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SIMPLE AND ACCURATE METHOD TO EVALUATE TYPE A STANDARD AND EXPANDED UNCERTAINTIES OF MEASUREMENT FOR THE LAPLACE DISTRIBUTED OBSERVATIONS

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Abstract

The article proposes and investigates a simple and accurate evaluation of the standard and expanded uncertainty of the Laplace population median. With the number of observations n, the known probability distribution describing the sample median for n-2 observations was used to approximate the uncertainty of the population median. The proposed approximation was tested by comparison with exact results for $n \leq 10$ and with the Monte Carlo method. It has been shown that the standard and expanded (confidence level $p = 0.90$, 0.95, and 0.99) uncertainties determined by the proposed approximation differ from values determined by MCM by less than about 1%. Using the median instead of the mean value as the measurement result provides a measurement uncertainty lower by about 25% when $n \geq 35$, and over 29% when $n \geq 70$.

Keywords: uncertainty of measurement, population, Laplace, median, distribution, approximation.

1. Introduction

1.1. General assumptions

The basic requirement of any measurement is to evaluate its uncertainty GUM [1]. The essence of the definition given in [1] is that the uncertainty characterizes the dispersion of the possible values of the measurand μ around the obtained result m. Therefore, in general, correct determining the measurement uncertainty requires the *probability density function* (PDF) $p(\mu|m)$ of the measurand value μ to be around the observed result m [1]. In this article, the evaluation of the standard and expanded uncertainties will be presented when applied to the processing of n>10 independent observations drawn from a population with a Laplace distribution. Therefore, at the beginning, the generally known formulas related to the Laplace distribution will be presented in brief and then used in the next parts of the article.

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For a Laplace or *double exponential* (DE) population, the PDF of random variable x is given by well-known function [2, 3]:

$$p_P(x;\mu,\sigma) = \frac{1}{2\sigma} e^{-\frac{|x-\mu|}{\sigma}} \quad (-\infty < x < \infty), \tag{1}$$

where μ ($\infty < \mu < \infty$) is the location-population median and σ ($\sigma < \infty$) are the scale-population mean absolute deviation parameters. For this distribution, the expectation is equal to the population median: $E(x) = \mu$, and variance is $\text{var}(x) = 2\sigma^2$. Of course, here σ does not mean the population standard deviation.

The use of the Laplace distribution has a long history, and it is one of the best-studied distributions in terms of its properties, including those related to parameter estimation, error analysis, and measurement uncertainty. There are many publications in which the properties of this distribution are presented very thoroughly in theoretical terms and a detailed analysis of various aspects related to this distribution is given [1, 2]. The Laplace distribution is used to describe various populations associated with the measurement of physical quantities when performing tests on various objects and processes [2–6]. For example, when the measurand is a difference of two independent exponentially distributed time intervals, the Laplace distribution is natural approximation of the measured observations [3]. The Laplace distribution can be used to modeling navigation errors and other processes related to measurements on the ground made from aircraft [2]. This distribution is also used when studying speech signals and signals distorted by impulse noise and the strength of flows in some materials [3]. An overview of various applications of the Laplace distribution for modeling various processes in various fields of physics, namely in image and speech recognition, ocean engineering, hydrology astronomers, finance, and others is presented in [4]. In such fields, the Laplace distribution can often provide a better model to describe observations of this kind than the normal distribution with the common variance [4]. A comprehensive approach to describe various aspects of road surface/elevation by using Laplace distribution is presented in [5].

In [6], it was found that uncertainties in many physical systems have impulsive properties and are therefore poorly modeled by Gaussian distributions, while the Laplace distribution model gives more adequate results. Obtained results showed that the introduction of such an estimator demonstrates marked resilience to large, un-modeled spikes in the measurements.

Therefore, the use of the Laplace model to analyze the development of measurement observations has not only theoretical significance but also practical applications.

It is well known [2, 3] that for n independent observations x_i ($i = 1, \ldots, n$) drawn from the Laplace population, the sample median m is the *maximum likelihood estimator* (MLE) of the population median μ, and an estimator of the population mean absolute deviation σ is a sample mean absolute deviation s [2, 3]:

$$m = \begin{cases} x^{(s)}_{(n-1)/2}, & n \text{ is odd}, \\ \dfrac{x^{(s)}_{n/2} + x^{(s)}_{n/2+1}}{2}, & n \text{ is even}, \end{cases} \qquad s = \frac{1}{n}\sum_{i=1}^{n} |x_i - m|, \tag{2}$$

where $x_i^{(s)}$ are the ordered observations.

