10.24425/acs.2024.149668

Archives of Control Sciences
Volume 34(LXX), 2024
No. 3*, pages 473–500*

Fuzzy sliding mode control based-fast finite-time projective synchronization for fractional-order chaotic systems

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This study explores the challenge of achieving a fast finite-time projective synchronization (FFTPS) in chaotic systems characterized by incommensurate fractional orders, unknown master-slave models, and uncertain external disturbances. Utilizing the principles of Lyapunov stability theory, two fuzzy sliding mode control (FSMC) schemes are proposed. Accordingly, two novel non-singular finite-time sliding surfaces are constructed. Fuzzy logic systems are utilized to provide an approximation of the continuous uncertain dynamics within the master-slave system. The sufficient conditions for both controllers are derived to ensure this robust FFTPS. Finally, the proposed controllers are validated through numerical simulations on two projective synchronization examples of fractional-order chaotic systems, demonstrating their feasibility.

Key words: finite-time projective synchronization, chaotic systems with fractional-orders, fuzzy systems, sliding mode control

1. Introduction

Chaotic systems appear to be dynamic systems that defy to be synchronized. In fact, two similar autonomous chaotic systems with very close initial conditions have trajectories which quickly become uncorrelated, despite each one charting the identical attractor within the phase space. The notion of chaotic synchronization was initially investigated in a 1990s paper authored by Pecora and Carroll [1]. Essentially, this method involves creating two chaotic systems that are linked together so that the behavior of one system (the master system) controls the behavior of the other system (the slave one). Controlling and synchronizing chaotic systems

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Received 7.06.2024. Revised 12.08.2024.

can provide practical benefits in secure communication, control of complex systems, and understanding natural and technological processes. It also contributes to the fundamental understanding of chaos and nonlinear dynamics [2–4]. Furthermore, in recent times, researchers have become increasingly interested in the synchronization and control challenges presented by integer-order chaotic systems. Various control methods, such as adaptive control [5, 6]; linear matrix inequality-based control [7]; sliding-mode control (SMC) [8–11]; fuzzy adaptive control [12–15]; and so on, have been explored in this context.

Fractional-order systems have garnered significant attention from the global research community [2, 4, 16]. The exploration of fractional-calculus began in 1695 when mathematician L'Hôpital posed a question to Leibniz about fractional-order derivation. Since then, fractional differential equations have been employed to model faithfully and accurately model various real-world systems [16–20], including but not being limited to electrical circuits, viscoelastic beams, diffusion equation models, as well as chaotic systems. Accordingly, the issue of the chaos control (i.e. chaos suppression) and synchronization of fractional-order chaotic systems have received substantial attention in the literature, as evidenced by numerous noteworthy studies, e.g. [21–28]. Nonetheless, in all of these studies, the stability of the closed-loop system is asymptotic (in the best exponential). This implies that the synchronization or chaos suppression (stabilization) can only be fully realized as time approaches infinity. It's worth highlighting that the rate of convergence and achieving rapid convergence are crucial aspects of synchronization and control effectiveness, particularly in certain practical applications within this domain.

That is why the study of finite-time (FT) control and FT synchronization has gained recently attention in the field of control due to their appealing attributes, including swift convergence, exceptional control efficiency, and the ability to effectively handle disturbances [29–31]. Over the past few years, numerous methodologies for achieving finite-time control, whether for chaos-suppression or achieving chaos synchronization, have emerged. These methods are based on the well-known sliding mode control (SMC) principle [32–38]. These strategies encompass terminal SMC [39–41], high-order SMC [42, 43], and super-twisting SMC [44, 45]. Nevertheless, each of these control techniques comes with its own set of limitations. For example, terminal SMC can encounter challenges such as the singularity problem and unwanted chattering in practical applications. In contrast, high-order SMC represents an elegant and robust control method, but it necessitates knowledge of all subsequent time derivatives of the designated sliding surface variable. Super-twisting SMC was introduced as a solution to these issues; however, it demands prior knowledge of the upper bounds of system uncertainties. Additionally, by combining universal function approximators, specifically the fuzzy logic systems and neural networks, the authors of [39,40,46–49] have achieved effective designs for FT control or FT synchronization for some classes of uncertain chaotic systems. These systems are distinctive due to their lack of known models and the existence of external disturbances. Importantly, in contrast to schemes based only on SMC, control and synchronization schemes based on the universal function approximators remove the requirement for prior information regarding the upper limits of system uncertainties.

Drawing inspiration from the aforementioned excellent works, this paper delves into the fast finite-time projective synchronization (FFTPS) problem for chaotic systems with noncommensurate fractional-orders and unknown master-slave models and external disturbances. For that, two novel fuzzy sliding mode control (FSMC) schemes are proposed. Here are the principal contributions made by this research:

- Unlike the respective works in [29,31–35,39–49] and [30,36,38,50], the proposed FSMC based-FFTPS schemes depend neither on the master-slave model nor the upper bounds for its uncertainties.
- Unlike many recent works [40,41,47–49], this paper provides a rigorous proof of FFTPS error convergence using Lyapunov theory.
- Notably, our second control scheme incorporating a useful integral term successfully solves the adverse chattering phenomenon, being a common issue in many existing finite-time controllers, e.g. [29–31,33–36,38,41,43,46,47,49,50].
- In contrast to the close related work [31–33, 35, 40, 41, 43, 45–49], our FSMCs guarantee that both the sliding-mode phase and the reaching phase are accomplished rapidly and in FT.
- Unlike [29, 30, 34, 36, 38–44, 48–50], the model of the master-slave system (MSS) under-consideration is assumed to be with non-commensurate fractional orders.

The remainder of this paper is structured as follows: Section 2 presents preliminary concepts and formulates the problem. Section 3 outlines the first control design approach, while Section 4 describes the second one. A theoretical comparison is shown in Section 5. Section 6 discusses the simulation results, and Section 7 concludes the paper.

2. Preliminaries and problem formulation

Below, we present helpful definitions, properties, and lemmas related to fractional calculus, finite-time stability, and fuzzy systems that will be employed in subsequent sections.

2.1. Fractional calculus

Caputo's definition of derivative operator will be employed extensively within this paper, because it offers well-established physical interpretations and meanings for the initial conditions in fractional differential equations [51].

The q-th fractional integral of a continuous function $f(t)$ is described as follows:

Definition 1. *The q-th fractional integral of a continuous function $f(t)$ is described as follows:*

$$J^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau)\,\mathrm{d}\tau, \tag{1}$$

where $\Gamma(q) = \int_0^{+\infty} t^{q-1} e^{-t}\,\mathrm{d}t$ *is the gamma function.*

Definition 2. *The Caputo fractional derivative is:*

$$D_t^q x(t) = J^{m-q} x^{(m)}(t) = \frac{1}{\Gamma(m-q)} \int_0^t (t-\tau)^{m-q-1} f^{(m)}(\tau)\,\mathrm{d}\tau, \tag{2}$$

where $m-1 < q < m$, $m \in Z^+$. In the remainder of this study, we will exclusively consider $0 < q < 1$.

Property 1. *Let $0 < q < 1$. Then, one has*

$$Dx(t) = D_t^{1-q} D_t^q x(t), \tag{3}$$

where $D = \dfrac{\mathrm{d}}{\mathrm{d}t}$.

Property 2. *Consider $x(t) \in R^n$ a state vector and a fractional differential system of order q in the Caputo sense [52–54]:*

$$D_t^q x(t) = f\left(x(t)\right), \tag{4}$$

where $f(x(t))$ is a Lipschitz function, i.e.

$$\| f(x(t)) - f(y(t)) \| \leqslant l \, \|x(t) - y(t)\| \tag{5}$$

with $l > 0$ is a Lipschitz's constant. Without sacrificing generality, let's assume that $f(0) = 0$. Consequently, we can derive the following:

$$\| f(x(t)) \| \leqslant l \, \|x(t)\| \,. \tag{6}$$

2.2. Some useful Lemmas for stability analysis

Lemma 1. *Consider* $0 < q < 2$, *and* δ_1, δ_2, ..., *and* δ_n *real numbers, then [55]:*

$$|\delta_1|^q + |\delta_2|^q + \ldots + |\delta_n|^q \geqslant \left(\delta_1^2 + \delta_2^2 + \ldots + \delta_n^2\right)^{q/2}. \tag{7}$$

Lemma 2. *Consider the nonlinear system* $\dot{x} = f(x,t)$ *with* $x \in R^n$ *being its state vector. Suppose the existence of a positively defined continuous functional* $V(x,t)$ *which satisfies [56]:*

$$\dot{V} + \alpha V + \beta V^{\gamma} \leqslant 0, \tag{8}$$

where $\alpha > 0$, $\beta > 0$, *and* $0 < \gamma < 1$.

