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A New Robust Confidence Interval for the Population Mean μ based on Winsorized Modified One Step M-Estimator and Winsorized Standard Deviation

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Abstract
In this paper, we propose a new confidence interval (CI) for the population mean μ based on robust estimators, which involves the application of the winsorized modified one-step M-estimator (WMOM) and winsorized standard deviation (WSD). The proposed method is modified for the Student's t confidence interval CI under the non-normal distribution. The performances of the proposed confidence interval were evaluated via a Monte-Carlo simulation study by considering the coverage ratio (CR) and the average length (AL) as performance criteria. The simulation study results show the superior performance of the proposed confidence interval (CI) over the classical parametric Student's t for data from a non-normal distribution. Two real data sets were analyzed, and the results agree to some extent with those of the simulation study. The results confirm the suitability of the proposed CI estimator for both normally and non-normally distributed data.

Keywords
Confidence interval, Robust estimator, Coverage ratio, Average length, Winsorized sample, Winsorized modified one step M-estimator, Winsorized standard deviation.

Introduction

The confidence interval (CI) is a crucial statistical method used to estimate population location and dispersion parameters, aiming to capture the true parameter value across repeated samples. CI defines a range of values that indicates the precision of parameter estimates. On the other hand, data handling is a very sensitive subject. While most data handling techniques produce optimized results, they typically only work with data that is normally distributed. Through the growth and development of technology, enormous amounts of data have been gathered (Brooks, 1985). However, normally distributed data is becoming increasingly rare today, and many statisticians have tried to develop models to describe their data and produce the desired results (Johnson, 1978; Sinsomboonthong et al., 2020).

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Data are often positively skewed in the biological sciences (Ghosh and Polansky, 2016), psychology (McDonald, 2014), environmental sciences Mudelsee and Alkio, 2007), and health sciences (Cain, et al., 2016). CI is an interval estimator that calculates the true parameter value from repeated samples. It provides a range of values indicating the accuracy of estimations for a specific parameter (Abu-Shawiesh and Saghir, 2019). The normal theory is typically used to form CI for (the mean (μ)) when concluding a specific population. However, when samples are drawn from non-normal populations, it is improper to build a CI using the normal theory.

The central limit theorem asserts that for large sample sizes (n), the distribution of sample means (XŻ) tends to become normal, even for non-normal or unknown distributions, as noted by Pek et al. (2017). As a result, a population mean (μ) should have CI that is unrestricted by the population's normality assumption (Miller and Penfield, 2005). There are thus many comparable mechanisms in the methodologies that can be used to obtain a satisfying coverage ratio or coverage probability (CR) and short interval average length (AL) in the presence of a skewed distribution and a small sample size, with transformation and bootstrap being two of the recommended approaches (Ghosh and Polansky, 2016).

As the literature shows, CI can usually be achieved for (μ). In practice, it is usually possible to work with smaller sample sizes, and in this case, Student's t CI can be used as a reference instead of the standard CI, provided that the data are normal. Furthermore, after extreme skewness or kurtosis in the data, the shape of the distribution is changed. It was done to verify the efficiency of the proposed CI in the presence of robust estimators that are free from distributional assumptions. According to Luh, and Guo (2001), the violation of the normality assumption can be quite common in applied research, especially with small sample sizes. In this regard, a robust and efficient alternative method would be useful to solve the problem. Robust estimators are therefore necessary because they are less affected by small or outlier deviations from the assumptions of the classical model Sindhumol et al., (2016). Johnson (1978) modified Student's t-CI for population mean μ using a new procedure to reduce the effects of a skewed distribution. Since then, many researchers have determined the Student's t confidence interval for μ in skewed distributions (see, for example, Sinsomboonthong et al., 2020; Akyuz and Abu-Shawiesh, 2020; Abu-Shawiesh and Saghir, 2019; Ialongo, 2019; Abu-Shawiesh, 2018; Pek et al., 2017; Visalakshi and Jeyaseelan, 2013; Abu-Shawiesh, 2022).

The modified one-step M-estimator (MOM), which is a highly robust estimator, is a superior substitute for the usage of the sample mean. Depending on the type of data distribution, the MOM estimator only trims the extreme data set. The amount of trimming at the two tails of a distribution with skewed data should not be equal. For instance, more of the distribution's right tail would be trimmed when the distribution was skewed to the right (Ochuko, 2015; Rousseeuw, and Croux, 1993). Accordingly, in this work, the CI was proposed using the robust location estimator (WMOM) as a substitute for the sample mean ($\overline{X}$) and the robust scale estimator winsorized standard deviation (WSD) as a substitute for sensitive sample standard deviation (S). Two types of data were used, those under normal distribution but with different values for the population standard deviation $\sigma = 5, 10, 15$ and those under the non-normal distributions generated from g–h distributions in different shapes ($g = 0$, $h = 0$ (normal), $g = 0$, $h = 0.5$ (symmetry with heavy tail), $g = 0.5$, $h = 0$ (skewed with normal tail), and $g = 0.5$, $h = 0.5$ (skewed with strong tail).

As a result, the study assesses the differences between the results obtained from the proposed confidence interval (CI) and the traditional/classical CI. The proposed CI demonstrated superior performance for non-normally distributed data, whereas the traditional CI performed well with normal data but inadequately with non-normal data.

The rest of this study is organized as follows: In Section 2, the winsorized modified one-step M-estimator (WMOM) is explained. The statistical methodology of deriving the proposed confidence interval for the population mean μ is given in Section 3. That section also includes a review of the classical Student's t-confidence interval. In Section 4, simulations are used to evaluate the performance of the proposed confidence interval in terms of coverage ratio (CR) and average length (AL). In Section 5, the results of the simulation study were discussed. In Section 6, we illustrate the use of the proposed confidence interval using two real-life datasets. Also to verify the strong performance of the new proposed CI, a comparison between the performance of several proposed methods in literature and the new proposed method WMOMSD$_{\text{WMOM}}$-t was performed. Section 7 limitation of the study. Finally, Section 8 reports our conclusions.

