as the input to static and dynamic analyses under live loads.

Calculation examples presented in Chapter 6 are executed in program UC-Form developed by the author in Scilab package. More details regarding the algorithm and instruction manual are present in [21]. Input data due to geometry and material can be imported to UC-Form from the MS Excel auxiliary file, which is a very convenient way of exchanging data between different programs. Calculation results are automatically presented in an interactive window containing current view of a structure and some switches to change the visibility of particular elements. Additionally the solution can be displayed in numerical form in Scilab console, so it is possible to use obtained data in other programs.

It should be emphasized, that not always the optimal form of a cable net can be found, because of geometrical reasons. Moreover, sometimes the optimal configuration is not satisfactory from the functional or esthetical point of view. In some cases only partial optimisation is possible. Then, it is crucial to identify the groups of elements in which the uniform prestress can be achieved. Usually a basic FDM analysis is a good source of information about initial tensile forces distribution and can help to choose the areas of similar force values and cross-sections used in following stages of design. Table 1 summarizes the essential properties of spiral strand wire ropes used in the calculation examples in Chapter 6.

Table 1. Properties of spiral strand wire ropes

Type	**ϕ [mm]**	**F_{Rd} [kN]**	**EA [kN]**	**μ [kg/m]**
SS16	16	154	27000	1.26
SS30	30	524	95000	4.29
SS115	115	7440	1180000	63.70

6. Calculation examples

6.1. Closed cable net covering circular area

First structure analysed in this paper is a closed cable net stretched on the circular ring with variable ordinates and cables arranged orthogonally. Tensile force value of 100 kN is assumed in all the elements in order to find the optimal configuration which is shown in Fig. 3. The orange line from node 222 to 223 is a symmetry axis which profile is used to compare different solutions.

Next, obtained element lengths L_{opt} were used to calculate the initial ones for the analysis under self-weight L_0 according to Eq. (5.1). A spiral strand wire rope SS16 was chosen for all the elements (Table 1). Figure 4 shows the comparison of central axis profiles of optimal and under self-weight configurations and it can be observed that they are nearly the same.

Figure 5 shows the comparison of force values distribution and Table 2 summarizes the extreme values of forces and force densities in both configurations. These values are very close and the sums of lengths are identical. Obtained configuration under self-weight can be regarded as optimal.

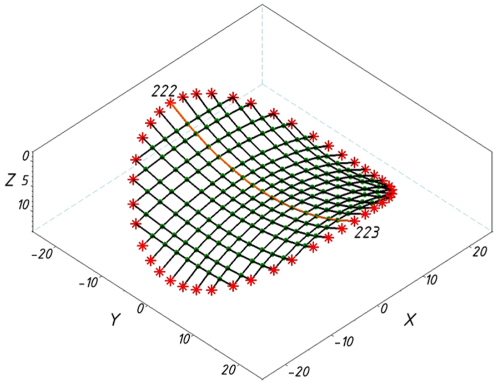

Fig. 3. Optimal configuration of the closed cable net with symmetry axis shown

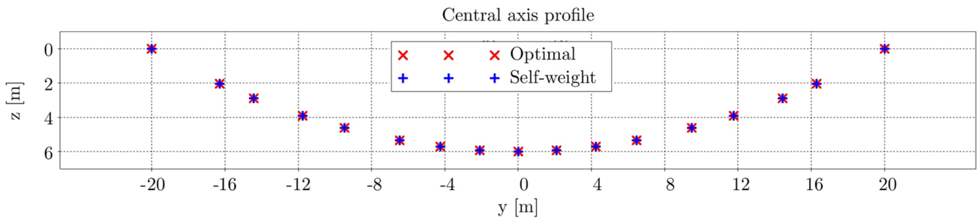

Fig. 4. Comparison of the central axis profiles of the optimal and under self-weight configurations