There are a lot of publications [6–10] related to the parameter estimation of the Laplace distribution by different methods, mainly MLE (maximum likelihood estimator) and MME (method of moment estimator). Namely, in [6] an estimator for a discrete-time scalar linear system with an additional Laplace measurement and process noise is introduced, and simulation results of the estimator are given. In [7], a new method of moment estimator was derived and the asymptotic normality of its distribution was presented, and this estimator was compared with the widely used maximum likelihood estimator. In [8], the approximations for the variance of the sample median only for small and moderate sample sizes and also exact formulas for the probability density function and for the variance of the median are given. In [9], both theoretical analysis (multivariate delta method) and a simulation study analyzed the effectiveness of the classical method of moments for estimating the parameters of symmetric generalized Laplace distributions in comparison with maximum likelihood estimation. To improve the efficiency, modifications to the method of moments were proposed, by taking absolute moments, which improved the performance of the method of moments. The results of research carried out to compare the accuracy of the maximum likelihood estimator with the classical method of moment determination by statistical simulations of quantities described by the Laplace distribution are presented in [10]. A comparison of the accuracy of estimators includes determining the systematic error, theoretical and simulated variances, as well as the mean square error and determining the coefficient of skewness, kurtosis, and histogram analysis is given in [10].

However, these studies, like many others, concern the properties of estimators m and s as random quantities at given values of μ and σ, and do not examine the properties μ and σ of distribution parameters at given estimator values of m and s. In the theory of estimation, the estimators *m and s* are the random quantities with appropriate PDFs: $p_m(m|\mu,\sigma)$, $p_s(s|\sigma)$, which depend on the population parameters μ, σ. The randomness of the m and s estimates can be interpreted as their possible values obtained by processing the observations by repeatedly drawing samples of size n from the population with the same location μ and scale σ parameters. It is well known [2, 3] that PDF $p_m(m\,|\mu,\sigma;k\,)$ of the sample median m is based on the PDF of order statistics [12] and depends on the n number of observations. For n odd ($n = 2k + 1$) and even ($n = 2k$), $k = 0, 1, 2, \ldots$) using normalized ratio $u = (m - \mu)/\sigma$, these PDFs are [2]:

$$p1_u(u;n) = \frac{n!}{\left[\left(\frac{n-1}{2}\right)!\right]^2} e^{-\frac{n+1}{2}|u|}\left(2-e^{-|u|}\right)^{\frac{n-1}{2}}, \tag{3}$$

$$p2_u(u;k) = \frac{n!e^{-n|u|}}{\left(\left(\frac{n}{2}-1\right)!\right)^2 2^{n-1}}\left[\sum_{i=0}^{\frac{n}{2}-2}\frac{(-1)^i C^i_{\frac{n}{2}-1}2^{\frac{n}{2}-1-i}}{\frac{n}{2}-1-i}\left(e^{(\frac{n}{2}-1-i)|u|}-1\right)+\frac{1}{n}+(-1)^{\frac{n}{2}-1}|u|\right]. \tag{4}$$

If it is known exactly that the population has a Laplace distribution, then the question arises of how effective the use of the median from the registered observations is compared to the use of the mean value? Theoretically, for the Laplace distribution (1), the median med $= X_{0.5}$ as $p = 50\%$ of the quantile at the point x_p have an asymptotic normal distribution and the variance of median depends on the PDF [11]: $\mathrm{var}(x_{\mathrm{med}}) = \sigma^2/n$. In contrast, the theoretical variance of the arithmetic mean value $\overline{x}$ of n observations taken from the population (1) is twice as large $\mathrm{var}(\overline{x}) = 2\sigma^2/n$. From the comparison of these two values, we can see that for a population with the Laplace distribution, the median has a variance theoretically 2 times smaller compared to the variance of the mean value. This means that to obtain the same standard deviation when using the median as the result of measurement, 2 times fewer observations are required than when using the mean value. Inversely, for the same number of observations, the use of the median provides theoretically about $\sqrt{2} \approx 1.41$, or about 41% less standard deviation than if the mean value is used. However, this concerns the quality of the estimator – the sample median. To compare the measurement uncertainty when the result is the median with the uncertainty when the result is the arithmetic mean, the dependence of the uncertainty on the number of observations should be carefully examined.

