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# Characteristics of solution to singular fractional differential equation with two Riemann-Stieltjes integral boundary value conditions

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**Abstract.** The purpose of this paper is to study unique solution and iterative sequence of approximate solution for uniformly approaching unique solution to a new class of singular fractional differential equations with two kinds of Riemann-Stieltjes integral boundary value conditions by using some fixed point theorems. Because of different properties of the nonlinear terms and complexity of the boundary conditions in equations, we first probe several fixed point theorems of sum-type operators which expand many existing works in this research area. It is essential to point out that some conditions in our works greatly simplify the proof process of fixed point theorems. By applying the operator conclusions obtained in this paper, some sufficient conditions that guarantee the existence and uniqueness of solutions to singular differential equations are obtained, two iterative schemes that uniformly converges to the unique solution are given which provide computational methods of approximating solutions. As applications, some examples are provided to illustrate our main results.

**Key words:** operator equation; mixed monotone operator; concave-convex operator; positive solution; fractional differential equation

## 1. INTRODUCTION

In this article, we study a new form of fractional differential equations with four different nonlinear terms under two different complex integral boundary conditions and gain the sufficient conditions which guarantee the unique nontrivial solution and approximating iterative schemes of unique solution. Namely, we discuss the following problem:

$$\begin{aligned} D_{0^+}^{\eta_1+\eta_2}x(t)+k_1f(t,x(t),x(t))+k_2g(t,x(t),x(t)) \\ +k_3\phi(t,x(t),x(t))+k_4\rho(t,x(t))=0, \end{aligned} \tag{1}$$

with two Riemann-Stieltjes integral boundary conditions

$$\begin{cases} x^{(i)}(0)=0, D_{0^+}^{\alpha_1}x(1)=\tau I_{0^+}^{\beta_1}x(\xi_1), D_{0^+}^{\eta_1+j}x(0)=0, \\ D_{0^+}^{\eta_1+\alpha_2}x(1)=\int_0^1 b(s)D_{0^+}^{\eta_1+\beta_2}x(s)\mathrm{d}A(s), \end{cases} \tag{2}$$

$$\begin{cases} x^{(j)}(0)=0, D_{0^+}^{\alpha_2}x(1)=\int_0^1 b(s)D_{0^+}^{\beta_2}x(s)\mathrm{d}A(s), \\ D_{0^+}^{\eta_2+i}x(0)=0, D_{0^+}^{\eta_2+\alpha_1}x(1)=\tau I_{0^+}^{\eta_2+\beta_1}x(\xi_1), \end{cases} \tag{3}$$

where $j=0,1,\cdots,m-2$, $i=0,1,\cdots,n-2$, $D_{0^+}^{\eta_1+\eta_2}$ stands for Riemann-Liouville (RL) fractional derivative, $n-1<\eta_1\leq n$, $m-1<\eta_2\leq m$, $\eta_1-\alpha_1-1>0$, $\tau>0$, $\beta_1>0$, $\alpha_2-\beta_2>0$, $\eta_2-\alpha_2-1>0$, $\eta_2-\beta_2-1>0$, $f,g,\phi\in C\big((0,1)\times(0,+\infty)\times(0,+\infty),[0,+\infty)\big)$, $\rho\in C\big((0,1)\times(0,+\infty),[0,+\infty)\big)$, $k_i>0$ $(i=1,2,3,4)$, $f(t,x,y)$, $g(t,x,y)$, $\phi(t,x,y)$, $\rho(t,x)$ are singular at $t=0,1$ and $y=0$.

As a matter of fact, FDEs with integral boundary value conditions are extensively applied in the description of a variety of practical situations and processes with memory and genetic characteristics ([1]-[7]) which explains why many authors have discussed existence, nonexistence and multiplicity questions for positive solutions to integral boundary value problems involving fractional derivative ([8]-[14]). For instance, the existence conclusions of solution to the following RL equation in [8] are obtained by mean of the fixed point index theory

$$\begin{cases} D_{0^+}^{\eta}y(t)+h(t)f(t,y(t)),\ \ 0<t<1,\ \ 3<\eta\leq 4, \\ y^{(i)}(0)=0,\ \ i=0,1,2,\ \ y(1)=\lambda\int_0^{\xi}y(s)\mathrm{d}s,\ \ 0\leq\xi\leq 1. \end{cases}$$

In [9], a Caputo FDE subject to integral boundary value conditions was investigated

$$\begin{cases} {}^{c}D_{0^+}^{\eta}y(t)+f(t,y(t),y'(t))=0,\ \ 0\leq t\leq 1,\ \ 2<\eta<3, \\ y(0)=y''(0)=0,\ \ y(1)=\lambda\int_0^1 y(s)\mathrm{d}s,\ \ 0<\lambda<2. \end{cases}$$

By using a fixed point theorem due to Avery and Peterson, sufficient conditions for the existence and multiplicity of positive solution to this system were given.

From literature [15]-[18], some existence, non-existence and multiplicity results of solution (or positive solution) when the nonlinear terms satisfy different requirements of superlinearity, sublinearity, and so forth. But the question of the unique solution and the computational methods of approximating solutions was not treated. Motivated by the above mentioned work, our work in this paper is to unfold the existence and iteration of unique positive solution to the problem Eq. (1) with the conditions (2) or (3) by using some fixed point theorems of sum-type operator.

The new features of this paper are as follows: (a) We provide several kinds of fixed point methods of sum-type operators to discuss one of the most considerable qualitative aspects "existence and uniqueness" of solution for the system Eq. (1) governed by the boundary conditions (2) or (3). The fixed point theorems of sum-type operators expand some existing results, such as [19]-[21]. (b) We study not only problem Eq. (1) with (2) or (3) admitting the existence and uniqueness of positive solutions but also two iterative schemes for dealing with approximate solutions that converge uniformly to the unique so-



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lution, which can provide computational methods of approximating solutions.

The rest of the content in this paper is organized as follows. In Section 2, some basic theory, works of Banach space and cone are reviewed. In Section 3, some conclusions of sum-type operators are presented. In Section 4, some theorems of existence-uniqueness solution for (1) with (2) or (3) are studied by the theoretical works of Section 3. In Section 5, some exact examples are given.

## 2. PRELIMINARIES

Let $(E, \|\cdot\|)$ be a real Banach space. The concept of cone, normal cone can be referred to [1]-[3]. Denoted a set $P_h = \{x \in E \mid \exists \mu, \nu > 0, \mu h \leq x \leq \nu h\}$.

**Definition 2.1** ([1]) Define an operator $A : P \times P \to P$, if $\forall u_i, v_i \in P\ (i = 1, 2)$, when $u_1 \leq u_2, v_1 \geq v_2$, there is $A(u_1, v_1) \leq A(u_2, v_2)$, i.e. $A(u, v)$ is increasing in $u$, and decreasing in $v$, then $A$ is called a mixed monotone operator. If $A(x, x) = x$, $x$ is called a fixed point of $A$.

**Definition 2.2** ([2]) Let $D \subset E$ be a convex subset, $A : D \to E$ be an operator. If $A$ satisfies $\forall x, y \in D,\ \ y \leq x$,

$$A(tx + (1-t)y) \leq tAx + (1-t)Ay,\ t \in (0, 1),$$

then $A$ is called a convex operator. If

$$A(tx + (1-t)y) \geq tAx + (1-t)Ay,\ t \in (0, 1),$$

then, $A$ is called a concave operator.

**Definition 2.3** ([3]) Define an operator $A : P \times P \to P$, if

$$A(tx) \geq tAx,\ \forall t \in (0, 1), x \in P.$$

then, $A$ is said to be sub-homogeneous operator.

**Lemma 2.4** ([4]) Let $P$ be a normal cone and operator $T : P \times P \to P$. If $T$ is a mixed monotone operator and

$(L_1)$ $\exists h \in P\ (h \neq \theta)$, such that $T(h, h) \in P_h$;

$(L_2)$ $\exists \varphi(t) \in (t, 1]$, such that $T(tx, t^{-1}y) \geq \varphi(t)T(x, y), \forall t \in (0, 1), x, y \in P$.

Then we have

$(C_1')$ $T : P_h \times P_h \to P_h$;

$(C_2')$ $\exists u_0, v_0 \in P_h$, such that $rv_0 \leq u_0 < v_0$, $u_0 \leq T(u_0, v_0) \leq T(v_0, u_0) \leq v_0, r \in (0, 1)$;

$(C_3')$ $T(x, x)$ has a unique solution $x^* \in P_h$;

$(C_4')$ for any initial values $x_0, y_0 \in P_h$, constructing the iterative sequences:

$$x_n = T(x_{n-1}, y_{n-1}),\quad y_n = T(y_{n-1}, x_{n-1}),\quad n = 1, 2, \cdots,$$

$x_n \to x^*,\ y_n \to x^*$, as $n \to \infty$.