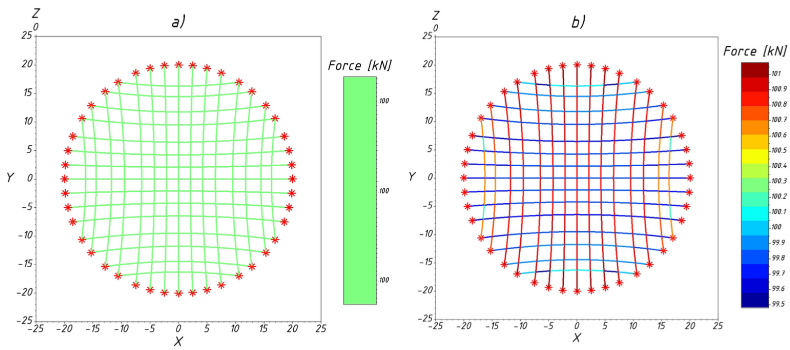

Fig. 5. Tensile force distribution in: a) optimal; b) under self-weight configuration of the closed net

Table 2. Comparison of optimal and under self-weight versions of the closed cable net

Version	$N_{\min}$ **[kN]**	$N_{\max}$ **[kN]**	$q_{\min}$ **[kN/m]**	$q_{\max}$ **[kN/m]**	ΣL_i **[m]**
Optimal	100.00	100.00	23.61	49.09	1028.40
Self-weight	99.49	101.08	23.53	49.59	1028.40

6.2. Open cable net covering circular area

In the next example an open cable net is analysed. It was developed from the previous one by removing some of the supports and adding edge cables instead. In this case edge cables should have higher prestress than inner cables in order to achieve possibly large area covered by the roof. This means that uniform prestress in the whole cable net cannot be obtained. Additionally, due to the particular cable net layout, there is no optimal solution, even if different force values are assumed for edge and inner cables (e.g. 1000 and 100 kN). In such case equilibrium configuration does not exist. Therefore only force values in inner cables are assumed equal to 100 kN. Edge cables are excluded from optimisation. In order to find the most advantageous geometry of the cable net three different versions of edge cables prestress are compared. In the first one shown in Fig. 6a initial force density values are the same in the whole cable net and equal to 20 kN/m. Second optimal version is shown in Fig. 6b and was obtained with initial force density values equal to 200 kN/m in edge cables and 20 kN/m in inner cables. In the last version initial force densities in the edge cables are 2000 kN/m and 20 kN/m in the inner cables. This optimal version is shown in Fig. 6c where the highest prestress in edge cables and the largest area covered by the net can be observed.

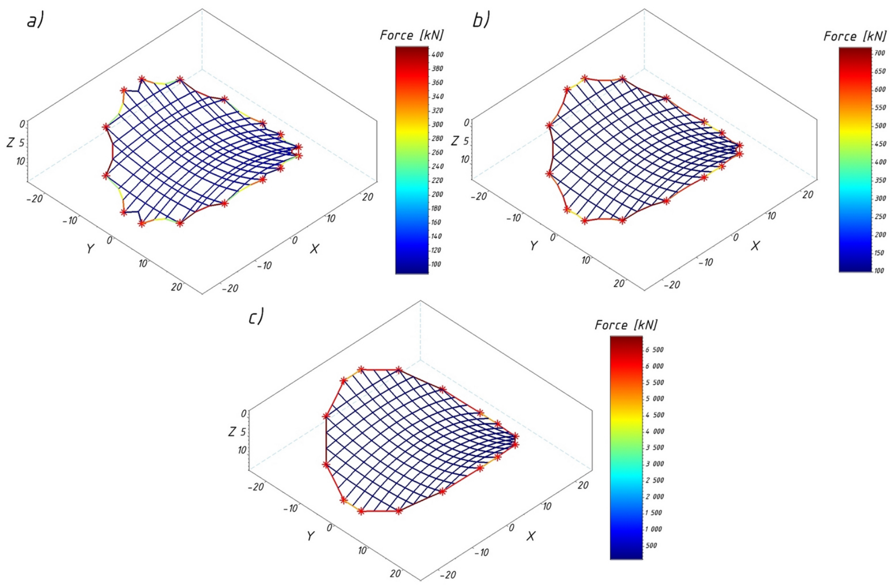

Fig. 6. Optimal configuration and force distribution of the open cable net: a) initial $q_{\text{inner}} = q_{\text{edge}} =$ 20 kN/m; b) initial $q_{\text{inner}} =$ 20 kN/m, $q_{\text{edge}} =$ 200 kN/m; c) initial $q_{\text{inner}} =$ 20 kN/m, $q_{\text{edge}} =$ 2000 kN/m

In this example it is obvious that comparing sums of all element lengths is unreliable because different prestress in edge cables mean different boundary conditions for the inner part. However such partially optimised cable net is rational because of the uniform prestress in all the inner cables.