Then, x is converges rapidly and in a FT to the equilibrium point. The convergence time is

$$T_s = t_0 + \frac{1}{\alpha(1-\gamma)} \ln\left(\frac{\alpha V^{\gamma}(x_0, t_0) + \beta}{\beta}\right). \tag{9}$$

2.3. Fuzzy systems

A fuzzy logic system (FLS) primarily comprises four elements: a knowledge base module, a fuzzification module, a fuzzy inference mechanism and a defuzzification module. Specifically, the knowledge base module contains a set of fuzzy rules with the following structure:

$$\begin{aligned} R^{(i)}: \quad & \text{IF } z_1 \text{ is } A_1^i \ \ldots \ \text{AND} \ \ldots \ \text{AND } z_n \text{ is } A_n^i, \\ & \text{THEN } F \text{ is } G^i, \quad \text{for } i = 1, 2, \ldots, N, \end{aligned} \tag{10}$$

where $Z^T = [z_1, \ldots, z_n] \in R^n$ represents the input vector of the FLS, while $F \in R$ corresponds to its generated output. N stands for the number of fuzzy rules.

The result of a FLS utilizing a center-average defuzzifier, a singleton fuzzifier, and a product inference can be succinctly formulated as:

$$F(Z) = \frac{\sum_{i=1}^{N} G^i \left(\prod_{j=1}^{n} \mu_{A_j^i}(z_j)\right)}{\sum_{i=1}^{N} \left(\prod_{j=1}^{n} \mu_{A_j^i}(z_j)\right)} = \theta^T \psi(Z), \tag{11}$$

with $\mu_{A_j^i}(z_j)$ being the membership function significance of the fuzzy variable z_j. $\theta^T = [G^1, G^2, \ldots, G^N]$ is the parameter' vector to be adjusted, and

$\psi^T = [\psi^1, \psi^2, \dots, \psi^N]$ with

$$\psi^i(Z) = \frac{\prod_{j=1}^{n} \mu_{A_j^i}\left(z_j\right)}{\sum_{i=1}^{N}\left(\prod_{j=1}^{n} \mu_{A_j^i}\left(z_j\right)\right)} \tag{12}$$

being the fuzzy basis function (FBF).

Lemma 3. [57]: *Let consider a continuous, real function $F(Z)$, defined within a bounded and closed set Ω_Z in R^n. For any positive constant ε, there is a FLS such that:*

$$\left|F(Z) - \theta^{*T}\psi(Z)\right| \leqslant \varepsilon, \tag{13}$$

where θ^* represents the best parameter vector which is theoretically assumed to be unknown.

2.4. Problem formulation

Our master system is a chaotic system with Caputo fractional-order dynamics, described by:

$$\begin{aligned} D_t^{q_1} x_1 &= f_1(x), \\ &\vdots \\ D_t^{q_n} x_n &= f_n(x) \end{aligned} \tag{14}$$

with $0 < q_i < 1$. $x = [x_1, \dots, x_n]^T \in R^n$ being its measurable pseudo-state vector, and $f_i(x) \in R$ uncertain functions.

We also consider the slave system to be a Caputo's fractional-order chaotic system:

$$\begin{aligned} D_t^{q_1} y_1 &= g_1(y) + u_1 + d_1(t, y), \\ &\vdots \\ D_t^{q_n} y_n &= g_n(y) + u_n + d_n(t, y), \end{aligned} \tag{15}$$

where $y = [y_1, \dots, y_n]^T \in R^n$ is its measurable pseudo-state vector. $g_i(y) \in R$ are uncertain functions. $u_i \in R$ is the slave system input and $d_i(t, y) \in R$ are external disturbances.

Define the synchronization error as $e_i = y_i - \lambda_i x_i$, with λ_i being a scaling factor. By taking the fractional-order time-derivative of e_i, one obtains

$$\begin{aligned} D_t^{q_1} e_1 &= g_1(y) - \lambda_1 f_1(x) + u_1 + d_1(t,y), \\ &\vdots \\ D_t^{q_n} e_n &= g_n(y) - \lambda_n f_n(x) + u_n + d_n(t,y). \end{aligned} \tag{16}$$

The main target of this research is to design two FSMC laws which adequately carry out a FFTPS between (14) and (15). Figure 1 illustrates the block-diagram of our synchronization scheme.

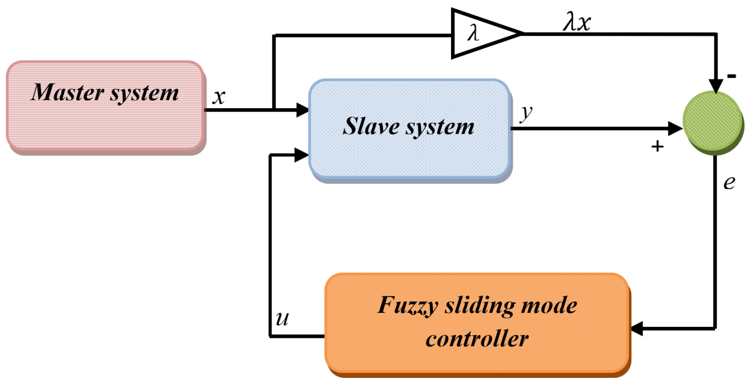

Figure 1: Block-diagram of our synchronization scheme

Remark 1. *In the sequel, we will use the state equations (16) to analyze the system stability and design two FSMC schemes. Via these dynamics, we can transform a projective synchronization problem to a stabilization one. So, our objective becomes the design of two FSMCs steering the synchronization errors to fastly converge in FT to zero.*

Remark 2. *It's crucial to emphasize that equations (14) and (15) can serve as models for various chaotic systems with fractional orders, including the Duffing oscillator, Chua's chaotic circuit, gyro system, Genesio-Tesi system, laser Lorenz system, and numerous others. Assumption 1 can be seen as a relatively mild condition, as the constants $\bar{d}_{di}$ and $\bar{d}_i$ are unknown. Furthermore, such a hypothesis is prevalent within the field of fractional-order control research.*

Remark 3. *In the sequel, unlike the close related work [41], one will design two FSMCs to achieve a FFTPS, where the MSS model is assumed to be unknown. Additionally, they ensure the attainment of both the reaching phase and the sliding-mode phase within a FT. It's crucial to emphasize that the work in [41]* *does not assure the fats achievement of FT convergence for synchronization errors.*

Remark 4. *Control and synchronization in chaotic systems are vital for enhancing stability, predictability, and efficiency in various applications, from engineering to biology to secure communications, and for advancing our understanding of complex systems.*

a) Why does one need control of chaotic systems?
Despite being deterministic, their complex dynamics make them difficult to predict and control. Control of chaotic systems is necessary for several reasons:

- *Stabilization:* In numerous practical applications, chaotic behavior is often unwanted. For example, in mechanical systems, chaos can cause unpredictable movements and increased wear. By controlling chaos, the system can be stabilized into a preferred state, such as a fixed point or a periodic orbit.
- *Synchronization:* In certain applications, it is crucial to synchronize the behavior of multiple chaotic systems. This is especially important in secure communications, where chaotic signals are employed to obscure information.
- *Improving system performance:* In certain situations, managing chaos can boost a system's performance. For instance, in power systems, controlling chaotic behavior can avert voltage collapse and enhance overall stability.
- *Application in engineering and technology:* Chaos control is applicable in diverse areas, including robotics, telecommunications, and biological systems.

b) What are the control requirements for chaotic systems?