## 3. FIXED POINT THEOREMS OF SUM OPERATOR

In this section, we use basic definition and lemma to investigate the properties and conclusions of a class of sum operators. Theorem 3.1 and Corollary 3.2 provide conclusions on the existence and uniqueness of solutions for a class of sum operators with parameters on $P$ and $P_h$. What's more, the study on the properties of operator equation solutions is given in Remark 3.3 when the operator satisfies different convexities. In fact, the operator theorems studied has generality, as the parameters and operator properties change, the conclusions obtained can be simplified into some theorems in some literature, which are given in Remark 3.4-3.5.

Define an operator $T_5 = k_1T_1 + k_2T_2 + k_3T_3 + k_4T_4$ by

$$T_5(x, y) = k_1T_1(x, y) + k_2T_2(x, y) + k_3T_3(x, y) + k_4T_4x,\ \forall x, y \in P.$$

**Theorem 3.1** Let $T_1, T_2, T_3 : P \times P \to P$ be mixed monotone operators, and $T_4 : P \to P$ be an increasing sub-homogeneous operator, $k_i > 0\ (i = 1, 2, 3, 4)$. Suppose that

$(H_1)$ For any $\lambda \in (0, 1), x, y \in P$, $\exists \psi_1(\lambda), \psi_2(\lambda) \in (\lambda, 1]$ such that

$$T_1(\lambda x, \lambda^{-1}y) \geq \psi_1(\lambda)T_1(x, y),\ T_2(\lambda x, \lambda^{-1}y) \geq \psi_2(\lambda)T_2(x, y),$$

and for any fixed $y \in P$, $T_3(\cdot, y)$ is concave; for any fixed $x \in P$, $T_3(x, \cdot)$ is convex;

$(H_2)$ $\exists \frac{1}{2} \leq a \leq 1$ such that $T_3(\theta, \widetilde{l}h) \geq aT_3(\widetilde{l}h, \theta)$, $\widetilde{l} \geq 1$;

$(H_3)$ $\exists h_1 \in P_h\ (h_1 \neq \theta)$ such that $T_1(h_1, h_1), T_2(h_1, h_1), T_3(h_1, h_1), T_4h_1 \in P_h$;

$(H_4)$ for any $x, y \in P$, $\exists p \in (0, 1)$ such that $k_1T_1(x, y) + k_2T_2(x, y) \geq \frac{p}{1-p}[k_3T_3(x, y) + k_4T_4x]$.

The following conclusions hold:

(c1) $T_5(h, h) \in P_h$;

(c2) $T_5 : P_h \times P_h \to P_h$;

(c3) $T_5(x, x) = x$ has a unique solution $x^* \in P_h$;

(c4) $\exists u_0, v_0 \in P_h, r \in (0, 1)$, such that $rv_0 \leq u_0 < v_0$,

$$u_0 \leq T_5(u_0, v_0) \leq T_5(v_0, u_0) \leq v_0;$$

(c5) for any initial values $x_0, y_0 \in P_h$, constructing the iterative sequences:

$$x_n = T_5(x_{n-1}, y_{n-1}),\quad y_n = T_5(y_{n-1}, x_{n-1}),\quad n = 1, 2, \cdots,$$

$x_n \to x^*,\ y_n \to x^*$, when $n \to \infty$.

**Proof** The proof process may be divided into three steps.

Step 1: We prove that $T_5 : P \times P \to P$, and $T_5$ is a mixed monotone operator. Due to $T_1, T_2, T_3 : P \times P \to P$, $T_4 : P \to P$, and $k_i > 0\ (i = 1, 2, 3, 4)$, it is easy to verify $T_5 : P \times P \to P$. For every $x_1, y_1, x_2, y_2 \in P$, let $x_1 \leq x_2, y_1 \geq y_2$, since $T_1, T_2, T_3$ are mixed monotone operators, and $T_4$ is an increasing operator, we conclude $T_5$ is a mixed monotone operator.

Step 2: We state that $T_5(h, h) \in P_h$. By the condition $(H_1)$, it is easy to obtain

$$\begin{aligned} T_1(\lambda^{-1}x, \lambda y) &\leq \psi_1(\lambda)^{-1}T_1(x, y), \\ T_2(\lambda^{-1}x, \lambda y) &\leq \psi_2(\lambda)^{-1}T_2(x, y), T_4(\lambda^{-1}x) \leq \lambda^{-1}T_4x. \end{aligned} \tag{4}$$

For any $\lambda \in (0, 1)$, we let $y = \lambda\lambda^{-1}y + (1-\lambda)\theta$, in view of the condition $(H_1)$ and the definition of convex operator in Definition 2.2, we have

$$\lambda T_3(x, \lambda^{-1}y) \geq T_3(x, y) - (1-\lambda)T_3(x, \theta). \tag{5}$$

Moreover, we can find a sufficiently large number $\widetilde{l}$ such that $x, y, \lambda^{-1}y \leq \widetilde{l}h$. By (5), $(H_2)$, mixed monotonicity of $T_3$, and the definition of concave operator in Definition 2.2, there exists

$\frac{1}{2} \le a \le 1$, such that

$$\begin{aligned}
T_3(\lambda x, \lambda^{-1}y) &= T_3(\lambda x + (1-\lambda)\theta, \lambda^{-1}y) \\
&\ge \lambda T_3(x, \lambda^{-1}y) + (1-\lambda)T_3(\theta, \lambda^{-1}y) \\
&\ge T_3(x,y) - (1-\lambda)T_3(x,\theta) + (1-\lambda)T_3(\theta, \lambda^{-1}y) \\
&\ge T_3(x,y) + (1-\lambda)[T_3(\theta, \widetilde{l}h) - T_3(\widetilde{l}h, \theta)] \\
&\ge T_3(x,y) + (1-\lambda)[T_3(\theta, \widetilde{l}h) - \frac{1}{a}T_3(\theta, \widetilde{l}h)] \\
&\ge [1 + (1-\lambda)(1-\frac{1}{a})]T_3(x,y) \\
&= [(2-\frac{1}{a}) + (\frac{1}{a}-1)\lambda]T_3(x,y) \\
&\ge \lambda T_3(x,y).
\end{aligned}$$

Then, for any $\lambda \in (0,1)$, it is easy to obtain

$$T_3(\lambda x, \lambda^{-1}y) \ge \lambda T_3(x,y), \quad T_3(\lambda^{-1}x, \lambda y) \le \lambda^{-1}T_3(x,y), \tag{6}$$

From $h_1 \in P_h$ of the condition $(H_3)$, there exists a constant $a_0 \in (0,1)$ such that

$$a_0 h \le h_1 \le a_0^{-1}h. \tag{7}$$

What's more, by means of $(H_3)$, there exist some constants $b_i \in (0,1)$ $(i = 1,2,\cdots,8)$ such that

$$\begin{aligned}
b_1 h \le T_1(h_1,h_1) \le b_2 h, \quad b_3 h \le T_2(h_1,h_1) \le b_4 h, \\
b_5 h \le T_3(h_1,h_1) \le b_6 h, \quad b_7 h \le T_4 h_1 \le b_8 h.
\end{aligned} \tag{8}$$

From (4)-(5), (7)-(8), and the monotonicity of $T_1$, the assertion follows

$$\begin{aligned}
T_1(h,h) \le T_1(a_0^{-1}h_1, a_0 h_1) \le \psi_1(a_0)^{-1}b_2 h, \\
T_1(h,h) \ge T_1(a_0 h_1, a_0^{-1}h_1) \ge \psi_1(a_0) b_1 h.
\end{aligned} \tag{9}$$

In the similar way, we obtain

$$\begin{aligned}
\psi_2(a_0)b_3 h \le T_2(h,h) \le \psi_2(a_0)^{-1}b_4 h, \\
a_0 b_5 h \le T_3(h,h) \le a_0^{-1}b_6 h, a_0 b_7 h \le T_4 h \le a_0^{-1}b_8 h.
\end{aligned} \tag{10}$$

From (9)-(10), it holds that $T_1(h,h), T_2(h,h), T_3(h,h), T_4 h \in P_h$. According to the definition of $T_5$ and (10)-(11), it is easy to obtain $T_5(h,h) \in P_h$.