Because of the largest area covered and regular inner cables layout the third version is selected for adding self-weight. Initial lengths for this analysis were calculated due to Eq. (5.1). The SS16 spiral strand wire rope is presumed for inner cables and SS115 for edge cables (Table 1). Symmetry axis profiles in all versions are compared in Fig. 7 and are very similar. The horizontal coordinates differ noticeably because of various levels of prestress in edge cables.

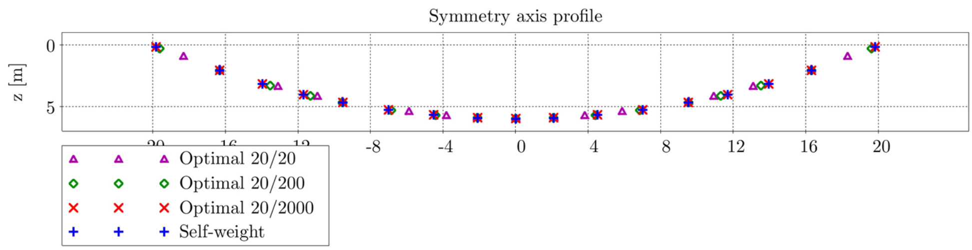

Fig. 7. Central axis profiles of three optimal and under self-weight cable nets

Sums of element lengths and extreme values of tensile forces are compared in Table 3. As it was expected, version under self-weight is very close to the third optimal version, so it can be regarded as initial configuration for the next analyses.

Table 3. Comparison of optimal and under self-weight versions of the open cable net

Version	$N_{\text{min,inner}}$ [kN]	$N_{\text{max,inner}}$ [kN]	$N_{\text{min,edge}}$ [kN]	$N_{\text{max,edge}}$ [kN]	ΣL_i [m]
Optimal 20/20	100.00	100.00	87.55	412.71	975.40
Optimal 20/200	100.00	100.00	502.54	718.21	993.61
Optimal 20/2000	100.00	100.00	5000.25	6950.86	1008.92
Self-weight	99.35	101.45	5019.81	6994.79	1008.92

6.3. Semiopen pentagonal roof

In the last example the semiopen, pentagonal cable net shown in Fig. 8 is analysed.

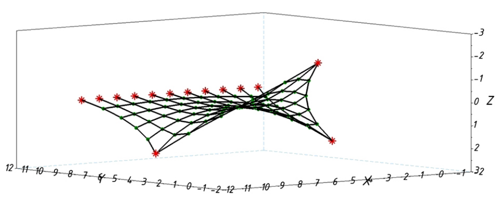

Fig. 8. Initial configuration of the semiopen, pentagonal cable net

There are four edge cables and one edge is fixed. In this case optimisation of the whole cable net is not possible due to geometrical limitations.

Two versions of partial optimisation are then performed. In both versions initial force densities in edge cables are 200 kN/m and in inner cables are 50 kN/m in order to achieve higher prestress in edge cables. In the first optimal configuration forces in all inner cables are 100 kN. Forces in edge cables and plan of the roof are shown in Fig. 9a. With the aid of element lengths from this solution the initial lengths for the analysis under self-weight are calculated with the use of Eq. (5.1). The SS16 spiral strand wire rope is used for inner cables and SS30 for edge cables (Table 1). Second optimal configuration is shown in Fig. 9b. Here only in the edge cables force values of 400 kN are imposed. In this configuration area covered by the roof is larger, so it seems to be more appropriate due to functional requirements.