- *Observability:* If these states cannot be accurately measured or estimated, designing a control system becomes difficult.
- *Controllability:* The chaotic system must be controllable, meaning it should be possible to influence its behavior through inputs or feedback mechanisms.
- *Model accurateness:* A precise mathematical model of the chaotic system is often required for designing an effective control law. Even slight inaccuracies in the model can lead to significant control errors.
- *Robustness:* The control law must be robust to uncertainties as well as external disturbances. Since chaotic systems are extremely sensitive to minor changes, the control strategy needs therefore to successfully manage these uncertainties.

- *Real-time processing*: In many cases, control must be implemented in real-time, requiring quick processing and response from the control algorithm. Therefore, the control law does not have to be very complicated.

c) What are sufficient conditions for chaos synchronization?
Chaos synchronization happens when two or more chaotic systems, whether they have similar or different models, eventually align their states despite starting from different initial conditions. Sufficient conditions for achieving chaos synchronization are:

- *Lyapunov stability*: A negative largest conditional Lyapunov exponent is needed. If the largest conditional Lyapunov exponent of the synchronization error dynamics is negative, the error will decrease to zero over time, resulting in synchronization.
- *Strength of coupling*: Adequate coupling strength between the chaotic systems is crucial. If the coupling is too weak, synchronization may not occur. The coupling function can be either linear or nonlinear, depending on the system and the wanted type of synchronization.
- *Adaptive control system*: Sometimes, an adaptive control system that dynamically adjusts the coupling parameters can ease synchronization, even under varying system conditions.

d) Synchronization of chaotic systems can fail under certain conditions, what are they?
Synchronization of chaotic systems might not succeed under the following conditions:

- *Weak coupling strength:* If the coupling strength between the chaotic systems is inadequate, synchronization will not occur, and the states of the systems will continue to diverge.
- *Positive Lyapunov exponent:* If the largest conditional Lyapunov exponent of the error dynamics is positive, the synchronization error will increase, obstructing synchronization.
- *Mismatched system parameters:* Significant differences in the parameters of the systems to be synchronized can hinder synchronization.
- *External disturbances*: High levels of noise or external disturbances can disrupt the synchronization process, leading to failure.
- *Time-delay*: Synchronization can become more challenging or even impossible if there is a time-delay in the coupling, depending on the length of the delay and the system dynamics.

3. Control design approach 1

Design a non-singular dynamic sliding-mode surface as:

$$S_i = D_t^{q_i-1}e_i + \lambda_{1i}\int_0^t \left|D_t^{q_i-1}e_i(\tau)\right|^r \operatorname{sign}\left(D_t^{q_i-1}e_i(\tau)\right)\mathrm{d}\tau + \lambda_{2i}\int_0^t D_t^{q_i-1}e_i(\tau)\,\mathrm{d}\tau \tag{17}$$

with $0 < r < 1$. λ_{1i}, and λ_{2i} are positive constants.

If $S_i = \dot{S}_i = 0$, one has the following dynamics:

$$D_t^{q_i-1}e_i + \lambda_{1i}\int_0^t \left|D_t^{q_i-1}e_i(\tau)\right|^r \operatorname{sign}\left(D_t^{q_i-1}e_i(\tau)\right)\mathrm{d}\tau + \lambda_{2i}\int_0^t D_t^{q_i-1}e_i(\tau)\,\mathrm{d}\tau = 0 \tag{18}$$

and

$$D_t^{q_i}e_i = -\lambda_{1i}\left|D_t^{q_i-1}e_i\right|^r \operatorname{sign}\left(D_t^{q_i-1}e_i\right) - \lambda_{2i}D_t^{q_i-1}e_i\,. \tag{19}$$

Theorem 1. *If $S_i = \dot{S}_i = 0$, then the synchronization errors e_i fastly vanish to zero in a FT.*

Proof. Consider $V_1 = \frac{1}{2}\sum_{i=1}^{n}\left(D_t^{q_i-1}e_i\right)^2$. Its time-derivative can be bounded by:

$$\begin{aligned}\dot{V}_1 &= \sum_{i=1}^{n}\left(D_t^{q_i-1}e_i\right)D_t^{q_i}e_i = -\sum_{i=1}^{n}\lambda_{2i}\left(D_t^{q_i-1}e_i\right)^2 - \sum_{i=1}^{n}\lambda_{1i}\left|D_t^{q_i-1}e_i\right|^{r+1}\\ &\leqslant -a_1\sum_{i=1}^{n}\left(D_t^{q_i-1}e_i\right)^2 - b_1\sum_{i=1}^{n}\left|D_t^{q_i-1}e_i\right|^{r+1},\end{aligned} \tag{20}$$

where $a_1 = \min\{\lambda_{2i}\}$, $b_1 = \min\{\lambda_{1i}\}$.

Using Lemma 1, (20) becomes

$$\dot{V}_1 \leqslant -2a_1\sum_{i=1}^{n}\frac{1}{2}\left(D_t^{q_i-1}e_i\right)^2 - 2^{\frac{1+r}{2}}b_1\left(\sum_{i=1}^{n}\frac{1}{2}\left|D_t^{q_i-1}e_i\right|^2\right)^{\frac{1+r}{2}} = -\alpha_1V_1 - \beta_1V_1^{\frac{1+r}{2}} \tag{21}$$

with $\alpha_1 = 2a_1$, $\beta_1 = 2^{\frac{1+r}{2}}b_1$.

Using Lemma 2, we can determine the settling time as

$$T_{s_1} = \frac{2}{\alpha_1(1-r)} \ln\left(\frac{\alpha_1 V_2^{\frac{1+r}{2}}(0) + \beta_1}{\beta_1}\right), \tag{22}$$

where $t_0 = 0$.

Based on the analysis provided, it can be deduced that the error variable $D_t^{q_i-1} e_i$ achieves a FT convergence to zero. It's evident from (19) that the error variable $D_t^{q_i} e_i$ achieves also a FT convergence. By using Property 2 (i.e. $|e_i| \leqslant l\left|D_t^{q_i-1} e_i\right|$), one can deduce the fast FT convergence of e_i. So, the proof of Theorem 1 ends here. □

From (16) and (17), one has

$$\dot{S}_i = F_i(y, x) + u_i + d_i(t, y) + \lambda_{1i}\left|D_t^{q_i-1} e_i\right|^r \operatorname{sign}\left(D_t^{q_i-1} e_i\right) + \lambda_{2i} D_t^{q_i-1} e_i\,, \tag{23}$$

where

$$F_i(y, x) = g_i(y) - \lambda_i f_i(x) \tag{24}$$

is an uncertain nonlinear function that can be approximated using the fuzzy system (11).

Assumption 1. *$|d_i(t,y)| \leqslant \bar{d}_i$, and $\left|D_t^{q_1} d_i(t,y)\right| \leqslant \bar{d}_{di}$, where $\bar{d}_i$ and $\bar{d}_{di}$ are unknown positive constants.*

According to [57], there is an optimal fuzzy system such that

$$F_i(y, x) = \theta_i^{*T} \psi_i(y,x) + \varepsilon_i(y,x) \tag{25}$$

with $\varepsilon_i(y,x)$ being the unavoidable margin of error in the fuzzy system approximation, and $|\varepsilon_i(y,x)| \leqslant \bar{\varepsilon}_i$ where $\bar{\varepsilon}_i$ is a positive constant.

Let write (25) as follows

$$F_i(y, x) = \theta_i^{*T} \psi_i(y) + \varepsilon_i(y,x) + \theta_i^{*T}\left[\psi_i(y,x) - \psi_i(y)\right]. \tag{26}$$

Substituting (26) into (23) yields

$$\dot{S}_i = \theta_i^{*T} \psi_i(y) + w_i(y,x,t) + u_i + \lambda_{1i}\left|D_t^{q_i-1} e_i\right|^r \operatorname{sign}\left(D_t^{q_i-1} e_i\right) + \lambda_{2i} D_t^{q_i-1} e_i\,, \tag{27}$$

where $w_i(y,x,t) = \varepsilon_i(y,x) + \theta_i^{*T}\left[\psi_i(y,x) - \psi_i(y)\right] + d_i(t,y)$. Evidently, $w_i(y,x,t)$ can be bounded as: $|w_i(y,x)| \leqslant \bar{w}_i$, where $\bar{w}_i$ is a positive constant.