Step 3: We verify that $T_5$ satisfies the condition $(L_2)$ in Lemma 2.4. Through $(H_4)$, it is easy to obtain that there exists $p \in (0,1)$, such that

$$(1-p)[k_1T_1(x,y)+k_2T_2(x,y)] \ge p[k_3T_3(x,y)+k_4T_4x], \ \forall x,y \in P.$$

Then by $(H_1)$, $\forall t \in (0,1)$, one observes

$$\begin{aligned}
T_5(tx,t^{-1}y) \ge& \psi(t)[k_1T_1(x,y)+k_2T_2(x,y)] + t[k_3T_3(x,y)+k_4T_4x] \\
=&[p\psi(t)+(1-p)t][k_1T_1(x,y)+k_2T_2(x,y)] \\
&+(1-p)(\psi(t)-t)[k_1T_1(x,y)+k_2T_2(x,y)] \\
&+t[k_3T_3(x,y)+k_4T_4x] \\
\ge&[p\psi(t)+(1-p)t][k_1T_1(x,y)+k_2T_2(x,y)] \\
&+p(\psi(t)-t)[k_3T_3(x,y)+k_4T_4x] \\
&+t[k_3T_3(x,y)+k_4T_4x] \\
=&[p\psi(t)+(1-p)t]T_5(x,y),
\end{aligned}$$

where $\psi(t) = \min\{\psi_1(t), \psi_2(t), t \in (0,1)\}$. Due to $p \in (0,1)$, $\psi(t) \in (t,1]$, it is easy to obtain $p\psi(t) + (1-p)t \in (t,1]$. Hence, there exists $\varphi(t) = p\psi(t) + (1-p)t \in (t,1]$ such that

$$T_5(tx,t^{-1}y) \ge \varphi(t)T_5(x,y).$$

Thus, according to Lemma 2.4, we get the conclusions (c1)-(c5).

**Corollary 3.2** Let $T_1, T_2, T_3 : P_h \times P_h \to P_h$ be mixed monotone operators, and $T_4 : P_h \to P_h$ be an increasing sub-homogeneous operator, $k_i > 0$ $(i = 1,2,3,4)$. Suppose that $(H_2)$ holds and

$(H_1^{'})$ for any $\lambda \in (0,1), x,y \in P_h$, $\exists \psi_1(\lambda), \psi_2(\lambda) \in (\lambda,1]$ such that

$$T_1(\lambda x, \lambda^{-1}y) \ge \psi_1(\lambda)T_1(x,y), \ T_2(\lambda x, \lambda^{-1}y) \ge \psi_2(\lambda)T_2(x,y),$$

and for any fixed $y \in P_h$, $T_3(\cdot,y)$ is concave; for any fixed $x \in P_h$, $T_3(x,\cdot)$ is convex;

$(H_4^{'})$ for any $x,y \in P_h$, $\exists p \in (0,1)$ such that $k_1T_1(x,y)$ $+k_2T_2(x,y) \ge \frac{p}{1-p}[k_3T_3(x,y)+k_4T_4x]$.

Thus, the conclusions (c1)-(c5).

**Proof** According to the definition of the set $P_h$, it can be inferred that $P_h \subset P$. It can be observed that the conditions in Corollary 3.2 are given on the subset $P_h$ of $P$. Therefore, according to the proof in Theorem 3.1, the same conclusion can be drawn.

**Remark 3.3** In Theorem 3.1, the properties of $T_1, T_2$ in the condition $(H_1)$ turn into the condition $(l_1)$ or $(l_2)$ or $(l_3)$:

$(l_1)$ for any $\lambda \in (0,1), x,y \in P$, $\exists \alpha_1, \alpha_2 \in (0,1)$ such that

$$\begin{aligned}
T_1(\lambda x, \lambda^{-1}y) \ge \lambda^{\alpha_1}T_1(x,y), \\
T_2(\lambda x, \lambda^{-1}y) \ge \lambda^{\alpha_2}T_2(x,y),
\end{aligned}$$

$(l_2)$ for any $\lambda \in (0,1), x,y \in P$, $\exists \eta_1(\lambda), \eta_2(\lambda) \in (0,1]$ such that

$$\begin{aligned}
T_1(\lambda x, \lambda^{-1}y) \ge [1+\lambda-\lambda^{\eta_1(\lambda)}]T_1(x,y), \\
T_2(\lambda x, \lambda^{-1}y) \ge [1+\lambda-\lambda^{\eta_2(\lambda)}]T_2(x,y),
\end{aligned}$$

$(l_3)$ for any $\lambda \in (0,1), x,y \in P$, $\exists \eta_3(\lambda), \eta_4(\lambda) > 0$ such that $\lambda[1+\eta_3(\lambda)] < 1, \lambda[1+\eta_4(\lambda)] < 1$ and

$$\begin{aligned}
T_1(\lambda x, \lambda^{-1}y) \ge \lambda[1+\eta_3(\lambda)]T_1(x,y), \\
T_2(\lambda x, \lambda^{-1}y) \ge \lambda[1+\eta_4(\lambda)]T_2(x,y),
\end{aligned}$$

then the conclusions (c1)-(c5) still hold.

**Remark 3.4** In Theorem 3.1, if we take $k_1 = k_4 = 1, k_2 = k_3 = 0$, or $k_1 = k_4 = 1, T_2 = T_3 = \theta$, and the property of $T_1$ in the condition $(H_1)$ turns into

$$\exists \alpha_1 \in (0,1) \text{ such that } T_1(\lambda x, \lambda^{-1}y) \ge \lambda^{\alpha_1}T_1(x,y),$$

the corresponding result can reduce to the Theorem 2.1 of [19].

**Remark 3.5** In Theorem 3.1, in the situation of $k_1 = k_2 = 1, k_3 = k_4 = 0$, or $k_1 = k_2, T_3 = T_4 = \theta$, the corresponding results is consistent with the result of [20].

## 4. UNIQUE SOLUTION TO THE FDES

In this part, using our main theoretical results, we study the existence, uniqueness solution and approximating iterative schemes of unique solution to Eq. (1) with the conditions

Eq. (2) or Eq. (3). The brief process of the research is as follows. Firstly, we discuss the equivalent integral equations of two singular differential equations (Lemma 4.1, Lemma 4.5), then the properties of the Green's function in the equivalent integral equations are obtained (Lemma 4.2). Thirdly, based on the obtained operator equation theorem (Theorem 4.1), the sufficient conditions for unique solution of two fractional differential equations are given (Theorem 4.3, Theorem 4.6).

In the following, we will work in $E=C[0,1]$ with the norm $\|x\|=\sup_{0\le t\le 1}|x(t)|$, and $P=\{x\in E: x(t)\ge 0, t\in[0,1]\}$. Evidently, $(E,\|\cdot\|)$ is a Banach space, $P$ is a normal cone and the normality constant is 1. Define

$$P_h=\{x\in C[0,1]:\exists J\in(0,1), Jh(t)\le x\le \frac{1}{J}h(t),\ t\in[0,1]\}.$$

For simplify, let $M_1=\Gamma(\eta_1)\Gamma(\eta_1+\beta_1)-\tau\Gamma(\eta_1)\Gamma(\eta_1-\alpha_1)\xi_1^{\eta_1+\beta_1-1}$, $M_2=\frac{1}{\Gamma(\eta_2-\alpha_2)}-\frac{\int_0^1 b(s)s^{\eta_2-\beta_2-1}dA(s)}{\Gamma(\eta_2-\beta_2)}$, $g_1(t,s)=\frac{t^{\eta_1-1}}{M_1}(1-s)^{\eta_1-\alpha_1-1}\Gamma(\eta_1+\beta_1)$, $g_2(t,s)=\frac{(t-s)^{\eta_1-1}}{\Gamma(\eta_1)}$, $g_3(t,s)=\tau\frac{\Gamma(\eta_1-\alpha_1)t^{\eta_1-1}(\xi_1-s)^{\eta_1+\beta_1-1}}{M_1}$

**Lemma 4.1** If $\sigma(t)=k_1f(t,x(t),x(t))+k_2g(t,x(t),x(t))+k_3\phi(t,x(t),x(t))+k_4\rho(t,x(t))\in C[0,1]$, $M_1>0, M_2>0$, the equation Eq. (1) with condition (2) has the following equivalent integral equation

$$x(t)=\int_0^1 K(t,s)\int_0^1 G(s,\tau)\sigma(\tau)d\tau ds,$$

where

$$G(t,s)=G_1(t,s)+t^{\eta_2-1}\int_0^1 G_2(\tau,s)b(\tau)dA(\tau),$$

$$G_1(t,s)=\begin{cases}\frac{t^{\eta_2-1}(1-s)^{\eta_2-\alpha_2-1}-(t-s)^{\eta_2-1}}{\Gamma(\eta_2)}, & 0\le s\le t\le 1,\\ \frac{t^{\eta_2-1}(1-s)^{\eta_2-\alpha_2-1}}{\Gamma(\eta_2)}, & 0\le t\le s\le 1,\end{cases}$$

$$G_2(t,s)=\begin{cases}\frac{(1-s)^{\eta_2-\alpha_2-1}t^{\eta_2-\beta_2-1}-(t-s)^{\eta_2-\beta_2-1}}{\Gamma(\eta_2)\Gamma(\eta_2-\beta_2)M_2}, & 0\le s\le t\le 1,\\ \frac{(1-s)^{\eta_2-\alpha_2-1}t^{\eta_2-\beta_2-1}}{\Gamma(\eta_2)\Gamma(\eta_2-\beta_2)M_2}, & 0\le t\le s\le 1,\end{cases}$$

$$K(t,s)=\begin{cases}g_1(t,s)-g_2(t,s)-g_3(t,s), & 0\le s\le t\le 1, s\le\xi_1,\\ g_1(t,s)-g_2(t,s), & 0\le\xi_1\le s\le t\le 1,\\ g_1(t,s)-g_3(t,s), & 0\le t\le s\le\xi_1\le 1,\\ g_1(t,s), & 0\le t\le s\le 1, \xi_1\le s.\end{cases}$$