The Winsorized Modified One Step M-estimator (WMOM)

The winsorized modified one-step M-estimator is used to construct robust confidence intervals when computational efficiency is important, especially for large datasets. It provides a flexible approach for dealing with data that does not follow a normal distribution and is very affected by outliers. By choosing the percentile at which we want to winsorize our data, we can control the level of robustness. This allows us to strike a balance between robustness and efficiency, depending on the specific characteristics of the data. From the distribution iid random sample $x_1, \ldots, x_n$ of size n is taken. Here are the specifics (Haddad, 2018). Meanwhile, the MOM estimator is written as follows:

$$\hat{\theta} = \sum_{i=i_1+1}^{n-i_2} \frac{X_{(i)}}{n - i_1 - i_2} \tag{1}$$

where $X_{(i)}$: i-th order statistics of the random sample. i_1: The number of $x_i s$ which satisfies the criterion.

$$\left(x_i - \hat{M}\right) < -2.24 \cdot Q_n \left(x_i - \hat{M}\right) < -2.24 \cdot Q_n \tag{2}$$

i_2: The number of $x_i s$ which satisfies the criterion

$$\left(x_i - \hat{M}\right) > 2.24 \cdot Q_n \tag{3}$$

n: Sample size. $\hat{M} = \text{med}\{X_1, \ldots, X_n\}$ and

$$Q_n = 2.2219 \cdot \{|x_i, x_j|; i < j\}_{(k)} \tag{4}$$

where $k = \binom{h}{2}$; $h = \left[\frac{n}{2}\right] + 1$.

The estimator Q_n is unbiased for σ in a normal distribution. Its influence function is smooth and efficient with a Gaussian distribution, making it suitable for both 82 percent breakdown point and asymmetric distributions. It is commonly used for its breakdown point and bounded influence function as referenced in (Rousseeuw, and Croux, 1993; Croux and Rousseeuw, 1992). It is considered as the most appropriate estimator for the study context (Huber,1981), accordingly Q_n is the most suitable assistant robust scale estimator with high breakdown properties.

Following the stipulations in equations (2) and (3), outliers found in samples are all removed, and then, data winsorization is performed. Hence, for each iid random sample $x_1, \ldots, x_n$, the winsorized sample is as expressed below:

$$\mathrm{W}_i = \left\{ \begin{array}{ll} x_{(i_1+1)}, & \text{if} \quad x_i \le x_{(i_1+1)} \\ x_i, & \text{if} \quad x_{(i_1+1)} \le x_i \le x_{(n-i_2)} \\ x_{(n-i_2)}, & \text{if} \quad x_i \ge x_{(n-i_2)} \end{array} \right\} \tag{5}$$

where:
i_1: The number of the smallest outliers in the data which satisfies the trimming criteria in equation (2).
i_2: The number of the largest outliers in the data which satisfies the trimming criteria in equation (3).

According to the above provisions, the winsorized standard deviation of W observations of the random variable and the winsorized MOM for the individual observations of the random variable are shown as follows:

$$\overline{X}_{\mathrm{WMOM}} = \frac{\sum_{i=1}^{n} \mathrm{W}_i}{n} \tag{6}$$

$$\mathrm{SD}_{\mathrm{WMOM}} = \sqrt{\frac{1}{n-1}\left[\sum_{i=1}^{n} \mathrm{W}_i^2 - n \cdot \overline{\mathrm{W}}^2\right]} \tag{7}$$

The standard error:

$$\mathrm{SE}_{\mathrm{WMOM}} = \frac{\mathrm{SD}_{\mathrm{WMOM}}}{\sqrt{n}} \tag{8}$$

where: n: The sample size after removing the outliers $(i_1 + i_2)$ from the two sides of the data the smallest i_1 and the largest i_2 and replacing them all with the following observations, the observation $x_{(i_1+1)}$ from the left side and $x_{(n-i_2)}$ from the right side.

A winsorized MOM is used in place of the sample mean. Syed-Yahaya, (2006) suggests that using different trimming criteria for MOM leads to the development of accurate scale estimators that can reduce the influence of contaminated observations. A typical robust scale estimate for the trimming criterion is the median absolute deviation $\mathrm{MAD}_n = 1.4826 med\{|X_i - med_j X_j|\}$, which is utilized in MOM. The bounded effect function and 50% breakdown threshold show that MAD has a simple explicit formula, is highly fast to calculate and is exceedingly robust. In this paper, we propose to develop explicit and 82% breakdown scale estimators that are more efficient. The distances 0.25 quantile served as the basis for the estimate $Q_n = 2.2219 \cdot \{|x_i, x_j|; i < j\}$ (Rousseeuw, and Croux, 1993; Croux and Rousseeuw, 1992).

Methodology and Notations

The methodology explains the strategies employed as well as the proposed robust techniques for calculating the confidence interval (μ) for normal and non-normal distributions. Accordingly, *iid* random sample $x_1, x_2, \ldots, x_n$ are of size n from a population with mean (μ) and standard deviation (σ). In this study, we attempt to calculate the interval estimate for (μ) with a given level of confidence. Consequently, several techniques have been presented in the literature that can be used to estimate CI for μ. Available techniques include the modified classical confidence interval t-Student and the nonparametric approach. The $(1-\alpha)100\%$ confidence intervals for the population mean μ are as follows:

The Classical Student's-t Confidence Interval

The classical method for finding the $(1-\alpha)100\%$ CI for the population mean μ is the most popular method among scholars because of its ease of use and simplicity. Suppose that $x_1, x_2, \ldots, x_n$ are random sample of size n from a normal distribution with mean μ and variance σ^2. The formation of the $(1-\alpha)100\%$ CI for (μ) is provided by Student's t-distribution CI (Student, 1908), when a small size n, $(n < 30)$ and (σ) the population standard deviation is unknown, this formula is set as follows:

$$\overline{X} \pm t_{(1-\frac{\alpha}{2}, n-1)} \frac{S}{\sqrt{n}} \tag{9}$$

From the above formula: $\overline{X}$ and S represent the sample mean and sample standard deviation, respectively, and $t_{(1-\frac{\alpha}{2}, n-1)}$ represents the critical value of the Student's t-distribution. As a perfect assumption, Student's t-distribution CI presupposes normality. As a result, when dealing with non-normal distributions, the Student's t CI may underperform, and its use may be inappropriate. Boos, and Hughes-Oliver, (2000) have pointed out the incompatibility and fragility of Students' t-CI when dealing with non-normal data.