1.2. The standard and expanded uncertainties of a population median as a measurand

In praxis, after a given experiment, *i.e.*, using a single sample that consists of n observations $x_1, x_2, \ldots, x_n$, the specific numeric values of estimators $m = m_e$ and $s = s_e$ (4) are determined. Therefore, after the measurement experiment the values m_e and s_e are known, i.e., not random. From the point of view of uncertainty, it is obvious that the same values of the estimates m_e and s_e, which were determined from a sample drawn from a population with parameters, say, μ_1 and σ_1, can also be obtained from the same type population with slightly different parameters μ_2 and σ_2. It means that sampling from the same type of population with slightly different parameters μ and σ may result in the same estimate values m_e and s_e. Then a question arises: what values μ, σ of population parameters may correspond to the estimates m_e, s_e obtained from the given measurement experiment [12, 13]. For the correct answer to this question after carrying out the measurement experiment, the population parameters μ, σ should be considered as random variables. Then, for simplicity, we will use the usual estimate: $m_e = m, s_e = s$. Hence having determined in a given experiment the numerical values of the sample median m and the absolute median deviation s to correctly describe the random population median μ, it is necessary to have its PDF $p_\mu(\mu|m, s; n)$. Only using this PDF, the values of standard and expanded uncertainties of the population median can be determined fully correctly. Namely, the Type A standard uncertainty $u_A(\mu|s; n)$ of the population median is:

$$u_A(\mu|s;n) = s_\mu(\mu|m,s;n) = \sqrt{\int_{-\infty}^{\infty} (\mu - m)^2 p_\mu(\mu|m,s;n)\, d\mu} \tag{5}$$

Due to the fact that the Laplace distribution (1) is described by a modulus function, deriving the distribution $p_\mu(\mu|m, s; n)$ of the population median, especially for numbers of observations from a few and more, is an extremely difficult task [13]. If in the case of a normal or uniform population, there are general expressions for the PDF of the location parameter for an arbitrary n, then for the Laplace population it is impossible, but it is possible to derive this distribution only for a specific sample size. For example, in [13] the exact PDF $p_w(w)$ for the normalized ratio $w = \dfrac{\mu - m}{s \cdot n}$ and DF $F_w(w)$ are derived for the number of observations $n = 3$, and $n = 5$. In [11], the exact PDFs $p_\tau(\tau; n)$ for the normalized ratio:

$$\tau = \frac{\mu - m}{s}. \tag{6}$$

of the population median were derived for the number of observations $2 \le n \le 10$.

From (5) using (6), the exact value of standard uncertainty $u_A(\mu)$ of population median can be determined using the estimated absolute median deviation s (1):

$$u_A(\mu) = \sigma_\mu = \sigma_\tau(n) \cdot s, \qquad \sigma_\tau(n) = \sqrt{\int_{-\infty}^{\infty} \tau^2 p_\tau(\tau;n) \mathrm{d}\tau}, \tag{7}$$

where $\sigma_\tau(n) = \sigma_\tau = u_A(\tau)$ is the standard deviation of the normalized population median (6).

The exact values of the standard deviation σ_τ of the normalized population median determined by (7) are given in [14]. In [12], it was shown that in the case of two-parameter populations, with the appropriate choice of estimators of location and scale parameters, the standard deviations of these parameters decrease proportionally to the square root of $n - 3$ ($\sim 1/\sqrt{n-3}$). It means

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that the standard uncertainty of the population median (and also the population absolute median deviation) can be determined completely correctly only when $n \geq 4$. Due to the dependency of standard uncertainty proportional to $\sim 1/\sqrt{n-3}$, it is advisable to modify the $\sigma_\tau(n)$ values by a multiple of $\sqrt{n-3}$, i.e., the modified value of standard uncertainty is:

$$\sigma_{u_A,\mathrm{mod}}(n) = \sigma_\tau(n) \cdot \sqrt{n-3}. \tag{8}$$

This modification ensures the stabilization of its value when n changes. Therefore, using the $\sigma_{u_A,\mathrm{mod}}(n)$ standard uncertainty of population median can be determined by:

$$u_A(\mu) = \sigma_{u_A,\mathrm{mod}}(n) \cdot \frac{s}{\sqrt{n-3}}. \tag{9}$$

For the confidence level p, the expanded uncertainty $U_{p,\tau}(\tau;n)$ of the normalized population median is a solution of the nonlinear equation:

$$U_{p,\tau}(\tau;n) = \mathrm{solve}\left\{F_\tau[U_{p,\tau}(\tau;n)] = \frac{p+1}{2}\right\}, \tag{10}$$

where $F_\tau(\tau;n)$ is the distribution function of the normalized population median. Thus, the expanded uncertainty of the population median is:

$$U_{p,\mu}(\mu) = U_{p,\tau}(\tau;n) \cdot s = k_{Up}(p;n) \cdot u_A(\mu), \tag{11}$$

where the coverage factor $k_{Up}(p;n)$ is:

$$k_{Up}(p;n) = \frac{U_{p,\mu}(\mu)}{u_A(\mu)} = \frac{U_{p,\tau}(\tau;n)}{\sigma_\tau(n)}. \tag{12}$$