**proof** Let $D_{0+}^{\eta_1}x(t)=u(t)$. Eq. (1) and some conditions of (2)

$$\begin{cases}D_{0+}^{\eta_1+j}x(0)=0,\quad j=0,1,\cdots,m-2,\\ D_{0+}^{\eta_1+\alpha_2}x(1)=\int_0^1 b(s)D_{0+}^{\eta_1+\beta_2}x(s)dA(s),\end{cases}$$

can turn into

$$\begin{cases}D_{0+}^{\eta_2}u(t)+\sigma(t)=0,\\ u^{(j)}(0)=0,\quad j=0,1,\cdots,m-2,\\ D_{0+}^{\alpha_2}u(1)=\int_0^1 b(s)D_{0+}^{\beta_2}u(s)dA(s).\end{cases}\tag{11}$$

Integrating $\eta_2$ times to the first formula of Eq. (11), we can obtain

$$u(t)=-I_{0+}^{\eta_2}\sigma(t)+\widetilde{c}_1t^{\eta_2-1}+\widetilde{c}_2t^{\eta_2-2}+\cdots+\widetilde{c}_mt^{\eta_2-m}.\tag{12}$$

From $u^{(j)}(0)=0$ $(j=0,1,\cdots,m-2)$, we see that $\widetilde{c}_m=\widetilde{c}_{m-1}=\cdots=\widetilde{c}_2=0$. Then by $D_{0+}^{\alpha}t^{\eta-1}=\frac{\Gamma(\eta)}{\Gamma(\eta-\alpha)}t^{\eta-\alpha-1}$ and $D_{0+}^{\alpha_2}u(1)=\int_0^1 b(s)D_{0+}^{\beta_2}u(s)dA(s)$, one gets

$$\begin{aligned}\widetilde{c}_1=&\int_0^1\frac{(1-s)^{\eta_2-\alpha_2-1}}{\Gamma(\eta_2)\Gamma(\eta_2-\alpha_2)M_2}\sigma(s)ds\\&-\int_0^1\int_0^s\frac{(s-\tau)^{\eta_2-\beta_2-1}}{\Gamma(\eta_2)\Gamma(\eta_2-\beta_2)M_2}b(s)\sigma(\tau)d\tau dA(s).\end{aligned}\tag{13}$$

Substituting (13) in (12), and from the definition of $M_2$, we can get

$$\begin{aligned}u(t)=&-\int_0^t\frac{(t-s)^{\eta_2-1}}{\Gamma(\eta_2)}\sigma(s)ds+\int_0^1\frac{t^{\eta_2-1}(1-s)^{\eta_2-\alpha_2-1}}{\Gamma(\eta_2)}\sigma(s)ds\\&+\int_0^1\int_0^1\frac{t^{\eta_2-1}(1-\tau)^{\eta_2-\alpha_2-1}s^{\eta_2-\beta_2-1}}{\Gamma(\eta_2)\Gamma(\eta_2-\beta_2)M_2}b(s)\sigma(\tau)d\tau dA(s)\\&-\int_0^1\int_0^s\frac{t^{\eta_2-1}(s-\tau)^{\eta_2-\beta_2-1}}{\Gamma(\eta_2)\Gamma(\eta_2-\beta_2)M_2}b(s)\sigma(\tau)d\tau dA(s)\\=&\int_0^1 G(t,s)\sigma(s)ds.\end{aligned}$$

i.e., $D_{0+}^{\eta_1}x(t)=u(t)=\int_0^1 G(t,s)\sigma(s)ds$.

Integrating $\eta_1$ times to the above formula, we have

$$x(t)=-I_{0+}^{\eta_1}u(t)+\widetilde{a}_1t^{\eta_1-1}+\widetilde{a}_2t^{\eta_1-2}+\cdots+\widetilde{a}_nt^{\eta_1-n}.\tag{14}$$

According to $x^{(i)}(0)=0$ $(i=0,1,\cdots,n-2)$, there is $\widetilde{a}_n=\widetilde{a}_{n-1}=\cdots=\widetilde{a}_2=0$. It follows from $D_{0+}^{\alpha_1}x(1)=\tau I_{0+}^{\beta_1}x(\xi_1)$ that

$$\begin{aligned}\widetilde{a}_1=&\int_0^1\frac{\Gamma(\eta_1+\beta_1)(1-s)^{\eta_1-\alpha_1-1}}{M_1}u(s)ds\\&-\int_0^{\xi_1}\frac{\tau\Gamma(\eta_1-\alpha_1)(\xi_1-s)^{\eta_1+\beta_1-1}}{M_1}u(s)ds,\end{aligned}\tag{15}$$

Replace $\widetilde{a}_1$ of (15) with (14) in the equation, there is

$$x(t)=\int_0^1 K(t,s)u(s)ds,$$

Then from $u(t)$ and $\sigma(t)$, we can get the conclusion.

**Lemma 4.2** If $M_1>0, M_2>0$, the following properties are established

(i) for any $t,s\in[0,1]$, $G(t,s), K(t,s)$ are continuous.

(ii) for any $t,s\in[0,1]$, we have

$$0\le l_1(s)t^{\eta_2-1}\le G(t,s)\le L_1t^{\eta_2-1},$$

where $L_1=\frac{1}{\Gamma(\eta_2)}+\int_0^1\frac{\tau^{\eta_2-\beta_2-1}}{\Gamma(\eta_2)\Gamma(\eta_2-\beta_2)M_2}b(\tau)dA(\tau)$, $l_1(s)=[1-(1-s)^{\alpha_2-\beta_2}](1-s)^{\eta_2-\alpha_2-1}\frac{(1-s)^{\eta_2-\alpha_2-1}[1-(1-s)^{\alpha_2}]}{\Gamma(\eta_2)}+\int_0^1\frac{\tau^{\eta_2-\beta_2-1}}{\Gamma(\eta_2)\Gamma(\eta_2-\beta_2)M_2}b(\tau)dA(\tau)$.

(iii) for any $t,s\in[0,1]$, we have

$$0\le l_2(s)t^{\eta_1-1}\le K(t,s)\le L_2t^{\eta_1-1},$$

where $L_2=\frac{\Gamma(\eta_1)\Gamma(\eta_1+\beta_1)}{M_1}$, $l_2(s)=\frac{\tau\Gamma(\eta_1-\alpha_1)\xi_1^{\eta_1+\beta_1-1}}{M_1}(1-s)^{\eta_1-\alpha_1-1}[1-(1-s)^{\alpha_1}]$.

**Proof** From the definition of $G(t,s),K(t,s)$, we can know that (i) holds.

From the definition of $G_1(t,s),G_2(t,s)$, for any $t,s\in[0,1]$, we compute $0\le \frac{t^{\eta_2-1}(1-s)^{\eta_2-\alpha_2-1}[1-(1-s)^{\alpha_2}]}{\Gamma(\eta_2)}\le G_1(t,s)\le \frac{t^{\eta_2-1}}{\Gamma(\eta_2)}$, $0\le \frac{(1-s)^{\eta_2-\alpha_2-1}t^{\eta_2-\beta_2-1}[1-(1-s)^{\alpha_2-\beta_2}]}{\Gamma(\eta_2)\Gamma(\eta_2-\beta_2)M_2}\le G_2(t,s)\le t^{\eta_2-\beta_2-1}\frac{1}{\Gamma(\eta_2)\Gamma(\eta_2-\beta_2)}$.

From the definition of $G(t,s)$, for any $t,s\in[0,1]$, it holds that

$$G(t,s)=G_1(t,s)+t^{\eta_2-1}\int_0^1 G_2(\tau,s)b(\tau)\mathrm{d}A(\tau)\ge l_1(s)t^{\eta_2-1},$$

and $G(t,s)\le L_1t^{\eta_2-1}$, i.e., (ii) holds.