Taking Young's inequality into account leads to

$$\begin{aligned} S_i \theta_i^{*T} \psi_i(y) + S_i w_i(y,x,t) &\leqslant k_{2i}|S_i|\,\|\psi_i(y)\|^2 + \frac{1}{4k_{2i}}|S_i|\,\left\|\theta_i^*\right\|^2 + \bar{w}_i|S_i| \\ &\leqslant k_{2i}|S_i|\,\|\psi_i(y)\|^2 + k_{1i}|S_i| \end{aligned} \tag{28}$$

with k_{2i} and k_{1i} being positive constants, where $k_{1i} > \frac{1}{4k_{2i}} \left\|\theta_i^*\right\|^2 + \bar{w}_i$.

Design our first FSMC law as follows:

$$\begin{aligned} u_i = &-\left(\rho_{1i} + k_{1i} + k_{2i} \left\|\psi_i(y)\right\|^2\right) \operatorname{sign}\left(S_i\right) - k_{3i} S_i \\ &- \lambda_{1i} \left|D_t^{q_i-1} e_i\right|^r \operatorname{sign}\left(D_t^{q_i-1} e_i\right) - \lambda_{2i} D_t^{q_i-1} e_i , \end{aligned} \tag{29}$$

where $k_{3i}, \rho_{1i} > 0$ are design constants.

Theorem 2. *Consider the MSS described by (14) and (15) with its control law (29) and assume that Assumption 1 holds. Then, this MSS achieves a FFTPS.*

Proof. Consider a quadratic Lyapunov function as $V_2 = \frac{1}{2} \sum_{i=1}^{n} S_i^2$. The time-derivative of V_2 along the solutions of (27) is

$$\begin{aligned} \dot{V}_2 = &\sum_{i=1}^{n} \Big(S_i \theta_i^{*T} \psi_i(y) + S_i w_i(y, x, t) + S_i u_i \\ &+ \lambda_{1i} S_i \left|D_t^{q_i-1} e_i\right|^r \operatorname{sign}\left(D_t^{q_i-1} e_i\right) + \lambda_{2i} S_i D_t^{q_i-1} e_i\Big) \\ \leqslant &\sum_{i=1}^{n} \Big(k_{2i} \left|S_i\right| \left\|\psi_i(y)\right\|^2 + k_{1i} \left|S_i\right| + S_i u_i \\ &+ \lambda_{1i} S_i \left|D_t^{q_i-1} e_i\right|^r \operatorname{sign}\left(D_t^{q_i-1} e_i\right) + \lambda_{2i} S_i D_t^{q_i-1} e_i\Big). \end{aligned} \tag{30}$$

Substituting (29) into (30) then using Lemma 1 results in

$$\dot{V}_2 \leqslant -\sum_{i=1}^{n} \rho_{1i} \left|S_i\right| - \sum_{i=1}^{n} k_{3i} S_i^2 \leqslant -\sqrt{2} \rho_1 V_2^{\frac{1}{2}} - 2k_3 V_2 = -\alpha_2 V_2 - \beta_2 V_2^{1/2} , \tag{31}$$

where $\rho_1 = \min\left\{\rho_{1i}\right\}$, $k_3 = \min\left\{k_{3i}\right\}$, $\alpha_2 = 2k_3$, $\beta_2 = \sqrt{2} \rho_1$.

According to Lemma 2, it can be derived that S_i converges rapidly to zero in a FT and the settling time is

$$T_{s_2} = \frac{2}{\alpha_2} \ln\left(\frac{\alpha_2 V_2^{\frac{1}{2}}(0) + \beta_2}{\beta_2}\right). \tag{32}$$

Based on the analysis provided above, it can be inferred that the MSS under-consideration achieves a projective synchronization within a FT, with a total settling time given by:

$$T = T_{s_1} + T_{s_2} = \frac{2}{\alpha_1(1-r)} \ln\left(\frac{\alpha_1 V_2^{\frac{1+r}{2}}(0) + \beta_1}{\beta_1}\right) + \frac{2}{\alpha_2} \ln\left(\frac{\alpha_2 V_2^{\frac{1}{2}}(0) + \beta_2}{\beta_2}\right).$$

So, the proof of Theorem 2 ends here. □

4. Control design approach 2

Let construct a non-singular dynamic sliding-mode surface as:

$$S_i = e_i + \lambda_{1i} \int_0^t |e_i(\tau)|^r \operatorname{sign}(e_i(\tau))\, d\tau + \lambda_{2i} \int_0^t e_i(\tau) d\tau, \tag{33}$$

where $0 < r < 1$, λ_{1i}, and λ_{2i} are positive constants.

During sliding mode behavior, one obtains

$$S_i = e_i + \lambda_{1i} \int_0^t |e_i(\tau)|^r \operatorname{sign}(e_i(\tau))\, d\tau + \lambda_{2i} \int_0^t e_i(\tau) d\tau = 0 \tag{34}$$

and

$$\dot{S}_i = \dot{e}_i + \lambda_{1i} |e_i|^r \operatorname{sign}(e_i) + \lambda_{2i} e_i = 0. \tag{35}$$

From (35), one obtains

$$\dot{e}_i = -\lambda_{1i} |e_i|^r \operatorname{sign}(e_i) - \lambda_{2i} e_i\,. \tag{36}$$

Theorem 3. *When $S_i = \dot{S}_i = 0$, the projective synchronization errors rapidly converge to zero in a FT.*

Proof. The obvious candidate is as usual the square shape $V_3 = \frac{1}{2}\sum_{i=1}^{n} e_i^2$. The time-derivative of V_3 is

$$\dot{V}_3 = -\sum_{i=1}^{n} \lambda_{1i} |e_i|^{r+1} - \sum_{i=1}^{n} \lambda_{2i} e_i^2 \leqslant -b_2 \sum_{i=1}^{n} |e_i|^{r+1} - a_2 \sum_{i=1}^{n} e_i^2\,, \tag{37}$$

where $a_2 = \min\{\lambda_{2i}\}$, $b_2 = \min\{\lambda_{1i}\}$.

Using Lemma 1 leads to

$$\dot{V}_3 \leqslant -\alpha_3 V_3 - \beta_3 V_3^{\frac{1+r}{2}} \tag{38}$$

with $\alpha_3 = 2a_2$, $\beta_3 = 2^{\frac{1+r}{2}} b_2$.

According to Lemma 2, the projective synchronization errors e_i can fastly converge to the origin in FT, with a settling time that can be estimated by:

$$T_{s_3} = \frac{2}{\alpha_3(1-r)} \ln\left(\frac{\alpha_3 V_3^{\frac{1+r}{2}}(0) + \beta_3}{\beta_3}\right). \tag{39}$$

This concludes the proof for Theorem 3. □

From (33), one has

$$D_t^{q_i} S_i = D_t^{q_i} e_i + \lambda_{1i} D_t^{q_i-1}\left(|e_i|^r \operatorname{sign}(e_i)\right) + \lambda_{2i} D_t^{q_i-1} e_i\,. \tag{40}$$

Using (16) and (40) gives

$$\begin{aligned}\dot{S}_i &= D_t^{1-q_i}(D_t^{q_i} S_i) = D_t^{1-q_i}\left(D_t^{q_i} e_i + \lambda_{1i} D_t^{q_i-1}\left(|e_i|^r \operatorname{sign}(e_i)\right) + \lambda_{2i} D_t^{q_i-1} e_i\right)\\ &= D_t^{1-q_i}\left(F_i(y,x) - k_{4i}S_i - k_{5i}\tanh\left(\frac{S_i}{\varepsilon_{ui}}\right)\right) + D_t^{1-q_i}\left(k_{4i}S_i + k_{5i}\tanh\left(\frac{S_i}{\varepsilon_{ui}}\right)\right)\\ &\quad + D_t^{1-q_i}u_i + D_t^{1-q_i}d_i(t,y) + \lambda_{1i}|e_i|^r \operatorname{sign}(e_i) + \lambda_{2i}e_i\,,\end{aligned} \tag{41}$$

where $k_{4i} > 0$ and $k_{5i} > 0$ are design constants and $\varepsilon_{ui} > 0$ is small design constant. tanh stands for the hyperbolic tangent function.