The Proposed Confidence Interval for the Population Mean

Construction of the proposed $(1-\alpha)100\%$ CI for the population mean μ involves the following steps:

Step 1: Compute $\overline{X}_{\text{WMOM}}$ based on equation (6).

Step 2: Compute SD_{WMOM} based on equation (7).

Step 3: Compute SE_{WMOM} based on equation (8).

Step 4: Construction of the modified $(1-\alpha)\,100\%$ CI for (μ)WMOMSD$_{\text{WMOM}}t$-distribution using the Winsorized modified one-step M-estimator $(\overline{X}_{\text{WMOM}})$ with winsorized standard deviation SD_{WMOM} as follows:

$$\text{CI} = \overline{X}_{\text{wmom}} \pm t_{(1-\frac{\alpha}{2},n-1)} \frac{\text{SD}_{\text{WMOM}}}{\sqrt{n}} \tag{10}$$

where $t_{(1-\frac{\alpha}{2},n-1)}$ represents the tabulated value of the Student's t-distribution.

The robust location MOM and the robust scale estimator Q_n possess advantageous statistical properties that affect WMOM and WSD estimators. These properties include reduced bias, increased efficiency, robustness, resistance to outliers, and consistency in non-normal data. These attributes position them as valuable alternatives to conventional estimators, especially when working with real-world datasets that show non-normality or include outliers.

Simulation Study

Considering the impossibility of a theoretical comparison between the traditional interval and the new proposed CI, a simulation study was performed in this study. MATLAB 2015 program was used to perform the simulation and generate the results.

The Design of the Simulation Study

For an iid random sample of size n, say $x_1, x_2, \ldots, x_n$, which comes from a non-normal distribution, the Student's-statistic distribution is not Student's t-distribution. In particular, the skewness of non-normally distributed data greatly affects the validity of Student's t-distribution, as can be seen in (Yanagihara and Yuan, 2005). The effect of skewness should be eliminated, and this can be achieved by modifying the t-statistic. Accordingly, several methods have been proposed to form the $(1-\alpha)100\%$ CI for μ to eliminate the skewness effect.

In this study, a novel $(1-\alpha)100\%$ robust WMOMSD$_{\text{WMOM}}$-t CI for μ was proposed, and under different conditions, the proposed method was investigated and compared in terms of its (CR) and (AL). The confidence interval was also examined for its strengths and weaknesses using different sample sizes ($n = 10, 20, 30, 40, 50, 100,$ and 150) with individual observations. Both the traditional and proposed CI were evaluated for their performance. For this purpose, a simulation procedure was developed. This was to test the effect of using robust scale and location estimators in approximating μ, and the procedure is as follows:

First Part Steps:

1. Generate 10,000 samples from the normal distribution with $\mu = 10$ and different values of the standard deviation such as $\sigma = 5, 10, 15$.
2. Calculate the $(\overline{X}_{\text{WMOM}})$ and the traditional sample mean $(\overline{X})$ for the 10,000 samples.
3. Calculate the $(\text{SD}_{\text{WMOM}})$ and $(\text{SE}_{\text{WMOM}})$ for the 10,000 samples.
4. Create the two CIs, namely traditional CI, and proposed CI.
5. Calculate the (CR) and the (AL) for the 10,000 CIs.

Second Part Steps:

1. Generate (generate) 10,000 data samples from the standard normal distribution Z.
2. Transform the standard normal data into random variables using the following equation (Mills, (1995), Badrinath and Chatterjee, (1988), Badrinath and Chatterjee, (1991); Hoaglin, (1985)):

$$X = \begin{cases} \dfrac{\exp(gZ)-1}{g}\exp(hZ^2/2), & g \neq 0 \\ Z\exp(hZ^2/2), & g = 0 \end{cases} \tag{11}$$

   Based on the above equation, the amount of skewness is regulated by the proportion of parameter g, while the amount of kurtosis is regulated by the proportion of parameter h. The combination of parameters used in the treatment of distributions of different shapes includes $g = 0$ and $h = 0$ (normal), $g = 0$ and $h = 0.5$ (symmetry with heavy tail), $g = 0.5$ and $h = 0$ (skewed with normal tail), and $g = 0.5$ and $h = 0.5$ (skewed with strong tail).
3. Calculate the values of $(\overline{X}_{\text{WMOM}})$ and $(\overline{X})$ for 10,000 samples.
4. Generate new 10,000 data samples with different distribution shapes from the g–h distribution.
5. Generate the two CIs consisting of the traditional and proposed CIs for the 10,000 data samples.
6. To obtain the estimated WMOM and $(\overline{X})$ values, calculate the (CR) and the (AL) for step (5).

Performance Evaluation

In this section, the performance of the two CIs (the traditional and the proposed) for g–h distributions is compared by using a Monte Carlo simulation study. A parametric confidence interval was used and compared with the proposed robust method. The comparison was used to determine the suitability of the proposed method for estimating the population mean (μ) for data from a non-normal distribution. Two CIs were compared in terms of their (CR) and (AL). Specifically, with unchanged (CR) length, a smaller average length (AL) means better CI. With an unchanged average length, a higher (CR) means a better CI. Different sample sizes ($n = 10, 20, 30, 40, 50, 100$, and 150) were randomly generated 10,000 times, and 95% CIs for μ were obtained for each sample set. Moreover, two formulas were applied to determine the (CR) and the (AL) of CI. The formulas are as follows:

$$\mathrm{CR} = \frac{\#(\mathrm{Lower} \leq \theta \leq \mathrm{Uper})}{10,000} \tag{12}$$