From the definition of $K(t,s)$, and $M_1>0$, when $0\le s\le t\le 1, s\le\xi_1$, we observe that

$$\begin{aligned}M_1\Gamma(\eta_1)K(t,s)&=\Gamma(\eta_1)\Gamma(\eta_1+\beta_1)t^{\eta_1-1}(1-s)^{\eta_1-\alpha_1-1}\\&\quad-\Gamma(\eta_1)\Gamma(\eta_1+\beta_1)(t-s)^{\eta_1-1}+\tau\Gamma(\eta_1)\Gamma(\eta_1-\alpha_1)\xi_1^{\eta_1+\beta_1-1}\\&\quad\cdot(t-s)^{\eta_1-1}-\tau\Gamma(\eta_1)\Gamma(\eta_1-\alpha_1)t^{\eta_1-1}(\xi_1-s)^{\eta_1+\beta_1-1}\\&\ge\tau\Gamma(\eta_1)\Gamma(\eta_1-\alpha_1)\xi_1^{\eta_1+\beta_1-1}t^{\eta_1-1}(1-s)^{\eta_1-\alpha_1-1}\\&\quad\cdot[1-(1-s)^{\alpha_1}]\ge 0,\end{aligned}$$

and

$$\begin{aligned}M_1\Gamma(\eta_1)K(t,s)\le&\Gamma(\eta_1)\Gamma(\eta_1+\beta_1)t^{\eta_1-1}(1-s)^{\eta_1-\alpha_1-1}\\\le&\Gamma(\eta_1)\Gamma(\eta_1+\beta_1)t^{\eta_1-1}.\end{aligned}$$

Similarly, when $0\le\xi_1\le s\le t\le 1$, $0\le t\le s\le\xi_1\le 1$, $0\le t\le s\le 1,\xi_1\le s$, we can draw the same conclusion. Hence, for any $t,s\in[0,1]$, we infer that (iii) holds.

**Theorem 4.3** Let $f,\ g,\ \phi: C\big((0,1)\times(0,+\infty)\times(0,+\infty),[0,+\infty)\big)$, $\rho: C\big((0,1)\times(0,+\infty),[0,+\infty)\big)$, and $f(t,u,v)$, $g(t,u,v),\phi(t,u,v),\rho(t,u)$ are singular at $t=0,1$ and $v=0$. Assume that

$(r_1)$ for $t\in(0,1),x,y\in(0,+\infty)$, for any fixed $t,y$, $f(t,x,y)$, $g(t,x,y),\phi(t,x,y),\rho(t,x)$ are non-decreasing in $x$; for any fixed $t,\ x,\ f(t,x,y),g(t,x,y),\phi(t,x,y)$ are non-increasing in $y$;

$(r_2)$ for any $\lambda,t\in(0,1)$, $x,y\in(0,+\infty)$, there exist $\psi_1(\lambda)\in(\lambda,1]$, $\psi_2(\lambda)\in(\lambda,1]$ such that

$$\begin{aligned}f(t,\lambda x,\lambda^{-1}y)\ge\psi_1(\lambda)f(t,x,y),\\ g(t,\lambda x,\lambda^{-1}y)\ge\psi_2(\lambda)g(t,x,y),\quad\rho(t,\lambda x)\ge\lambda\rho(t,x),\end{aligned}$$

and for fixed $t,\ y,\ \phi(t,\cdot,y)$ is concave; for fixed $t,\ x,\ \phi(t,x,\cdot)$ is convex;

$(r_3)$ $\exists\frac{1}{2}\le a\le 1$ such that $\phi(t,\theta,\widetilde{l}h)\ge a\phi(t,\widetilde{l}h,\theta)$, where $\widetilde{l}\ge 1,h(t)=t^{\eta_1-1}$;

$(r_4)$ $\int_0^1 s^{\eta_2-1}\int_0^1\psi_1(\tau^{\eta_1-1})^{-1}f(\tau,1,1)\mathrm{d}\tau\mathrm{d}s<+\infty$, $\int_0^1 s^{\eta_2-1}\int_0^1\psi_2(\tau^{\eta_1-1})^{-1}g(\tau,1,1)\mathrm{d}\tau\mathrm{d}s<+\infty$, $\int_0^1 s^{\eta_2-1}\int_0^1\tau^{1-\eta_1}\phi(\tau,1,1)\mathrm{d}\tau\mathrm{d}s<+\infty$, $\int_0^1 s^{\eta_2-1}\int_0^1\tau^{1-\eta_1}\rho(\tau,1)\mathrm{d}\tau\mathrm{d}s<+\infty$;

$(r_5)$ $\exists p\in(0,1)$ such that $k_1f(t,x,y)+k_2g(t,x,y)\ge\frac{p}{1-p}[k_3\phi(t,x,y)+k_4\rho(t,x)]$.

Then (T1) there exist $u_0,v_0\in P_h$, $r\in(0,1)$ such that $rv_0\le u_0<v_0$, and

$$\begin{aligned}u_0\le&\int_0^1K(t,s)\int_0^1G(s,\tau)[k_1f(\tau,u_0(\tau),v_0(\tau))+k_2g(\tau,u_0(\tau),\\&v_0(\tau))+k_3\phi(\tau,u_0(\tau),v_0(\tau))+k_4\rho(\tau,u_0(\tau))]\mathrm{d}\tau\mathrm{d}s\\\le&\int_0^1K(t,s)\int_0^1G(s,\tau)[k_1f(\tau,v_0(\tau),u_0(\tau))+k_2g(\tau,v_0(\tau),\\&u_0(\tau))+k_3\phi(\tau,v_0(\tau),u_0(\tau))+k_4\rho(\tau,v_0(\tau))]\mathrm{d}\tau\mathrm{d}s\le v_0;\end{aligned}$$

(T2) the equation Eq. (1) with the boundary value condition (1) has a unique positive solution $x^*$ in $P_h$, where $h(t)=t^{\eta_1-1}$;

(T3) for any initial values $x_0,\ y_0\in P_h$, constructing successively the iterative sequences

$$\begin{aligned}x_{n+1}(t)=&[\int_0^1K(t,s)\int_0^1G(s,\tau)k_1f(\tau,x_n(\tau),y_n(\tau))+k_2g(\tau,\\&x_n(\tau),y_n(\tau))+k_3\phi(\tau,x_n(\tau),y_n(\tau))+k_4\rho(\tau,x_n(\tau))]\mathrm{d}\tau\mathrm{d}s,\\y_{n+1}(t)=&\int_0^1K(t,s)\int_0^1G(s,\tau)[k_1f(\tau,y_n(\tau),x_n(\tau))+k_2g(\tau,\\&y_n(\tau),x_n(\tau))+k_3\phi(\tau,y_n(\tau),x_n(\tau))+k_4\rho(\tau,y_n(\tau))]\mathrm{d}\tau\mathrm{d}s,\end{aligned}$$

$n=0,1,2,\cdots,x_{n+1}\to x^*,\ y_{n+1}\to x^*$, when $n\to\infty$.

**Proof** Define some operators $T_1,T_2,T_3:P_h\times P_h\to E,\ T_4:P_h\to E$ by

$$\begin{aligned}T_1(x,y)(t)&=\int_0^1K(t,s)\int_0^1G(s,\tau)f(\tau,x(\tau),y(\tau))\mathrm{d}\tau\mathrm{d}s,\\T_2(x,y)(t)&=\int_0^1K(t,s)\int_0^1G(s,\tau)g(\tau,x(\tau),y(\tau))\mathrm{d}\tau\mathrm{d}s,\\T_3(x,y)(t)&=\int_0^1K(t,s)\int_0^1G(s,\tau)\phi(\tau,x(\tau),y(\tau))\mathrm{d}\tau\mathrm{d}s,\\T_4x(t)&=\int_0^1K(t,s)\int_0^1G(s,\tau)\rho(\tau,x(\tau))\mathrm{d}\tau\mathrm{d}s.\end{aligned}$$

Hence, from Lemma 4.1, $x$ is the solution of equation Eq. (1)-(2) if and only if $x$ is the solution of $x=k_1T_1(x,x)+k_2T_2(x,x)+k_3T_3(x,x)+k_4T_4x$. We divide the proof into five steps.

Step 1: Due to $f(t,x,y)$, $g(t,x,y)$, $\phi(t,x,y)$, $\rho(t,x)\ge 0$, and $G(t,s),K(t,s)\ge 0$ of Lemma 4.2, it is easy to verify $T_1,T_2,T_3:P_h\times P_h\to P$, $T_4:P_h\to P$. Moreover, it can be verified that $T_1,T_2,T_3$ are monotone operators, and $T_4$ is non-decreasing operator by monotonicity of $f,g,\phi$ and $\rho$ in the condition $(r_1)$.