Assumption 2. *Suppose that*

a) $\left|D_t^{1-q_i} d_i(t,y)\right| \leqslant d_{di}^*$,

b) $\left|D_t^{1-q_i}\left(F_i(y,x) - k_{4i}S_i - k_{5i}\tanh\left(\frac{S_i}{\varepsilon_{ui}}\right)\right)\right| \leqslant \bar{F}_i(y,x)$,

where $d_{di}^* > 0$ *is an unknown constant and* $\bar{F}_i(y,x) > 0$ *an unknown function.*

Remark 5. *Assumption 2 can be seen a relatively mild condition, because* d_{di}^* *and* $\bar{F}_i(y,x)$ *are already unknown. Furthermore, such an assumption is frequently made in control literature [27].*

In accordance with reference [57], there is always a fuzzy system capable of optimally estimating the nonlinear function $\bar{F}_i(y,x)$:

$$\bar{F}_i(y,x) = \theta_i^{*T}\psi_i(y,x) + \varepsilon_i(y,x) \tag{42}$$

with $|\varepsilon_i(y,x)| \leqslant \bar{\varepsilon}_i$, where $\bar{\varepsilon}_i > 0$ is a constant. θ_i^* is an unknown constant parameters' vector.

One can rewrite (42) as follows

$$\bar{F}_i(y,x) = \theta_i^{*T}\psi_i(y) + \varepsilon_i(y,x) + \theta_i^{*T}\left[\psi_i(y,x) - \psi_i(y)\right]. \tag{43}$$

Using Assumption 2, (41) and (43) leads to

$$
\begin{aligned}
S_i\dot{S}_i &\leqslant |S_i|\,\bar{F}_i(y,x) + S_i D_t^{1-q_i} u_i + |S_i|\, d_{di}^* + S_i D_t^{1-q_i}\left(k_{4i}S_i + k_{5i}\tanh\left(\frac{S_i}{\varepsilon_{ui}}\right)\right) \\
&\quad + S_i\big(\lambda_{1i}|e_i|^r \operatorname{sign}(e_i) + \lambda_{2i}e_i\big) \\
&\leqslant |S_i|\,\theta_i^{*T}\psi_i(y) + |S_i|\, w_i(y,x,t) + S_i D_t^{1-q_i} u_i + S_i D_t^{1-q_i}\left(k_{4i}S_i + k_{5i}\tanh\left(\frac{S_i}{\varepsilon_{ui}}\right)\right) \\
&\quad + S_i\left(\lambda_{1i}|e_i|^r \operatorname{sign}(e_i) + \lambda_{2i}e_i\right). \qquad (44)
\end{aligned}
$$

where $w_i(y,x,t) = |\varepsilon_i(y,x)| + \left|\theta_i^{*T}\left[\psi_i(y,x) - \psi_i(y)\right]\right| + d_{di}^*$. It is straightforward to prove that $w_i(y,x,t)$ is upper bounded such that: $|w_i(y,x)| \leqslant \bar{w}_i$, with $\bar{w}_i$ being a positive constant.

Using Young's inequality leads to this outcome:

$$
\begin{aligned}
|S_i|\,\theta_i^{*T}\psi_i(y) + |S_i|\, w_i(y,x,t) &\leqslant k_{2i}|S_i|\,\|\psi_i(y)\|^2 + \frac{1}{4k_{2i}}|S_i|\,\left\|\theta_i^*\right\|^2 + \bar{w}_i|S_i| \\
&\leqslant k_{2i}|S_i|\,\|\psi_i(y)\|^2 + k_{1i}|S_i| \qquad (45)
\end{aligned}
$$

with k_{2i} and k_{1i} being positive constants, where $k_{1i} > \dfrac{1}{4k_{2i}}\left\|\theta_i^*\right\|^2 + \bar{w}_i$.

Design our second FSMC law as:

$$
\begin{aligned}
u_i &= -D_t^{q_i-1}\left(\left(\rho_{1i} + k_{1i} + k_{2i}\|\psi_i(y)\|^2 + k_{3i}|S_i|\right)\operatorname{sign}(S_i)\right) - k_{4i}S_i \\
&\quad - k_{5i}\tanh\left(\frac{S_i}{\varepsilon_{ui}}\right) - \lambda_{1i}D_t^{q_i-1}\left(|e_i|^r\operatorname{sign}(e_i)\right) - \lambda_{2i}D_t^{q_i-1}e_i \qquad (46)
\end{aligned}
$$

with ρ_{1i} and k_{3i} are positive constants.

Substituting (45) and (46) into (44) yields

$$
S_i\dot{S}_i \leqslant -\rho_{1i}|S_i| - k_{3i}S_i^2\,. \qquad (47)
$$

Theorem 4. *Consider the MSS described by (14) and (15) with its control law (46) and assume that Assumption 2 holds. Therefore this MSS achieves a FFTPS.*

Proof. Consider $V_4 = \dfrac{1}{2}\sum_{i=1}^{n} S_i^2$, where its derivative versus time is bounded as:

$$
\dot{V}_4 \leqslant -\sum_{i=1}^{n}\rho_{1i}|S_i| - \sum_{i=1}^{n}k_{3i}S_i^2 \leqslant -\sqrt{2}\rho_1 V_4^{1/2} - 2k_3V_4 = -\alpha_4V_4 - \beta_4V_4^{1/2}\,, \quad (48)
$$

where $\rho_1 = \min\{\rho_{1i}\}$ and $k_3 = \min\{k_{3i}\}$, $\alpha_4 = 2k_3$, $\beta_4 = \sqrt{2}\rho_1$.

Similarly, by Lemma 2, we can infer that S_i quickly approaches zero within a FT. The settling time is therefore:

$$T_{s_4} = \frac{2}{\alpha_4} \ln\left(\frac{\alpha_4 V_4^{\frac{1}{2}}(0) + \beta_4}{\beta_4}\right). \tag{49}$$

Based on the analysis provided above, it can be inferred that the proposed controller solves the FFTPS problem of our MSS, where the total convergence time is:

$$T = T_{s_3} + T_{s_4} = \frac{2}{\alpha_3(1-r)} \ln\left(\frac{\alpha_3 V_3^{\frac{1+r}{2}}(0) + \beta_3}{\beta_3}\right) + \frac{2}{\alpha_4} \ln\left(\frac{\alpha_4 V_4^{\frac{1}{2}}(0) + \beta_4}{\beta_4}\right).$$

Thus, this concludes the proof. □

5. Comparison between the control schemes

Table 1 shows the comparison of our control schemes with other neighboring control ones proposed in the literature. Table 2 summarizes the key differences between our FSMC schemes proposed in this paper.

Table 1: Comparison with other neighboring control schemes proposed in the literature

	Is the model of the MSS unknown?	Are the master and slave models identical?	Is the stability analysis rigorous?	Does the chattering phenomenon exist in the control signal?	Does the singularity problem exist?	Does the proposed control law guarantee the FT convergence in both sliding-mode and reaching phase?	Does the MSS possess incommensurate fractional-order?
Our scheme 1	YES	NO	YES	YES	NO	YES	YES
Our scheme 2	YES	NO	YES	NO	NO	YES	YES
[29]	NO	YES	NO	YES	YES	NO	NO
[30]	NO	YES	YES	YES	YES	YES	NO
[31]	NO	YES	YES	YES	NO	NO	YES
[32]	NO	NO	YES	NO	NO	NO	YES
[33]	NO	NO	YES	YES	NO	NO	NO
[34]	NO	NO	YES	YES	NO	YES	NO
[35]	NO	NO	NO	YES	NO	NO	YES
[36]	NO	YES	YES	YES	NO	NO	NO

Table 1 [cont.]