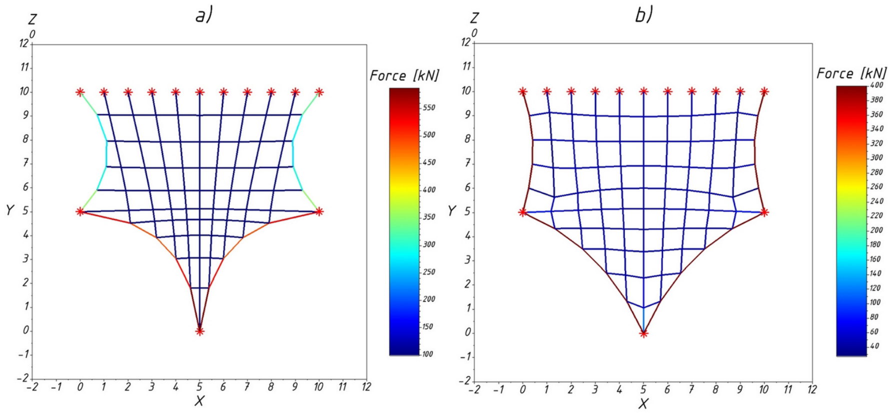

Fig. 9. Optimal configurations of the semiopen cable net due to: a) inner cables; b) edge cables

Figure 10 shows the comparison of two optimal shapes in plan and sideview. In the blue version only edge cables are optimised, in the green one – only inner cables.

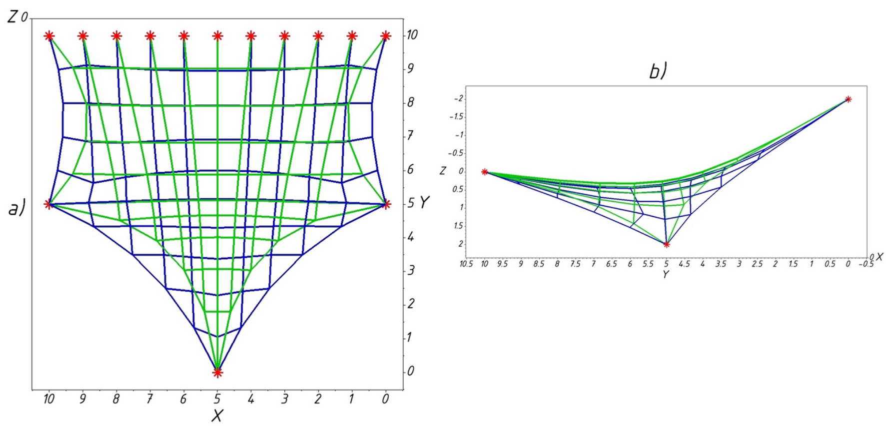

Fig. 10. Two optimal configurations of the semiopen cable net in: a) plan and b) sideview

Like in the previous case, the analysis with self-weight was also conducted with the same cable sections assumed and initial lengths calculated on the basis of optimal ones.

Table 4 summarizes force values in inner and edge cables in optimal and under self-weight configurations. Comparison shows that they are similar so it means that adding dead load with initial element lengths obtained from the optimal version does not change the optimal force distribution significantly.

Table 4. Comparison of optimal and under self-weight versions of the open cable net

Version	$N_{\mathbf{min,inner}}$ **[kN]**	$N_{\mathbf{max,inner}}$ **[kN]**	$N_{\mathbf{min,edge}}$ **[kN]**	$N_{\mathbf{max,edge}}$ **[kN]**
Optimal – inner	100.00	100.00	270.58	586.94
Self-weight – opt. inner	99.79	100.69	271.00	590.43
Optimal – edge	27.83	117.42	400.00	400.00
Self-weight – opt. edge	28.05	117.69	399.22	402.45