$$\mathrm{AL} = \frac{\sum_{i=1}^{10,000} (U_i - L_i)}{10,000} \tag{13}$$

Results and Discussion

The results of the simulations for all cases studied are shown in Tables 1 to 11. For both methods, the performance of 95% CI for μ is as follows: Tables 1–3 illustrate that the traditional confidence interval CI outperforms the suggested $\mathrm{WMOMSD_{WMOM}}$-t CI in terms of (CR). The values in the traditional CI tables are closer to 95% when compared to the proposed $\mathrm{WMOMSD_{WMOM}}$-t CI. Nevertheless, proposed $\mathrm{WMOMSD_{WMOM}}$-t CI exhibits shorter (AL) than the traditional CI, indicating a more precise estimation of the population mean μ. A notable trend observed in the tables is that larger sample sizes necessitate shorter interval lengths for enhanced accuracy in parameter estimation. Additionally, broader (AL) is required as the standard deviation σ increases. The narrower width of proposed $\mathrm{WMOMSD_{WMOM}}$-t CI leads to greater accuracy for larger standard deviations.

To illustrate this superiority of the confidence interval of the new proposal in terms of (AL), graphs (Fig. 1 to Fig. 3) have been drawn representing all previous cases, all showing a clear superiority in terms of performance.

For non-normally distributed data are obtained from the g–h distribution with different distribution shapes. Accordingly, the traditional and proposed CI $\mathrm{WMOMSD_{WMOM}}$-t were compared for the non-normally distributed data, and the details can be seen in Tables 4 to 7 as follows:

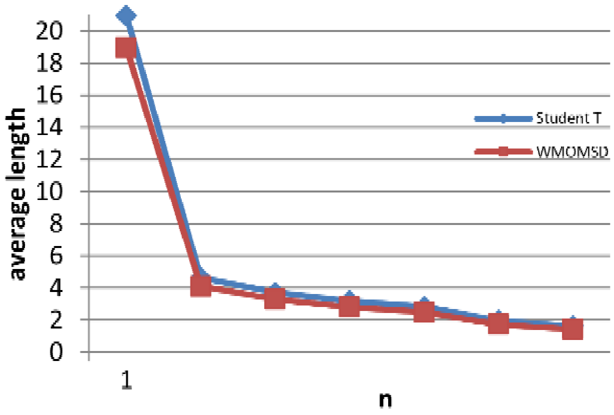

Fig. 1. (AL) of 95% CI for μ for the normal distribution with $\mu = 10$ and $\sigma = 5$

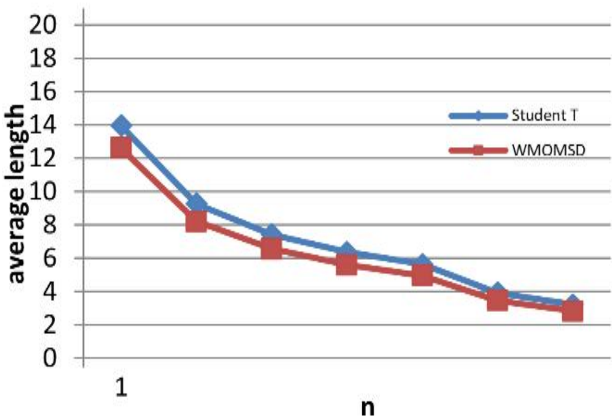

Fig. 2. (AL) of 95% CI for μ for the normal distribution with $\mu = 10$ and $\sigma = 10$

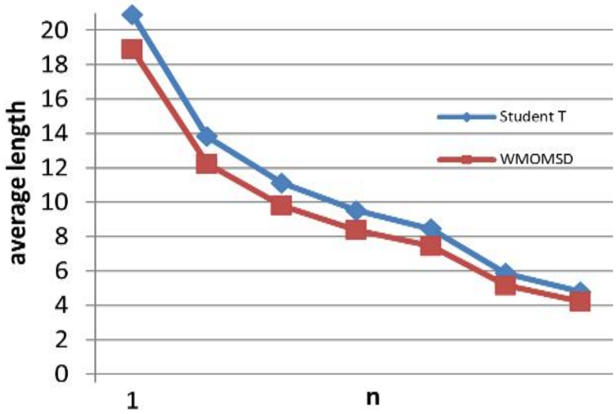

Fig. 3. (AL) of 95% CI for μ for the normal distribution with $\mu = 10$ and $\sigma = 15$

Table 1
(CR) and (AL) of 95% CI for μ for the normal distribution with $\mu = 10$ and $\sigma = 5$.

CI Methods	Performance Measure	N						
		10	20	30	40	50	100	150
Student-t	CR	0.9532	0.9502	0.9533	0.9499	0.9470	0.9425	0.9442
	AL	20.956	4.6220	3.7059	3.1742	2.8113	1.9544	1.5972
$\text{WMOMSD}_{\text{WMOM}}$-t	CR	0.9007	0.9012	0.9045	0.9053	0.9050	0.8976	0.9008
	AL	18.9360	4.0900	3.267	2.7946	2.4752	1.7178	1.4051

Table 2
(CR) and (AL) of 95% CI for μ for the normal distribution with $\mu = 10$ and $\sigma = 10$.

CI Methods	Performance Measure	N						
		10	20	30	40	50	100	150
Student-t	CR	0.9477	0.9527	0.9481	0.9491	0.9473	0.9478	0.9484
	AL	13.9423	9.2618	7.4091	6.3489	5.6272	3.9111	3.1936
$\text{WMOMSD}_{\text{WMOM}}$-t	CR	0.8958	0.9054	0.9046	0.9041	0.9008	0.9023	0.9021
	AL	12.593	8.1978	6.5369	5.5836	4.9454	3.4405	2.8093

Table 3
(CR) and (AL) of 95% CI for μ for the normal distribution with $\mu = 10$ and $\sigma = 15$.