	Is the model of the MSS unknown?	Are the master and slave models identical?	Is the stability analysis rigorous?	Does the chattering phenomenon exist in the control signal?	Does the singularity problem exist?	Does the proposed control law guarantee the FT convergence in both sliding-mode and reaching phase?	Does the MSS possess incommensurate fractional-order?
[37]	NO	NO	NO	YES	NO	YES	NO
[38]	NO	NO	YES	YES	NO	YES	NO
[39]	NO	NO	NO	NO	NO	YES	NO
[40]	NO	NO	NO	YES	NO	YES	NO
[41]	NO	NO	NO	YES	NO	NO	NO
[42]	NO	YES	NO	YES	NO	YES	NO
[43]	NO	NO	YES	YES	NO	NO	NO
[44]	NO	NO	NO	NO	YES	YES	NO
[45]	YES	NO	NO	YES	NO	NO	NO
[46]	NO	NO	YES	YES	NO	NO	NO
[47]	NO	NO	NO	YES	YES	NO	YES
[48]	NO	YES	NO	NO	NO	NO	NO
[49]	NO	NO	NO	YES	NO	NO	NO
[50]	NO	NO	YES	YES	NO	YES	NO

Table 2: Key differences between the first control scheme and the second one

	First control scheme	Second control scheme
Type of sliding surface	Dynamic **fractional-order** surface	Dynamic **integer-order** surface
Presence of chattering in the control signal	Due to the presence of the sign function, the control law (29) experiences a chattering issue.	The control law (46) addresses the chatter problem thanks to the inclusion of a fractional order integral.
Number of design parameters	7 parameters	10 parameters.
Assumptions used in the control design	$a)$ $\lvert d_i(t,y)\rvert \leqslant \bar{d}_i$, and $\left\lvert D_t^{q_1} d_i(t,y)\right\rvert \leqslant \bar{d}_{di}$,	$a)$ $\left\lvert D_t^{1-q_i} d_i(t,y)\right\rvert \leqslant d^*_{di}$, $b)$ $\left\lvert D_t^{1-q_i}\left(F_i(y,x) - k_{4i}S_i - k_{5i}\tanh\left(\frac{S_i}{\varepsilon_{ui}}\right)\right)\right\rvert \leqslant \bar{F}_i(y,x)$

6. Simulation results

To confirm the effectiveness of the established control strategies, we present two simulation examples below.

Example 1. Consider the fractional-order Lü system [58]:

Master system:

$$\begin{aligned} D_t^{q_1} x_1 &= a_l(x_2 - x_1), \\ D_t^{q_2} x_2 &= c_l x_2 - x_1 x_3\,, \\ D_t^{q_3} x_3 &= x_1 x_2 - b_l x_3],. \end{aligned} \tag{50}$$

Slave system:

$$\begin{aligned} D_t^{q_1} y_1 &= a_l(y_2 - y_1) + u_1 + d_1(t, y), \\ D_t^{q_2} y_2 &= c_l y_2 - y_1 y_3 + u_2 + d_2(t, y), \\ D_t^{q_3} y_3 &= y_1 y_2 - b_l y_3 + u_3 + d_3(t, y), \end{aligned} \tag{51}$$

where $a_l = 35$, $b_l = 3$ and $c_l = 28$, and $(q_1, q_2, q_3) = (0.95,\ 0.98,\ 0.99)$. The initial states are selected at random, as: $x(0) = [-0.5, 1.2, 2]^T$ and $y(0) = [1, 2, -3]$. The disturbances are taken as: $d_1(t, y) = d_2(t, y) = d_3(t, y) = 0.15\sin(3t) - 0.2\cos(5t)$. Three non-adaptive fuzzy systems (i.e. $\theta_i^T \psi_i(Y)$, with $i = 1, 2, 3$) are constructed and they have as input the vector $y = [y_1,\ y_2,\ y_3]^T$. As described in reference [59], three membership functions (comprising two trapezoidal and one triangular) are generated, with uniform distribution across the intervals $[-50\ 50]$ for each input variable in these fuzzy systems.

For the first control scheme, the values of the design parameters have been taken as follows:
$k_{11} = k_{12} = k_{13} = 4.5$, $k_{21} = k_{22} = k_{23} = 10$, $k_{31} = k_{32} = k_{33} = 0.1$, $r = 0.4$, $\lambda_{11} = \lambda_{12} = \lambda_{13} = 5$, $\lambda_{21} = \lambda_{22} = \lambda_{23} = 1$, and $\rho_{11} = \rho_{12} = \rho_{13} = 0.5$.

For our second control scheme, the values of the design parameters were set as follows:
$k_{11} = k_{12} = k_{13} = 0.5$, $k_{21} = k_{22} = k_{23} = 6$, $k_{31} = k_{32} = k_{33} = 2.7$, $k_{41} = k_{42} = k_{43} = 1.5$, $k_{51} = k_{52} = k_{53} = 1$, $r = 0.4$, $\lambda_{11} = \lambda_{12} = \lambda_{13} = 0.01$, $\lambda_{21} = \lambda_{22} = \lambda_{23} = 0.01$, and $\rho_{11} = \rho_{12} = \rho_{13} = 0.5$.

We examine two projective synchronization cases.

Case 1: Complete synchronization (i.e. for $\lambda_i = 1$)

The simulation results are depicted in Fig. 2 and Fig. 3. Figures 2a–2c and 3a–3c show the states of the MSS. These figures demonstrate the successful attainment of an accurate and fast complete synchronization. Figure 2d and 3d provide the curves of the control signals generated by both controllers, respectively.

Obviously, these control signals are bounded and within the admissible range. Furthermore, the second controller has substantially diminished the chattering effect.

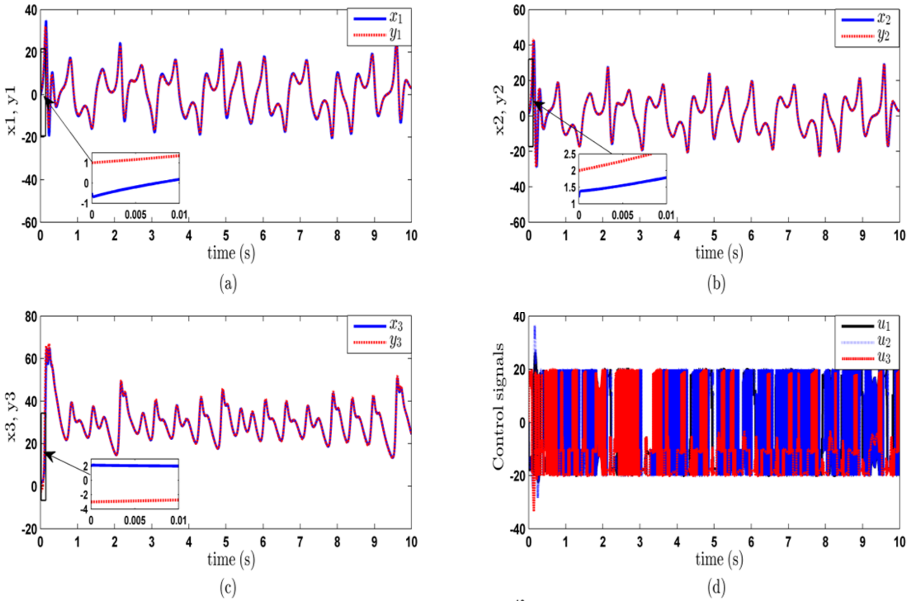

Figure 2: Synchronization results and the control signal generated by controller 1 (Example 1)

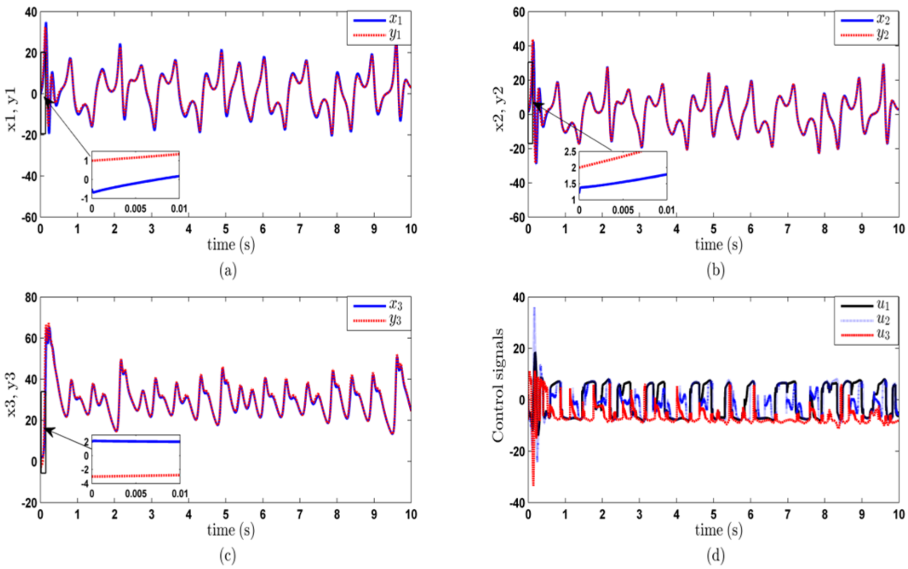

Figure 3: Synchronization results and the control signal generated by controller 2 (Example 1)

Case 2: Anti-phase synchronization (i.e. for $\lambda_i = -1$)

The anti-phase synchronization simulation results are plotted in Fig. 4 and Fig. 5. Clearly evident in these graphs is the rapid anti-phase synchronization of the slave states with those of the master, even when uncertain nonlinear dynamics and disturbances are present.