7. Conclusions

In this article the form-finding procedure for cable nets derived from the Force Density Method is proposed. With the use of the classic and extended, nonlinear version of the method it is possible to find the initial shape of a cable net under self-weight and prestress which is close to optimal configuration. It can be regarded as very advantageous for further static and dynamic analyses as self-weight and prestress are the loads present in each design situation in the structure. Such approach means that the natural state of a cable net is optimal configuration which is only disturbed by the live loads changing during exploitation. Thanks to use of catenary element formulation, geometry and tensile force distribution are very close to the exact solution in contrary to parabolic formulation. In this paper it was proved that minimum weight configuration can be achieved by enforcing uniform prestress in the whole cable net. Sometimes it is more beneficial to optimise only a part of a cable net (e.g. in open cable nets) to achieve better functional and economical features as it was shown in Chapters 6.2 and 6.3. Maximising area covered by the roof can be obtained by increasing prestress in edge cables. It leads to greater sum of lengths for the whole cable net but also allows the uniform prestress and using one cross-section for all the inner cables. Adding self-weight to optimal or partially optimal cable net always changes the configuration. In order to preserve the optimal one initial cable lengths can be calculated on the basis of the optimal ones. It was shown in three different examples that configuration under self-weight and prestress obtained this way is sufficiently close to the optimal one and therefore can be further treated as a starting point for design process. Algorithm presented in this paper supplements completely geometrical form-finding by the Force Density Method with the elastic properties and self-weight of cables. It also enables finding optimal configurations by imposing uniform force distribution.

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Znajdowanie kształtu optymalnych siatek cięgnowych obciążonych ciężarem własnym przy użyciu metody gęstości sił

Słowa kluczowe: znajdowanie kształtu, siatka cięgnowa, krzywa łańcuchowa, gęstość siły

Streszczenie:

W pracy przedstawiono procedurę, dzięki której optymalny kształt siatki cięgnowej jest wykorzystany do uzyskania wstępnej konfiguracji pod działaniem sił sprężenia i ciężaru własnego. Ze względu na nieliniowość geometryczną siatek cięgnowych ich konfiguracja pod wpływem różnych schematów obciążeń może dość znacznie się różnić. Dlatego optymalne projektowanie tego typu konstrukcji zgodnie z zaleceniem Lewis [1] powinno opierać się na stanie początkowym, w którym działają jedynie obciążenia obecne we wszystkich kombinacjach obciążeń, czyli siły sprężenia oraz ciężar własny. Ten stan jest również wyróżniony w Eurokodzie [18] jako oddzielny przypadek obciążeń. W niniejszym artykule zaproponowano ścieżkę postępowania, w której na początku przeprowadza się proces znajdowania kształtu konstrukcji, następnie znajduje się konfigurację minimalizującą ciężar własny, a na końcu wprowadza się własności sprężyste cięgien i ciężar własny tak, aby uzyskana geometria była zbliżona do tej optymalnej. W tym celu wykorzystuje się Metodę Gęstości Sił w wersji podstawowej, czyli liniowej i w wersjach nieliniowych. W artykule zaprezentowano najważniejsze informacje na temat Metody Gęstości Sił według Scheka [3]. Jest to jedna z popularniejszych metod znajdowania kształtu początkowego siatek cięgnowych. Opiera się na układzie równań równowagi węzłów, który dzięki wprowadzeniu pojęcia gęstości siły jako stosunku siły podłużnej do długości elementu, jest układem równań liniowych, z którego można uzyskać poszukiwane współrzędne węzłów konstrukcji. W tym celu należy narzucić konkretne wartości gęstości sił w każdym z elementów cięgnowych, co powoduje powstanie nowej konfiguracji konstrukcji. Pokazano również główną ideę i podstawowe równania Rozszerzonej Metody Gęstości Sił zaproponowanej przez autorkę w pracach [21] i [22]. Dzięki tej wersji możliwe jest uwzględnienie ciężaru własnego cięgien luźnych (przy użyciu krzywej łańcuchowej) oraz napiętych. Równocześnie dzięki tej metodzie wprowadza się sprężyste własności materiału cięgien, a zatem z czysto geometrycznego zadania Metody Gęstości Sił przechodzi się do modelu numerycznego dobrze odwzorowującego własności mechaniczne konstrukcji. W następnej części pracy zaproponowano zadanie optymalizacji polegające na poszukiwaniu minimalnego ciężaru siatki cięgnowej. W przypadku założenia o jednorodności materiału oznacza to poszukiwanie minimalnej objętości. Pokazano, że zadanie to sprowadza się do znajdowania siatki cięgnowej o równomiernym rozkładzie sił podłużnych, które