CI Methods	Performance Measure	n						
		10	20	30	40	50	100	150
Student-t	CR	0.9454	0.9466	0.9481	0.9508	0.9490	0.9480	0.9466
	AL	20.9039	13.820	11.109	9.5052	8.4461	5.8580	4.7879
$\text{WMOMSD}_{\text{WMOM}}$-t	CR	0.8902	0,8969	0.897	0.9082	0.9050	0.9029	0.8993
	AL	18.884	12.2253	9.7977	8.3781	7.4314	5.1505	4.2116

Table 4
(CR) and (AL) of 95% CI for the population mean of g–h distribution with $g = 0$, $h = 0$

CI Methods	Performance Measure	n						
		10	20	30	40	50	100	150
Student-t	CR	0.9488	0.9493	0.9503	0.9514	0.9520	0.9504	0.9482
	AL	1.3908	0.9231	0.7402	0.6351	0.5628	0.3908	0.3195
$\text{WMOMSD}_{\text{WMOM}}$-t	CR	0.8998	0.9022	0.9056	0.9059	0.9049	0.9063	0.9056
	AL	1.2577	0.8181	0.6528	0.5593	0.4953	0.3440	0.2812

Table 5
(CR) and (AL) of 95% CI for population mean of g–h distribution with $g = 0.5$, $h = 0$

CI Methods	Performance Measure	n						
		10	20	30	40	50	100	150
Student-t	CR	0.9213	0.9262	0.9354	0.9398	0.9394	0.9442	0.9434
	AL	1.6109	1.0864	0.8787	0.7569	0.6722	0.4692	0.3845
$\text{WMOMSD}_{\text{WMOM}}$-t	CR	0.8481	0.8517	0.8555	0.8559	0.8536	0.8523	0.8506
	AL	1.209	0.7773	0.6203	0.5309	0.4700	0.3262	0.2667

Table 6

(CR) and (AL) of 95% CI for population mean of g–h distribution with $g = 0$, $h = 0.5$

CI Methods	Performance Measure	n						
		10	20	30	40	50	100	150
Student-t	CR	0.9710	0.9671	0.9665	0.9656	0.9657	0.9622	0.9615
	AL	3.7726	2.7948	2.3820	2.1216	1.9410	1.4435	1.2431
$\text{WMOMSD}_{\text{WMOM}}$-t	CR	0.902	0.8933	0.8956	0.89838	0.8942	0.8943	0.8968
	AL	1.6958	1.0661	0.8478	0.7259	0.6434	0.4482	0.3669

Table 7

(CR) and (AL) of 95% CI for population mean of g–h distribution with $g = 0.5$, $h = 0.5$

CI Methods	Performance Measure	n						
		10	20	30	40	50	100	150
Student-t	CR	0.7185	0.7259	0.7489	0.7454	0.7548	0.7737	0.7903
	AL	4.8407	3.8535	3.4756	3.1035	2.8736	2.2282	1.9611
$\text{WMOMSD}_{\text{WMOM}}$-t	CR	0.8717	0.8670	0.8686	0.8704	0.8687	0.8729	0.8740
	AL	1.6703	1.0443	0.8291	0.7103	0.6291	0.4383	0.3590

Table 8

(CR) of 95% CI for the population mean of g–h distribution with $g = 0$, $h = 0$

CI Methods	Performance Measure	n						
		10	20	30	40	50	100	150
Student-t	CR	0.9488	0.9493	0.9503	0.9514	0.9520	0.9504	0.9482
$\text{WMOMSD}_{\text{WMOM}}$-t	CR	0.9487	0.9492	0.9503	0,9514	0.9520	0.9504	0.9482

Table 9

(CR) of 95% CI for the population mean of g–h distribution with $g = 0.5$, $h = 0$

CI Methods	Performance Measure	n						
		10	20	30	40	50	100	150
Student-t	CR	0.8821	0.7991	0.7125	0.6185	0.5394	0.2380	0.0940
$\text{WMOMSD}_{\text{WMOM}}$-t	CR	0.9321	0.9192	0.9135	0.8959	0.8815	0.7990	0.7331

Table 10

(CR) of 95% CI for the population mean of g–h distribution with $g = 0$, $h = 0.5$

CI Methods	Performance Measure	n						
		10	20	30	40	50	100	150
Student-t	CR	0.9476	0.9474	0.9489	0.9498	0.9529	0.9495	0.9490
$\text{WMOMSD}_{\text{WMOM}}$-t	CR	0.9488	0.9492	0.9506	0.9517	0.9521	0.9503	0.9484

Table 11

(CR) of 95% CI for the population mean of g–h distribution with $g = 0.5$, $h = 0.5$

CI Methods	Performance Measure	n						
		10	20	30	40	50	100	150
Student-t	CR	0.2138	0.0251	0.0034	0.0002	0	0	0
$\text{WMOMSD}_{\text{WMOM}}$-t	CR	0.9318	0.9190	0.9131	0.8953	0.8809	0.7993	0.7335

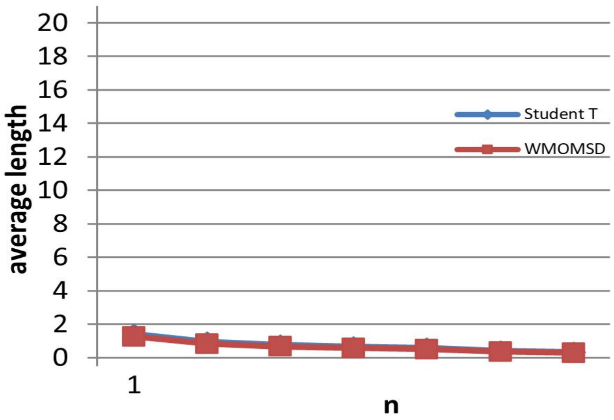

Fig. 4. (AL) of 95% CI for the population mean of g–h distribution with $g = 0$, $h = 0$ No citation in text