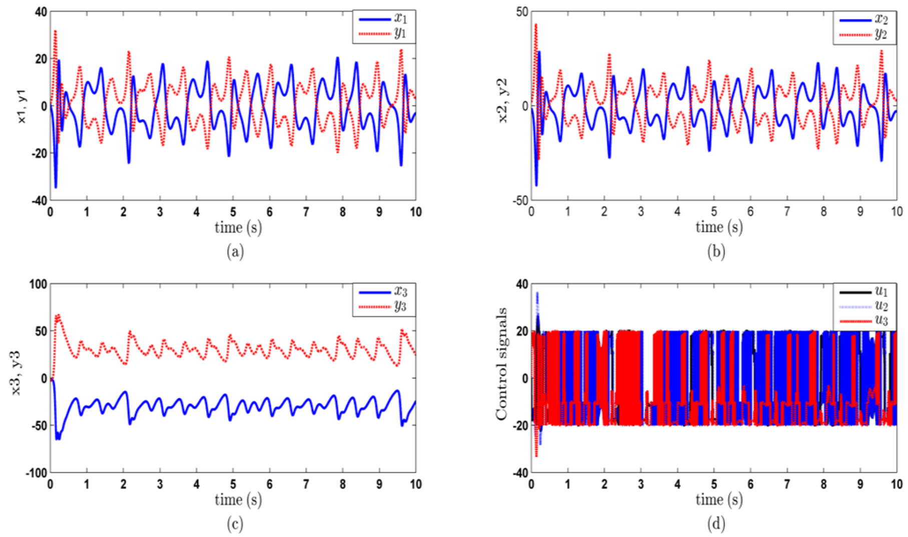

Figure 4: Anti-phase synchronization results and the control signal generated by controller 1 (Example 1)

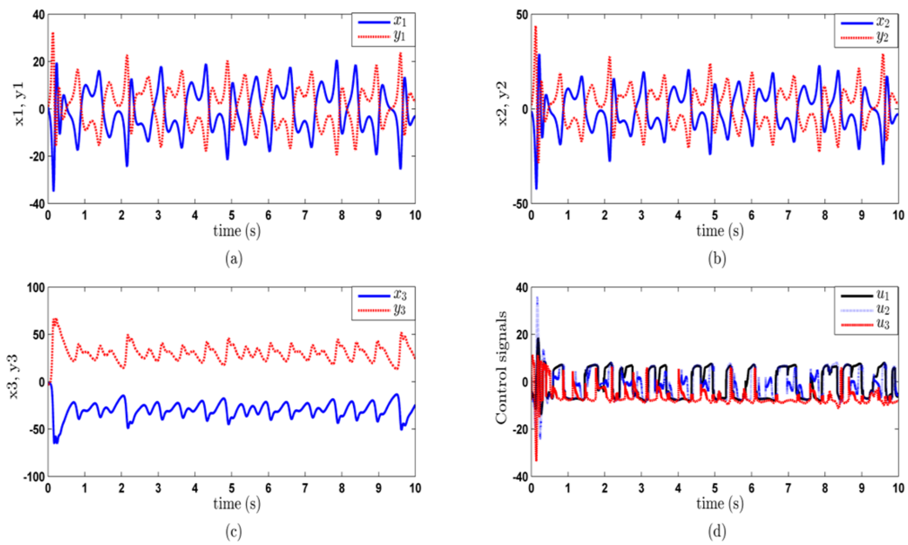

Figure 5: Anti-phase synchronization results and the control signal generated by controller 2 (Example 1)

Example 2. Consider the chaotic laser Lorenz system [60].

Master system:

$$\begin{aligned} D_t^{q_1} x_1 &= \delta_c (x_2 - x_1), \\ D_t^{q_2} x_2 &= a_c x_1 - x_1 x_3 - x_2 , \\ D_t^{q_3} x_3 &= x_1 x_2 - r_c x_3 , \end{aligned} \tag{52}$$

where $\delta_c = 10$, $a_c = 28$, and $r_c = 8/3$.

Slave system:

$$\begin{aligned} D_t^{q_1} y_1 &= \delta_c (y_2 - y_1) + u_1 + d_1(t, y), \\ D_t^{q_2} y_2 &= a_c y_1 - y_1 y_3 - y_2 + u_2 + d_2(t, y), \\ D_t^{q_3} y_3 &= y_1 y_2 - r_c y_3 + u_3 + d_3(t, y), \end{aligned} \tag{53}$$

where $q_1 = 0.98$, $q_2 = 0.99$, $q_3 = 1$, and $d_i(t, y) = 0.2 \sin(3t) + 0.2 \cos(3t)$. The initial state conditions are arbitrarily selected as: $x(0) = [-2,\ -3.2,\ 10.9]$ and $y(0) = [-2.9,\ -3.9,\ -11.5]^T$.

Three non-adaptive fuzzy systems (i.e. $\theta_i^T \psi_i(Y)$, with $i = 1, 2, 3$) are constructed and they have as input the vector $y = [y_1,\ y_2,\ y_3]^T$. As described in reference [59], three membership functions (comprising two trapezoidal and one triangular) are generated, with uniform distribution across the intervals $[-50\ 50]$ for each input variable in these fuzzy systems.

The design parameters for **controller 1** are selected as follows: $k_{11} = k_{12} = k_{13} = 0.4$, $k_{21} = k_{22} = k_{23} = 5$, $k_{31} = k_{32} = k_{33} = 0.1$, $r = 0.7$, $\lambda_{11} = \lambda_{12} = \lambda_{13} = 0.1$, $\lambda_{21} = \lambda_{22} = \lambda_{23} = 0.1$, and $\rho_{11} = \rho_{12} = \rho_{13} = 0.1$.

Those of the **second controller** are taken as: $k_{11} = k_{12} = k_{13} = 0.2$, $k_{21} = k_{22} = k_{23} = 1$, $k_{31} = k_{32} = k_{33} = 4$, $k_{41} = k_{42} = k_{43} = 0.2$, $k_{51} = k_{52} = k_{53} = 2$, $r = 0.7$, $\lambda_{11} = \lambda_{12} = \lambda_{13} = 0.1$, $\lambda_{21} = \lambda_{22} = \lambda_{23} = 0.1$, and $\rho_{11} = \rho_{12} = \rho_{13} = 0.1$.

Similarly, two cases are considered here.

Case 1: Complete synchronization (i.e. when $\lambda_i = 1$)

The simulation results are presented in Fig. 6 and Fig. 7. Figures 6a–6c and Figs. 7a–7c display the curves of MSS state variables. These figures demonstrate the successful attainment of an accurate and fast complete synchronization. Figure 6d and 7d exhibit the curves of the control signals produced by our two controllers, respectively. Evidently, these control signals are bounded and within the admissible range. Additionally, the signals produced by the second controller are devoid of chattering.