Step 2: We illustrate that $T_1,T_2,T_3:P_h\times P_h\to P_h$, and $T_4:P_h\to P_h$. Since $\phi(t,x,\cdot)$ is convex, we conclude

$$\phi(t,x,y)\le\lambda\phi(t,x,\lambda^{-1}y)+(1-\lambda)\phi(t,x,\theta),\ \ \lambda\in(0,1),$$

For $x,y\in P_h$, there exists sufficiently large $\widetilde{l}$ such that $x,y,\lambda^{-1}y\le\widetilde{l}h$. Since $\phi(t,\cdot,y)$ is concave, by the condition $(r_3)$, and the monotonicity of $\phi$, we infer that

$$\begin{aligned}\phi(t,\lambda x,\lambda^{-1}y)&\ge\lambda\phi(t,x,\lambda^{-1}y)+(1-\lambda)\phi(t,\theta,\lambda^{-1}y)\\&\ge\phi(t,x,y)+(1-\lambda)(\phi(t,\theta,\widetilde{l}h)-\phi(t,\widetilde{l}h,\theta))\\&\ge[1+(1-\lambda)(1-\frac{1}{a})]\phi(t,x,y)\\&=[(2-\frac{1}{a})+(\frac{1}{a}-1)\lambda]\phi(t,x,y)\ge\lambda\phi(t,x,y).\end{aligned}$$

Then, we can deduce

$$\begin{aligned} &\phi(t,\lambda x,\lambda^{-1}y)\geq\lambda\phi(t,x,y),\\ &\phi(t,\lambda^{-1}x,\lambda y)\leq\lambda^{-1}\phi(t,x,y). \end{aligned}\tag{16}$$

In view of $(r_2)$, we deduce

$$\begin{aligned} &f(t,\lambda^{-1}x,\lambda y)\leq\psi_1(\lambda)^{-1}f(t,x,y),\\ &g(t,\lambda^{-1}x,\lambda y)\leq\psi_2(\lambda)^{-1}g(t,x,y),\\ &\rho(t,\lambda^{-1}x)\leq\lambda^{-1}\rho(t,x). \end{aligned}\tag{17}$$

Taking $x=y=1$ in (16)- (17) and the condition $(r_2)$, we have

$$\begin{aligned} &f(t,\lambda,\lambda^{-1})\geq\psi_1(\lambda)f(t,1,1), f(t,\lambda^{-1},\lambda)\leq\psi_1(\lambda)^{-1}f(t,1,1),\\ &g(t,\lambda,\lambda^{-1})\geq\psi_2(\lambda)g(t,1,1), g(t,\lambda^{-1},\lambda)\leq\psi_2(\lambda)^{-1}g(t,1,1),\\ &\phi(t,\lambda,\lambda^{-1})\geq\lambda\phi(t,1,1), \phi(t,\lambda^{-1},\lambda)\leq\lambda^{-1}\phi(t,1,1),\\ &\rho(t,\lambda)\geq\lambda\rho(t,1),\quad \rho(t,\lambda^{-1})\leq\lambda^{-1}\rho(t,1). \end{aligned}$$

For $x,y\in P_h$, there exists $J\in(0,1)$ such that $Jh\leq x,y\leq J^{-1}h$. Then, we obtain

$$\begin{aligned} f(t,x(t),y(t))&\leq f(t,J^{-1}t^{\eta_1-1},Jt^{\eta_1-1})\leq f(t,J^{-1}t^{1-\eta_1},Jt^{\eta_1-1})\\ &\leq\psi_1(t^{\eta_1-1})^{-1}\psi_1(J)^{-1}f(t,1,1),\\ f(t,x(t),y(t))&\geq f(t,Jt^{\eta_1-1},J^{-1}t^{\eta_1-1})\geq f(t,Jt^{\eta_1-1},J^{-1}t^{1-\eta_1})\\ &\geq\psi_1(t^{\eta_1-1})\psi_1(J)f(t,1,1). \end{aligned}$$

In the similar way, the following inequalities hold $\psi_2(t^{\eta_1-1})\psi_2(J)g(t,1,1)\leq g(t,x(t),y(t))\leq\psi_2(t^{\eta_1-1})^{-1}\psi_2(J)^{-1}g(t,1,1)$, $t^{\eta_1-1}J\phi(t,1,1)\leq\phi(t,x(t),y(t))\leq t^{1-\eta_1}J^{-1}\phi(t,1,1)$, $t^{\eta_1-1}J\rho(t,1)\leq\rho(t,x(t))\leq t^{1-\eta_1}J^{-1}\rho(t,1)$.

Using Lemma 4.2, and the condition $(r_4)$, one observes

$$\begin{aligned} T_1(x,y)(t)\leq&\int_0^1 L_2t^{\eta_1-1}\int_0^1 L_1s^{\eta_2-1}\psi_1(\tau^{\eta_1-1})^{-1}\psi_1(J)^{-1}\\ &\cdot f(\tau,1,1)\mathrm{d}\tau\mathrm{d}s=L_1L_2\psi_1(J)^{-1}t^{\eta_1-1}\int_0^1 s^{\eta_2-1}\\ &\cdot\int_0^1\psi_1(\tau^{\eta_1-1})^{-1}f(\tau,1,1)\mathrm{d}\tau\mathrm{d}s<+\infty. \end{aligned}\tag{18}$$

Similarly, it gives

$$T_2(x,y)(t)<+\infty,\ T_3(x,y)(t)<+\infty,\ T_4x(t)<+\infty.$$

Assume that $J_1\in(0,1)$ and

$$\begin{aligned} J_1<\min\Big\{&\Big(L_1L_2\psi_1(J)^{-1}\int_0^1 s^{\eta_2-1}\int_0^1\psi_1(\tau^{\eta_1-1})^{-1}f(\tau,1,1)\mathrm{d}\tau\\ &\mathrm{d}s\Big)^{-1},\psi_1(J)\int_0^1 l_2(s)s^{\eta_2-1}\int_0^1 l_1(\tau)\psi_1(\tau^{\eta_1-1})f(\tau,1,1)\mathrm{d}\tau\mathrm{d}s\\ &\Big(L_1L_2\psi_2(J)^{-1}\int_0^1 s^{\eta_2-1}\int_0^1\psi_2(\tau^{\eta_1-1})^{-1}g(\tau,1,1)\mathrm{d}\tau\mathrm{d}s\Big)^{-1},\\ &\psi_2(J)\int_0^1 l_2(s)s^{\eta_2-1}\int_0^1 l_1(\tau)\psi_2(\tau^{\eta_1-1})g(\tau,1,1)\mathrm{d}\tau\mathrm{d}s\\ &\Big(L_1L_2J^{-1}\int_0^1 s^{\eta_2-1}\int_0^1\tau^{1-\eta_1}\phi(\tau,1,1)\mathrm{d}\tau\mathrm{d}s\Big)^{-1},\\ &J\int_0^1 l_2(s)s^{\eta_2-1}\int_0^1 l_1(\tau)\tau^{\eta_1-1}\phi(\tau,1,1)\mathrm{d}\tau\mathrm{d}s\\ &\big(L_1L_2J^{-1}\int_0^1 s^{\eta_2-1}\int_0^1\tau^{1-\eta_1}\rho(\tau,1)\mathrm{d}\tau\mathrm{d}s)^{-1},\\ &J\int_0^1 l_2(s)s^{\eta_2-1}\int_0^1 l_1(\tau)\tau^{\eta_1-1}\rho(\tau,1)\mathrm{d}\tau\mathrm{d}s\Big\}. \end{aligned}$$

By means of $J_1$, it holds that

$$J_1t^{\eta_1-1}\leq T_1(x,y)(t)\leq J_1^{-1}h(t),$$

i.e., $T_1:P_h\times P_h\to P_h$. Similarly, $T_2,T_3:P_h\times P_h\to P_h$, $T_4:P_h\to P_h$.

Step 3: From the assumption $(r_2)$, there exist $\psi_1(\lambda)\in(\lambda,1]$, $\psi_2(\lambda)\in(\lambda,1]$ such that

$$\begin{aligned} T_1(\lambda x,\lambda^{-1}y)(t)\geq&\int_0^1 K(t,s)\int_0^1 G(s,\tau)\psi_1(\lambda)\\ &\cdot f(\tau,x(\tau),y(\tau))\mathrm{d}\tau\mathrm{d}s=\psi_1(\lambda)T_1(x,y)(t),\\ T_2(\lambda x,\lambda^{-1}y)(t)\geq&\int_0^1 K(t,s)\int_0^1 G(s,\tau)\psi_2(\lambda)\\ &\cdot g(\tau,x(\tau),y(\tau)))\mathrm{d}\tau\mathrm{d}s=\psi_2(\lambda)T_2(x,y)(t). \end{aligned}$$

What's more, for fixed $t\in(0,1),y\in P$, for any $\iota\in(0,1)$, $x_1,x_2\in P$, we can find

$$\begin{aligned} &T_3(\iota x_1+(1-\iota)x_2,y)(t)\geq\int_0^1 K(t,s)\int_0^1 G(s,\tau)[\iota\phi(\tau,x_1(\tau),\\ &\quad y(\tau))+(1-\iota)\phi(\tau,x_2(\tau),y(\tau))]\mathrm{d}\tau\mathrm{d}s\\ &=\iota\int_0^1 K(t,s)\int_0^1 G(s,\tau)\phi(\tau,x_1(\tau),y(\tau))\mathrm{d}\tau\mathrm{d}s\\ &\quad+(1-\iota)\int_0^1 K(t,s)\int_0^1 G(s,\tau)\phi(\tau,x_2(\tau),y(\tau))\mathrm{d}\tau\mathrm{d}s\\ &=\iota T_3(x_1,y)+(1-\iota)T_3(x_2,y)(t). \end{aligned}$$