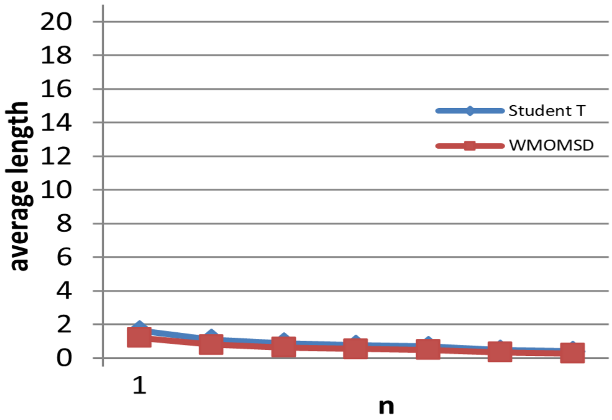

Fig. 5. (AL) of 95% CI for the population mean of g–h distribution with $g = 0.5$, $h = 0$

For $g = 0$ and $h = 0$ in the case of normal distribution, the (CR) of $\text{WMOMSD}_{\text{WMOM}}$-t CI seems to be slightly lower than that of traditional CI. The values (AL) for the proposed CI are smaller than the values (AL) of the traditional CI for all sample sizes. On the other hand, the (CR) of $\text{WMOMSD}_{\text{WMOM}}$-t seems to be slightly lower than 0.95 for a distribution of $g = 0.5$ and $h = 0$. Moreover, the value (CR) seems to remain unchanged at any sample size n. Moreover, the values of proposed CIs (AL) are smaller than those of traditional CIs. This indicates the superiority of proposed CIs in estimating the population mean.

With a distribution of $g = 0$ and $h = 0.5$, the (CR) of $\text{WMOMSD}_{\text{WMOM}}$-t seems to be much smaller than 95%. On the other hand, the (CR) of the traditional CI is significantly close to 95%. Moreover, the proposed CI shows better performance as evidenced by its achieved (AL) value for all sample sizes. Concerning the distributions $g = 0.5$ and $h = 0.5$, the performance of the proposed CI seems to be more robust, as evidenced by the obtained values of (CR) and (AL) for all sizes n. On the other hand, the performance of traditional parametric CI was poor in the case of the distributions $g = 0.5$ and $h = 0.5$.

In addition, the performance of the confidence interval of the new proposal in terms of (AL), graphs have been drawn, in all previous cases, all showing a clear superiority in performance.

The arithmetic mean is calculated in the performance test using robust statistics, and the estimation is done in two ways: with and without the use of robust statistics. A CIs for the population mean were then calculated using normal data. The arithmetic means and standard deviation statistics were then computed. According to the data, both CIs appear to have equivalent values (AL). Surprisingly, (CR) in both CIs appears to differ. This shows that the suggested CI is far stronger than its traditional CI, particularly when dealing with skewed distributions. Tables 8 through 11 contain more information.

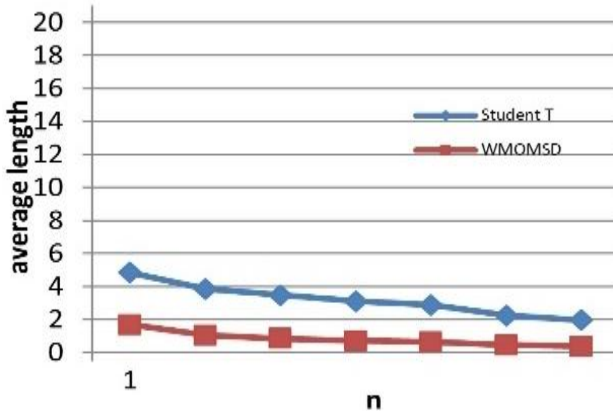

Fig. 6. (AL) of 95% CI for the population mean of g–h distribution with $g = 0$, $h = 0.5$

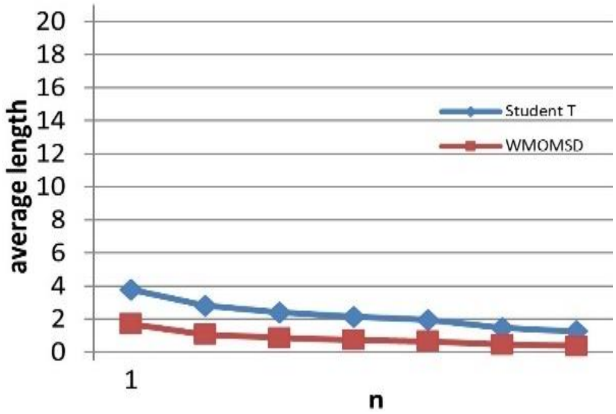

Fig. 7. (AL) of 95% CI for the population mean of g–h distribution with $g = 0.5$, $h = 0.5$

Applications using Real Data

In this section, two real-life data sets will be analyzed to illustrate the application of all considered confidence intervals CIs for the population mean (μ). These two data sets are from normal and skewed distributions.

Melting Points of Beeswax Data

In this study, data from previous studies were used from (Jonathan et al., (1960); Panichkitkosolkul (2015); Sinsomboonthong et al, (2020). Accordingly, Table 12 shows the data on the melting points of beeswax from 59 sources. Following the data analysis in (Sinsomboonthong et al, (2020), it was assumed that the data are normally distributed.

For (μ), 95% CI for the breaking load data was examined, and CI and resulting length can be seen in Table 13. As shown, (μ) of the melting points of beeswax data formed using the traditional and $\text{WMOMSD}_{\text{WMOM}}$-t CI results in slightly different length values for the estimated 95% confidence interval. Therefore, the results of this example of actual normal data shown in Table 13 are consistent with those of the simulation study.

Urinary Tract Infections (UTI) Data

Table 14 shows the data on urinary tract infections in male patients (UTIS) in days. Various researchers (e.g., Santiago and Smith, 2013; Aslam et al., 2014; Azam et al., 2017; Sinsomboonthong et al., 2020) have analyzed these data. In examining the data, (Sinsomboonthong et al. (2020) found that the data were not normally distributed; rather, the data were exponentially distributed, with their recovery positively skewed.

The estimated 95% CIs for the average use of urinary tract infection data (UTI) can be seen in Table 15. The $\text{WMOMSD}_{\text{WMOM}}$-t method was used, which resulted in smaller values of length, as opposed to using traditional parametric CI, which resulted in larger values. The dataset used was positively skewed, and therefore the results shown in Table 15 are consistent with those of the simulation results for the positively skewed distribution.