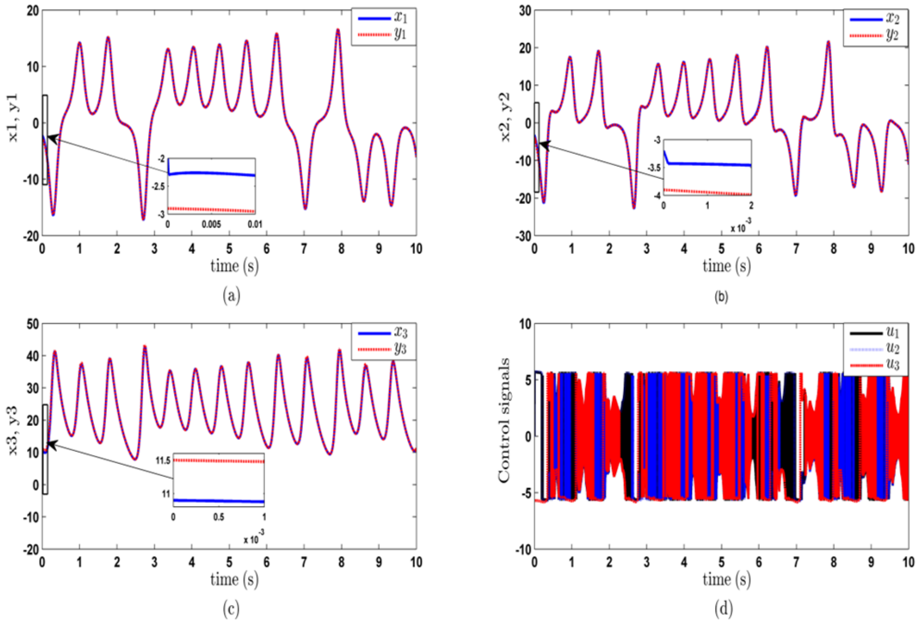

Figure 6: Synchronization results and the control signal generated by controller 1 (Example 2)

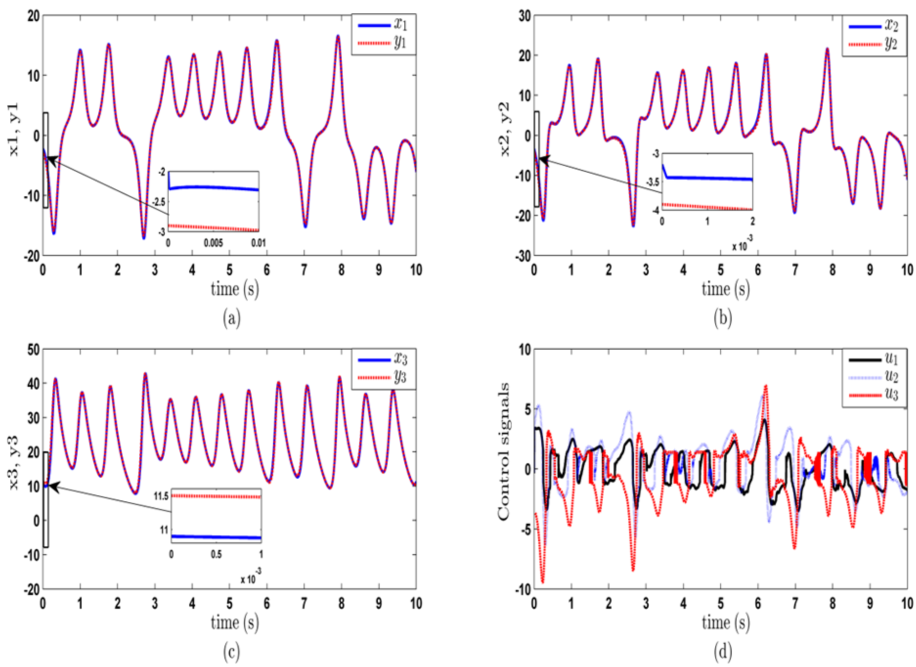

Figure 7: Synchronization results and the control signal generated by controller 2 (Example 2)

Case 2: Anti-phase synchronization (i.e. when $\lambda_i = -1$)

The anti-phase synchronization simulation results are plotted in Fig. 8 and Fig. 9. It can be obviously observed from these images that the slave state variables

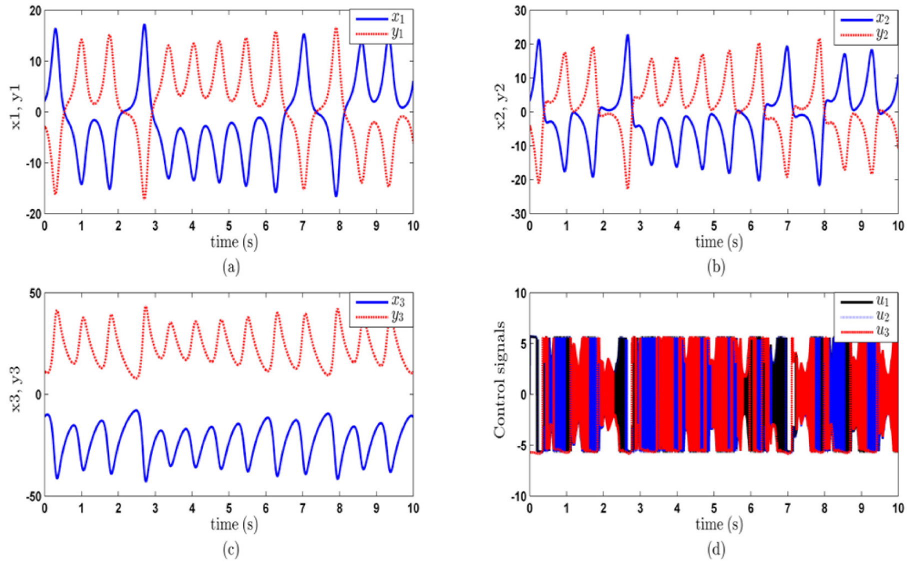

Figure 8: Anti-phase synchronization results and the control signal generated by controller 1 (Example 2)

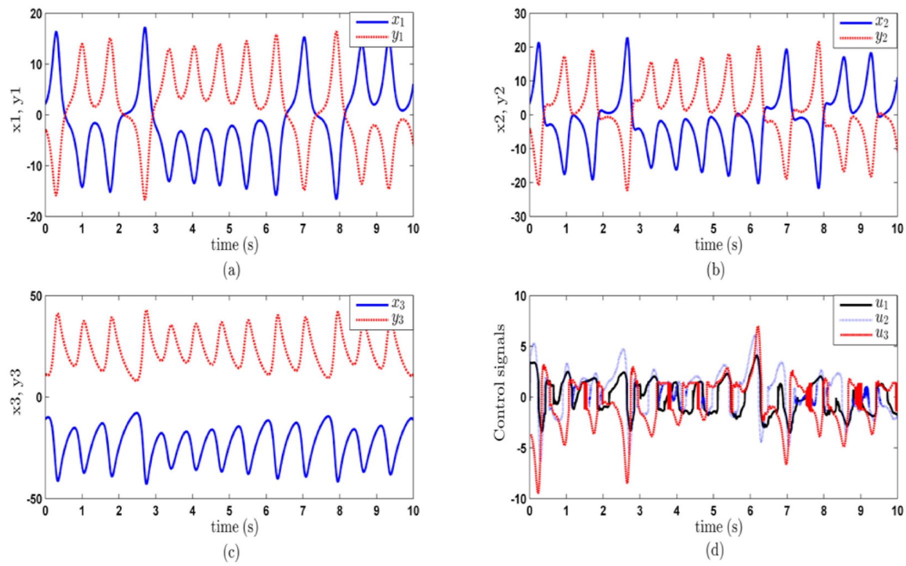

Figure 9: Anti-phase synchronization results and the control signal generated by controller 2 (Example 2)

are successfully antiphase-synchronized with those of the master one. Furthermore, the second controller has notably minimized the chattering effect in the obtained results.

7. Conclusions

In this paper, by designing two FSMC schemes, the problem of FFTPS of uncertain fractional-order chaotic systems has been solved. The considered class of chaotic systems has been supposed to be with non-commensurate fractional-orders, unknown dynamics and external disturbances. Of primary importance, a thorough and rigorous analysis grounded in Lyapunov theory has been conducted to establish both the criteria for finite-time stability and the convergence proof for FFTPS errors. A set of simulation results have been presented to highlight the effectiveness of the proposed control schemes.

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