For fixed $t\in(0,1),x\in P$, for any $\iota\in(0,1),y_1,y_2\in P$, it gives

$$\begin{aligned} &T_3(x,\iota y_1+(1-\iota)y_2)(t)\leq\int_0^1 K(t,s)\int_0^1 G(s,\tau)[\iota\phi(\tau,x(\tau),\\ &\quad y_1(\tau))+(1-\iota)\phi(s,x(s),y_2(s))]\mathrm{d}\tau\mathrm{d}s\\ &=\iota\int_0^1 K(t,s)\int_0^1 G(s,\tau)\phi(\tau,x(\tau),y_1(\tau))\mathrm{d}\tau\mathrm{d}s\\ &\quad+(1-\iota)\int_0^1 K(t,s)\int_0^1 G(s,\tau)\phi(\tau,x(\tau),y_2(\tau))\mathrm{d}\tau\mathrm{d}s\\ &=\iota T_3(x,y_1)+(1-\iota)T_3(x,y_2)(t). \end{aligned}$$

Hence, for fixed $y\in P$, $T_3(\cdot,y)$ is concave; for fixed $x\in P$, $T_3(x,\cdot)$ is convex. Besides, for any $\lambda\in(0,1)$, we derive that

$$\begin{aligned} T_4(\lambda x)(t)\geq&\int_0^1 K(t,s)\int_0^1 G(s,\tau)\lambda\rho(\tau,x(\tau))\mathrm{d}\tau\mathrm{d}s\\ =&\lambda T_4x(t), \end{aligned}$$

i.e., $T_4$ is a sub-homogeneous operator.

Step 3: By the condition $(r_3)$, there exists $\frac{1}{2}\leq a\leq 1$ such that

$$\begin{aligned}T_3(\theta,\widetilde{l}h)(t) &= \int_0^1 K(t,s)\int_0^1 G(s,\tau)\phi(\tau,\theta,\widetilde{l}h(\tau))\mathrm{d}\tau\mathrm{d}s\\ &\geq \int_0^1 K(t,s)\int_0^1 G(s,\tau)a\phi(\tau,\widetilde{l}h(\tau),\theta)\mathrm{d}\tau\mathrm{d}s\\ &= aT_3(\widetilde{l}h,\theta)(t),\end{aligned}$$

which prove the condition $(H_2)$ of Corollary 3.2.

Step 5: It can be concluded from $(r_5)$ that

$$\begin{aligned}&k_1T_1(x,y)(t)+k_2T_2(x,y)(t) = \int_0^1 K(t,s)\int_0^1 G(s,\tau)[k_1f(\tau,\\ &\quad x(\tau),y(\tau))+k_2g(\tau,x(\tau),y(\tau))]\mathrm{d}\tau\mathrm{d}s\\ &\geq \frac{p}{1-p}\int_0^1 K(t,s)\int_0^1 G(s,\tau)[k_3\phi(\tau,x(\tau),y(\tau))\\ &\quad +k_4\rho(\tau,x(\tau))]\mathrm{d}\tau\mathrm{d}s\\ &= \frac{p}{1-p}[k_3T_3(x,y)(t)+k_4T_4x(t)],\ p\in(0,1),\forall t\in(0,1),\end{aligned}$$

which implies $(H_4')$ of Corollary 3.2.

Based on the above five steps, from Corollary 3.2, the conclusions (T1)-(T3) hold.

**Remark 4.4** In Theorem 4.3, the properties of $T_2$ in the condition $(r_2)$ and $(r_4)$ turn into $\forall\lambda\in(0,1), x,y\in(0,+\infty)$ such that $g(t,\lambda x,\lambda^{-1}y)\geq\lambda g(t,x,y)$, and $\int_0^1 s^{\eta_2-1}\int_0^1\tau^{1-\eta_1}g(\tau,1,1)\mathrm{d}\tau\mathrm{d}s<+\infty$, the condition $(r_5)$ changes into $\exists p\in(0,1)$ such that $k_1f(t,x,y)\geq\frac{p}{1-p}[k_2g(t,x,y)+k_3\phi(t,x,y)+k_4\rho(t,x)]$, then from Corollary 3.8, the conclusions (T1)-(T3) still hold.

**Lemma 4.5** If $\sigma(t)=k_1f(t,x(t),x(t))+k_2g(t,x(t),x(t))+k_3\phi(t,x(t),x(t))+k_4\rho(t,x(t))\in C[0,1]$, $M_1>0, M_2>0$, the equation (1) with condition (3) has the following equivalent integral equation

$$\begin{aligned}x(t) = &\int_0^1 G(t,s)\int_0^1 K(s,\tau)[k_1f(\tau,x(\tau),x(\tau))+k_2g(\tau,x(\tau),\\ &x(\tau))+k_3\phi(\tau,x(\tau),x(\tau))+k_4\rho(\tau,x(\tau))]\mathrm{d}\tau\mathrm{d}s,\end{aligned}$$

where $G(t,s),K(t,s)$ defined as Lemma 4.1.

**Theorem 4.6** Let $f,g,\phi : C\big((0,1)\times(0,+\infty)\times(0,+\infty),[0,+\infty)\big), \rho : C\big((0,1)\times(0,+\infty),[0,+\infty)\big)$, and $f(t,u,v)$, $g(t,u,v),\phi(t,u,v),\rho(t,u)$ are singular at $t=0,1$ and $v=0$. Assume that $(r_1)$-$(r_2)$, $(r_5)$ hold and

$(r_9)$ $\exists\frac{1}{2}\leq a\leq 1$ such that $\phi(t,\theta,\widetilde{l}h)\geq a\phi(t,\widetilde{l}h,\theta)$, where $\widetilde{l}\geq 1, h(t)=t^{\eta_2-1}$;

$(r_{10})$ $\int_0^1 s^{\eta_1-1}\int_0^1\psi_1(\tau^{\eta_2-1})^{-1}f(\tau,1,1)\mathrm{d}\tau\mathrm{d}s<+\infty$,
$\int_0^1 s^{\eta_1-1}\int_0^1\psi_2(\tau^{\eta_2-1})^{-1}g(\tau,1,1)\mathrm{d}\tau\mathrm{d}s<+\infty$,
$\int_0^1 s^{\eta_1-1}\int_0^1\tau^{1-\eta_2}\phi(\tau,1,1)\mathrm{d}\tau\mathrm{d}s<+\infty$,
$\int_0^1 s^{\eta_1-1}\int_0^1\tau^{1-\eta_2}\rho(\tau,1)\mathrm{d}\tau\mathrm{d}s<+\infty$;

Then (T6) there exist $u_0,v_0\in P_h$, $r\in(0,1)$ such that $rv_0\leq u_0<v_0$, and

$$\begin{aligned}u_0\leq &\int_0^1 G(t,s)\int_0^1 K(s,\tau)[k_1f(\tau,u_0(\tau),v_0(\tau))+k_2g(\tau,u_0(\tau),\\ &v_0(\tau))+k_3\phi(\tau,u_0(\tau),v_0(\tau))+k_4\rho(\tau,u_0(\tau))]\mathrm{d}\tau\mathrm{d}s\\ \leq &\int_0^1 G(t,s)\int_0^1 K(s,\tau)[k_1f(\tau,v_0(\tau),u_0(\tau))+k_2g(\tau,v_0(\tau),\\ &u_0(\tau))+k_3\phi(\tau,v_0(\tau),u_0(\tau))+k_4\rho(\tau,v_0(\tau))]\mathrm{d}\tau\mathrm{d}s\leq v_0;\end{aligned}$$

(T7) the equation (1) with the boundary value condition (3) has a unique positive solution $x^*$ in $P_h$, where $h(t)=t^{\eta_2-1}$;

(T8) for any initial values $x_0,\ y_0\in P_h$, constructing successively the iterative sequences

$$\begin{aligned}x_{n+1}(t) = &\int_0^1 G(t,s)\int_0^1 K(s,\tau)[k_1f(\tau,x_n(\tau),y_n(\tau))+k_2g(\tau,\\ &x_n(\tau),y_n(\tau))+k_3\phi(\tau,x_n(\tau),y_n(\tau))+k_4\rho(\tau,x_n(\tau))]\mathrm{d}\tau\mathrm{d}s,\\ y_{n+1}(t) = &\int_0^1 G(t,s)\int_0^1 K(s,\tau)[k_1f(\tau,y_n(\tau),x_n(\tau))+k_2g(\tau,\\ &y_n(\tau),x_n(\tau))+k_3\phi(\tau,y_n(\tau),x_n(\tau))+k_4\rho(\tau,y_n(\tau))]\mathrm{d}\tau\mathrm{d}s,\end{aligned}$$

$n=0,1,2,\cdots,x_{n+1}\to x^*,\ y_{n+1}\to x^*$, when $n\to\infty$.