Evaluation Results

To verify the strong performance of the new proposal CI $\text{WMOMSD}_{\text{WMOM}}$-t, a comparison between the performance of proposals in (Abu-Shawiesh, et al., 2022) and the new proposal $\text{WMOMSD}_{\text{WMOM}}$-t was performed. According to Abu-Shawiesh, et al., (2022),

Table 12
Melting points of beeswax data

No.	X	No.	X	No.	X	No.	X	No.	X	No.	X	No.	X
1	63.39,	10	63.3	19	63.68	28	63.6	37	64.12	46	63.1	55	63.27
2	63.36	11	63.83	20	63.78	29	63.51	38	63.83	47	63.53	56	63.43
3	63.69	12	63.61	21	63.08	30	63.31	39	63.86	48	63.92	57	63.86
4	64.27	13	63.92	22	64.21	31	63.30	40	63.50	49	63.40	58	63.13
5	63.60	14	63.58	23	63.34	32	64.40	41	63.03	50	64.24	59	63.12
6	63.41	15	63.43	24	63.88	33	63.34	42	63.45	51	63.50		
7	63.51	16	63.63	25	63.13	34	63.78	43	63.56	52	63.36		
8	62.85	17	63.93	26	63.84	35	63.92	44	63.27	53	63.05		
9	63.56	18	63.36	27	63.66	36	64.42	45	63.92,	54	63.50		

Table 13
Lower Limit (LL) and Upper Limits (UL) for the 95% CI for the population mean (load at failure data)

Confidence Interval	Estimated (C I) Limits		Length
	LL	UL	
Student-t	63.4925	63.6723	0.1798
$\text{WMOMSD}_{\text{WMOM}}$-t	63.4777	63.6226	0.1449

Table 14
Urinary tract infection (UTI) data

No.	X	No.	X	No.	X	No.	X	No.	X	No.	X
1	0.14583	10	0.03333	19	0.15139	28	0.70833	37	0.15625	46	0.11944
2	0.12014	11	0.32639	20	0.03472	29	0.07431	38	0.03819	47	0.22222
3	0.08681	12	0.18403	21	0.40069	30	0.11458	39	0.01389	48	1.08889
4	0.13889	13	0.08681	22	0.52569	31	0.04514	40	0.24653	49	0.05208
5	0.04861	14	0.64931	23	0.23611	32	0.15278	41	0.57014	50	0.29514
6	0.40347	15	0.12500	24	0.02500	33	0.00347	42	0.12014	51	0.05208
7	0.14931	16	0.53472	25	0.07986	34	0.13542	43	0.29514	52	0.24653
8	0.02778	17	0.02778	26	0.35972	35	0.33681	44	0.46806	53	0.03819
9	0.12639	18	0.25000	27	0.27083	36	0.14931	45	0.01736	54	0.04514

Table 15
Lower and Upper Limits for the 95% CI for the mean (use of psychotropic drugs)

Confidence Interval	Estimated (C I) Limits		Length
	LL	UL	
Student-t	0.1526	0.2668	0.1130
$\text{WMOMSD}_{\text{WMOM}}$-t	0.1396	0.2042	0.0660

several methods have been proposed to construct the $(1-\alpha)100\%$ CI for the population mean (μ) to remove the effect of skewness and non-normal data by modifying the t-statistic. For example, the Johnson t-approach proposes new modified CI based on the estimator of the third central moment of the population (μ_3). The Chen-t approach in its new modified CI for the population mean μ depends on the estimate of the skewness coefficient $\hat{\gamma} = \frac{\hat{\mu}_3}{S^3}$. The Mad-t approach of Shi and Kibria improved traditional CI using the sample mean absolute deviation (MAD). Abu-Shawiesh et al., (2018) proposed modified CI based on the mean absolute deviation from the sample median with their AADM t approach. The confidence interval is based on the resampling approach using the bootstrap method. The last approach is the proposed robust DMSDDM t-approach where the authors use the mean standard deviation of deciles SDDM and standard error of the mean standard deviation of deciles (SE_{DM}). Further explanation can be found in Abu-Shawiesh et al., (2022).

Real data representing the results of the final scores of 40 players in the long jump in meters (International Olympic Committee, 2019) were taken and applied to the proposals in Abu-Shawiesh et al., (2022) and simultaneously applied to the new proposal $\text{WMOMSD}_{\text{WMOM}}$-t, with the results in (Abu-Shawiesh et al., 2022) as shown in Table 16 and the results for the new proposal $\text{WMOMSD}_{\text{WMOM}}$-t, as shown in Table 17. Looking at the results in the two tables, it can be seen that the new proposal $\text{WMOMSD}_{\text{WMOM}}$-t

Table 16
The 95% confidence intervals for the population mean of final scores for long jump distance in meters.

Method	Estimated CI Limits		Width
	Lower Limit	Upper Limit	
Student-t	7.5528	7.7962	0.2434
Johnson-t	7.5512	7.7945	0.2434
Chen-t	7.5668	7.7822	0.2154
Mad-t	7.5793	7.7697	0.1903
AADM-t	7.5587	7.7903	0.2316
DMSD_{DM}-t	7.6242	7.8029	0.1787
Bootstrap-pct	7.5553	7.7798	0.2245

Table 17
The 95% confidence intervals for the population mean of the final scores for long jump distance in meters.