można rozwiązać przy użyciu nieliniowej wersji Metody Gęstości Sił. Przyjmując uzyskaną konfigurację konstrukcji jako optymalną, należy wykorzystać uzyskane długości poszczególnych elementów do zdefiniowania długości początkowych, czyli przed odkształceniem sprężystym. Te długości wraz z danymi dotyczącymi przyjętego przekroju i materiału stanowią dane wejściowe do analizy z ciężarem własnym. Uzyskaną w ten sposób geometrię konstrukcji i rozkład sił, a także zaproponowane wstępnie przekroje cięgien można wykorzystać jako dane wejściowe do dalszych analiz statycznych i dynamicznych pod działaniem pozostałych obciążeń.

W celu sprawdzenia, czy zaproponowana ścieżka postępowania jest prawidłowa, przeanalizowano trzy przykłady obliczeniowe. Pierwszym jest siatka zamknięta, rozpięta na okrągłym obwodzie o zmiennych rzędnych. Drugim jest analogiczna siatka otwarta, która powstała przez usunięcie niektórych podpór z siatki zamkniętej i wprowadzenie cięgien brzegowych pomiędzy pozostałymi podporami. Trzeci przykład to siatka półotwarta o rzucie pięciokątnym i podporach na rożnych wysokościach. W pierwszym przykładzie bez problemu uzyskano konfigurację optymalną z siłami w cięgnach równymi 100 kN. Po wykorzystaniu tej konfiguracji do znalezienia długości początkowych cięgien przeprowadzono analizę uwzględniającą ciężar własny i własności sprężyste. Założono konkretny typ cięgna i uzyskano geometrię z dużą dokładnością odzwierciedlającą konfigurację optymalną. Również rozkład siłjest zbliżony do założonego. Analizując siatkę okrągłą, otwartą okazało się, że nie ma możliwości uzyskania optymalnej konfiguracji całej siatki. Uzyskano zatem trzy wersje siatki częściowo optymalnej, gdzie równomierny rozkład sił uzyskano jedynie w cięgnach wewnętrznych, a cięgna brzegowe zostały napięte w różnym stopniu. Do dalszej analizy wybrano wersję, która przekrywa największą powierzchnię, czyli wersję z maksymalnymi siłami naciągu w cięgnach brzegowych. Ponownie uzyskano bardzo zbliżone wartości współrzędnych oraz sił w konfiguracji optymalnej i obciążonej ciężarem własnym. W ostatnim przykładzie również nie uzyskano pełnej konfiguracji optymalnej. Przeprowadzono oddzielnie optymalizację ze względu na cięgna wewnętrzne oraz zewnętrzne i porównano uzyskane kształty konstrukcji. Wykorzystano wyniki obu wersji do przyłożenia ciężaru własnego i własności sprężystych. Rozkłady sił w uzyskanych konfiguracjach były bardzo zbliżone do wersji optymalnych. Z przedstawionych przykładów wynika, że możliwe jest uzyskanie konfiguracji siatki cięgnowej z uwzględnieniem własności mechanicznych oraz ciężaru własnego cięgien, która jest zbliżona do konfiguracji optymalnej, czyli o minimalnym ciężarze własnym. W zależności od kształtu siatki i wzajemnego ułożenia cięgien optymalizacja może być wykonana dla całego układu lub tylko dla wybranych fragmentów, w których panują zbliżone wartości sił naciągu. Tak uzyskana konfiguracja wstępna pod działaniem początkowych obciążeń jest dobrym punktem wyjścia do dalszych analiz statycznych i dynamicznych. Zakłada się, że rzeczywiste sytuacje obciążeniowe będą tylko czasowo wytrącały konstrukcję z konfiguracji optymalnej, aby powrócić do niej w sytuacjach obciążenia początkowego.

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