## 5. APPLICATIONS

Consider the following equation:

$$\begin{cases} D_{0+}^{\frac{41}{12}}x(t)+2[(x+4)^{\frac{1}{3}}+4x^{-\frac{1}{4}}+t^{-\frac{1}{8}}(1-t)^{-\frac{1}{9}}+3]\\ \qquad +(x+3)^{\frac{1}{4}}+3x^{-\frac{1}{4}}+t^{-\frac{1}{9}}(1-t)^{-\frac{1}{10}}\\ \qquad +\frac{1}{2}[-\frac{3}{5}e^{-x}+\frac{9}{4}e^{-y}+t^{-\frac{1}{10}}(1-t)^{-\frac{1}{11}}+1]\\ \qquad +\frac{1}{3}[(x+1)^{\frac{1}{6}}+t^{-\frac{1}{12}}(1-t)^{-\frac{1}{13}}]=0,\\ x(0)=0,\quad D_{0+}^{\frac{5}{3}}x(0)=0,\quad D_{0+}^{\frac{1}{7}}x(1)=2I_{0+}^{\frac{1}{8}}x(\frac{1}{2}),\\ D_{0+}^{\frac{27}{15}}x(1)=\int_0^1 s^{\frac{1}{2}}D_{0+}^{\frac{11}{6}}x(s)\mathrm{d}s^2,\end{cases} \tag{19}$$

where $\eta_1=\frac{5}{3},\ \eta_2=\frac{7}{4},\ \alpha_1=\frac{1}{7},\ \alpha_2=\frac{1}{5},\ \beta_1=\frac{1}{8},\ \beta_2=\frac{1}{6},\ \tau=2,\ b(s)=s^{\frac{1}{2}},\ A(s)=s^2,\ \xi_1=\frac{1}{2},\ k_1=2,\ k_2=1,\ k_3=\frac{1}{2},\ k_4=\frac{1}{3}$, and

$$\begin{aligned}f(t,x,y) &= (x+4)^{\frac{1}{3}}+4y^{-\frac{1}{4}}+t^{-\frac{1}{8}}(1-t)^{-\frac{1}{9}}+3,\\ g(t,x,y) &= (x+3)^{\frac{1}{4}}+3y^{-\frac{1}{4}}+t^{-\frac{1}{9}}(1-t)^{-\frac{1}{10}},\\ \phi(t,x,y) &= -\frac{3}{5}e^{-x}+\frac{9}{4}e^{-y}+t^{-\frac{1}{10}}(1-t)^{-\frac{1}{11}}+1,\\ \rho(t,x) &= (x+1)^{\frac{1}{6}}+t^{-\frac{1}{12}}(1-t)^{-\frac{1}{13}}.\end{aligned}$$

Then Eq. (19) has unique positive solution $x^*\in P_h$, where $h(t)=t^{\frac{2}{3}}$. In addition, the other conclusions in Theorem 4.3 hold.

**Proof** The proof process can be divided into the following steps.

$(\widetilde{a1})$ $\eta_2-\alpha_2-1=\frac{13}{4}-\frac{11}{5}-1>0,\ \eta_2-\beta_2-1=\frac{13}{4}-\frac{7}{6}-1>0,\ \alpha_2-\beta_2=\frac{1}{5}-\frac{1}{6}>0$, and $M_1=\Gamma(\frac{5}{3})\Gamma(\frac{5}{3}+\frac{1}{8})-2\Gamma(\frac{5}{3})\Gamma(\frac{5}{3}-\frac{1}{7})\frac{1}{2}^{\frac{5}{3}+\frac{1}{8}-1}>0$, $M_2=\frac{1}{\Gamma(\frac{7}{4}-\frac{1}{5})}-\frac{\int_0^1 s^{\frac{1}{2}}s^{\frac{7}{4}-\frac{1}{6}-1}\mathrm{d}s^2}{\Gamma(\frac{7}{4}-\frac{1}{6})}>0$.

$(\widetilde{b1})$ from the definition of $f,g,\phi,\rho$, we can get that $f(t,x,y),g(t,x,y),\phi(t,x,y),\rho(t,x)$ are non-decreasing in $x$, $f(t,x,y),g(t,x,y),\phi(t,x,y)$ are non-increasing in $y$, and $f(t,u,v),g(t,u,v),\rho(t,u)$ are singular at $t=0,1$ and $v=0$, $\phi(t,u,v)$ is singular at $t=0,1$.

$(\widetilde{c1})$ there exist $\psi_1(\lambda)=\lambda^{\frac{1}{3}}, \psi_2(\lambda)=\lambda^{\frac{1}{4}}$ such that

$$\begin{aligned} f(t,\lambda x,\lambda^{-1}y) &\geq \lambda^{\frac{1}{3}}[(x+4)^{\frac{1}{3}}+4y^{-\frac{1}{4}}+t^{-\frac{1}{8}}(1-t)^{-\frac{1}{9}}+3] \\ &= \psi_1(\lambda)f(t,x,y), \\ g(t,\lambda x,\lambda^{-1}y) &\geq \lambda^{\frac{1}{4}}[(x+3)^{\frac{1}{4}}+3y^{-\frac{1}{4}}+t^{-\frac{1}{9}}(1-t)^{-\frac{1}{10}}] \\ &= \psi_2(\lambda)g(t,x,y). \end{aligned}$$

From $\phi''_{xx}(t,x,y)=-\frac{3}{5}e^{-x}\leq 0$, $\phi''_{yy}(t,x,y)=\frac{9}{4}e^{-y}\geq 0$, we can know that $\phi(t,\cdot,y)$ is concave; $\phi(t,x,\cdot)$ is convex. For any $\lambda,t\in(0,1)$, $x,y\in(0,+\infty)$, one gets

$$\rho(t,\lambda x)\geq \lambda[(x+1)^{\frac{1}{6}}+t^{-\frac{1}{12}}(1-t)^{-\frac{1}{13}}]\geq \lambda\rho(t,x).$$

$(\widetilde{d1})$ Let $x,y\leq M_3$, $M_3$ is a sufficiently large number. Take $a=\frac{1}{9}$, we deduce $\phi(t,\theta,M_3)\geq \frac{1}{9}[-\frac{3}{5}e^{-M_3}+\frac{9}{4}+t^{-\frac{1}{10}}(1-t)^{-\frac{1}{11}}+1]=a\phi(t,M_3,\theta)$.

$(\widetilde{e1})$ We can get $\int_0^1 s^{\eta_2-1}\int_0^1\psi_1(\tau^{\eta_1-1})^{-1}f(\tau,1,1)\mathrm{d}\tau\mathrm{d}s=\int_0^1 s^{\frac{3}{4}}\int_0^1\tau^{-\frac{2}{9}}[5^{\frac{1}{3}}+\tau^{-\frac{1}{8}}(1-\tau)^{-\frac{1}{9}}+7]\mathrm{d}\tau\mathrm{d}s<+\infty$. Similarly, $\int_0^1 s^{\eta_2-1}\int_0^1\psi_2(\tau^{\eta_1-1})^{-1}g(\tau,1,1)\mathrm{d}\tau\mathrm{d}s<+\infty$, $\int_0^1 s^{\eta_2-1}\int_0^1\tau^{1-\eta_1}\phi(\tau,1,1)\mathrm{d}\tau\mathrm{d}s<+\infty$, $\int_0^1 s^{\eta_2-1}\int_0^1\tau^{1-\eta_1}\rho(\tau,1)\mathrm{d}\tau\mathrm{d}s<+\infty$;

$(\widetilde{f1})$ Let $p=\frac{3}{4}$. We obtain

$$\begin{aligned} k_1f(t,x,y)+k_2g(t,x,y) &\geq 3\cdot\frac{1}{2}[-\frac{3}{5}e^{-x}+\frac{9}{4}e^{-y}+t^{-\frac{1}{10}} \\ &\cdot(1-t)^{-\frac{1}{11}}+1]+3\cdot\frac{1}{3}[(x+1)^{\frac{1}{6}}+t^{-\frac{1}{12}}(1-t)^{-\frac{1}{13}}] \\ =&\frac{p}{1-p}[k_3\phi(t,x,y)+k_4\rho(t,x)]. \end{aligned}$$

According to Theorem 4.3, Eq. (19) has unique positive solution $x^*\in P_h$, where $h(t)=t^{\frac{2}{3}}$. and $(T1),(T3)$ hold.

## 6. CONCLUSIONS

This paper mainly discusses the existence and uniqueness of solution as well as the iterative sequences for uniformly approximating the unique solution for two types of singular FDEs with RS integral boundary conditions. The main conclusions are as follows. (i) Considering that nonlinear terms in abstract differential equations in some real field have different convexity and monotonicity, in order to better study these equations, this paper discuss the fixed point theorems of sum operators that contain differential properties. We obtain the corresponding properties of solution to operator equation with parameters $T_5(x,y)=k_1T_1(x,y)+k_2T_2(x,y)+k_3T_3(x,y)+k_4T_4x$ on cone $P$ and set $P_h$. Based on different parameter values and operator properties, the operator conclusions obtained generalize some existing literature results. (ii) Using the conclusions of nonlinear operators in the study, two types of fractional differential equations are discussed. Several sufficient conditions of unique solutions, the existence of maximum and minimum solutions, and the iterative approximation sequences of unique solutions are given.

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