Method	Estimated CI Limits		Width
	Lower Limit	Upper Limit	
WMOMSD$_{\text{WMOM}}$-t	7.6253	7.8022	0.1768

Table 18
The mercury contamination in largemouth bass. No citation in text

1.23	0.98	0.04	1.08	0.59	0.84	0.75	0.19	0.43	0.50	0.27	0.21	0.34	0.68
1.16	1.33	0.34	0.94	1.20	0.25	0.49	0.65	0.18	0.81	0.16	0.50	0.49	
0.05	0.56	0.19	0.044	1.10	0.27	0.34	0.49	0.83	0.04	0.71	0.40	0.27	
0.15	0.73	0.77	0.63	0.41	0.34	0.87	0.17	0.56	0.10	0.28	0.52	0.19	

Table 19
The summary statistics from Excel are displayed below. No citation in text

Descriptive Measurements	n	Mean	Median	Wmom	Stdev	Wstdev	Minimum	Maximum	Q1	Q3
Values	53	0.525	0.49	0.502	0.3486	0.3067	0.04	1.33	0.24	0.78

is superior to all the new proposals in Abu-Shawiesh et al., (2022), with the length of the confidence intervals being less than all the lengths in Table 16. This comparison shows the strength of the new proposal in terms of performance, as well as the effectiveness of the WMOM and WSD estimators used in the new proposal when the distributions are non-normal.

Example Using Real Data

The findings of a study to investigate the mercury poisoning of the large-mouth bass were published in Large, et al, (1993). Fish from 53 Florida lakes were chosen for the sample, and the amount of mercury in the muscle tissue was calculated (ppm). The following are the mercury concentration values:

Figure 8 displays the histogram representing the data on mercury concentration. The graph illustrates a positively skewed distribution, which is not normal. Our objective is to determine an estimated 95% confidence interval (CI) for the population mean μ.

The advantage of Formula (9) is that it does not require the assumption of normality due to the large sample size ($n = 53$). However, according to Table 20, the new proposed CI based on Formula 10 outperforms the recommended confidence interval.

The Assumptions and Limitations

Winsorization could unintentionally alter the actual data distribution, particularly when outliers are not incorrect but instead reflect genuine variation within the population. The decision on what level of winsorization to use (i.e., how much to trim or replace) is subjective and has the potential to affect outcomes. Various

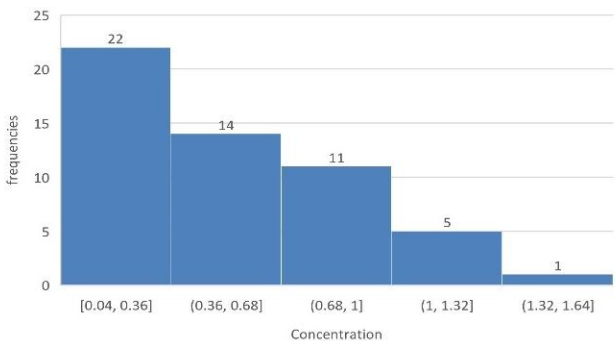

Fig. 8. Mercury concentration in largemouth bass

Table 20
The 95% confidence intervals for the population mean of mercury contamination in largemouth bass.

Method	Estimated CI Limits		Width
	Lower Limit	Upper Limit	
Student-t	0.4311	0.6188	0.1877
WMOMSD$_{\text{WMOM}}$-t	0.4194	0.5845	0.1651

degrees of trimming can result in varying outcomes, causing the results to be dependent on this parameter. Winsorization is based on the assumption of data having a symmetric distribution, which may not be true in all practical situations. One side of the distribution may have a higher concentration of extreme values, which could affect the results by causing skewness.

Using winsorized mean and winsorized standard deviation can change statistical inference, impacting hypothesis testing and confidence intervals. Confidence intervals calculated using winsorized statistics may not accurately represent the actual variability present in the data. Winsorization has the potential to hide the true patterns or trends in the data, which could result in misinterpretation or missing out on key insights in certain situations. The decision to winsorize could impact the reproducibility of outcomes, particularly if the reasoning for winsorization is not clearly explained or comprehended by those trying to duplicate the study (Yuen, 1971; Chambers et al, 2000).

In conclusion, while winsorized mean and winsorized standard deviation, along with their confidence intervals, can be useful tools for handling outliers and non-normal data, researchers should carefully consider the underlying assumptions, limitations, and potential impacts on findings before applying these methods in their analyses. It's essential to assess the robustness of results to different level selections and to interpret findings in light of the inherent uncertainties introduced by winsorization.

Concluding Remarks

This study proposed a CI method, namely $\text{WMOMSD}_{\text{WMOM}}$-t, and this method is essentially a modified Student's t CI that adopts (WMOM) in place of the traditional sample mean and ($SDWMOM$) as a substitute for the sensitive sample standard deviation. In many cases, simulation results prove the superiority of new CI over available estimators for observations from non-normal distributions. In particular, for observations coming from a normal distribution, a CI of $\text{WMOMSD}_{\text{WMOM}}$-t produces a smaller average length. However, for nominal values, the (CR) of the new robust CI is usually larger than that of the proposed CI. This shows the superiority of the new robust CI method over the traditional CI in terms of (CR) and (AL). This is because the (CR) of $\text{WMOMSD}_{\text{WMOM}}$-t CI is usually slightly lower than the nominal level. In addition, the (CR) of the new robust confidence interval is slightly lower compared to that of Student's t. However, the (AL) values of the proposed CI are lower than those of traditional CI. Therefore, the proposed CI has a better performance. Two real data sets were analyzed to illustrate the findings of the study and the simulation results were verified. Furthermore, the proposed confidence interval is easy to compute and can be recommended for practitioners in several fields of industry, engineering, medical, and physical sciences.

Expanding the confidence interval of the proposed $\text{WMOMSD}_{\text{WMOM}}$-t CI using winsorized mean and winsorized standard deviation is a great method to increase the robustness of a statistical estimate. Combining this approach with other statistical measures such as the median and mode can provide a more comprehensive view of the data distribution, especially when the data may not have a normal distribution. Since the mean, median, and mode of data are all equal when it is normal. It is a good idea to assess the newly suggested CI's effectiveness in estimating the populations of median and mode. For example, winsorizing the dataset before calculating the median might modify the winsorization approach for estimating the median. Reducing their impact on the median estimate involves substituting fewer extreme numbers for the extreme ones. Winsorizing may also be used to the mode to reduce the effect of outliers on the mode computation and provide a more representative estimate of central tendency. We will take this suggestion into account to enhance the flexibility and effectiveness of our method in the future work of this research.

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