1.3. Problems with deriving exact PDF for the population median

As was shown in [13, 14], for observation numbers over 5, the expressions for the PDF $p_\tau(\tau;n)$ of the normalized median population become increasingly complex. In this regard, [6] stated: "The derivation of the exact distributions becomes quite tedious as n increases". For example, the expression for the PDF of the median population for $n = 10$ observations takes up a whole page written in small symbols [14]. Thus, even if we have PDF expressions for $n > 10$, due to their enormous complexity, the practical use of such formulas to determine the standard and expanded uncertainties is also extremely difficult. To solve the problem related to the large sample size, various approximations and asymptotic PDFs and DFs have been proposed and studied [13, 15, 17], which are mainly used to determine confidence intervals of the population median when estimates (2) are experimentally determined.

Namely, in [13], a few approximated methods, used to determine confidence intervals of the population median and median absolute deviation, are studied. Only for $n = 3$ and 5, the exact PDFs of the normalized quantity

$$W_n = \frac{\hat{\theta} - \theta}{n \cdot \hat{\sigma}}$$

(using the notation $\hat{\theta} = \mathrm{median}(x_i)$ of the estimated median and

$$\hat{\sigma} = \frac{1}{n}\sum_{i=1}^{n} |x_i - \hat{\theta}|$$

is the absolute median deviation) are given in [13]. Also, asymptotic and approximate distributions are investigated in [13]. It was stated that "asymptotic distributions are not adequate approximations for moderate sample sizes". To improve the asymptotic distribution W_n, the approximation based on the ratio of two independent normal variables (the so-called Normal/Normal approximation) is investigated in [13]. Another approximation, the so-called Median/χ^2 approximation in the form of the ratio of the median to an independent chi-square variable, based on the fact that the exact density of the sample median can be determined analytically and since a chi-square (χ^2) approximation for $\hat{\sigma}$ is better than a normal approximation, is investigated in [13]. It was shown that last approximation gives better results in comparison with the so-called Normal/Normal approximation. A few numerical examples for $n = 3, 5, 9, 15$ and 33 related to determining of the cumulative probability $P[n^{\frac{3}{2}} W_n = \sqrt{n} \cdot \tau < z]$ for the some given values of z are also presented in [13]. The results presented and given in the Appendix showed that such approximations are asymptotically correct but they do not always provide sufficient accuracy.

In [15], the conditional confidence intervals were constructed using appropriate so-called ancillary statistics. For the arbitrary sample size, to determine the approximate value of confidential intervals, the MCM can be used. In [16] the construction of the confidence intervals for DE distribution based on simulated data is studied. The results are compared with the Student confidence intervals. The results obtained are illustrated in the example of $n = 10$ observations. Unfortunately, it seems that some numerical values in this example should be corrected. It should also be noted here that a large number of publications [17–21] concern the interval estimation related to the censored samples.

The following research aims to propose a simple and accurate method for approximate determination of the standard and expanded uncertainty of measurement, which results in the median of sample being taken from the Laplace population, and also to investigate the accuracy of the proposed method using a Monte Carlo method. In addition, the goal is also to prove the effectiveness of the median compared to the arithmetic mean value in terms of measurement uncertainty.

2. Proposed approximation of PDFs of population median by PDFs of sample median for $n - 2$ sample size for the uncertainty evaluation

From the general properties of estimators [11], it follows that as the number of observations increases ($n \to \infty$), the PDF $p_m(m|\mu, \sigma; n)$ of the parameter estimator m at known values of the population parameters μ and σ the PDF $p_\mu(\mu|m, s; n)$ of the population parameter μ at a known value of the estimators m and s become increasingly close. For example, it is well known that for a normal population $N(\mu, \sigma)$, the PDF of the normalized arithmetic mean value $\overline{x} : u = (\overline{x} - \mu)/\sigma$ with known μ and σ is also normal, while the distribution of the normalized ratio $t = (\mu - \overline{x})/s$ of μ with known values of the estimators $\overline{x}$ and $s = \text{stdev}(x)$ is the Student's t-distribution. But when the number of observations increases ($n \to \infty$), the Stude'nt's distribution becomes closer and closer to normal distribution. Besides, when the sample median$m$ (2) is the result of the measurement, then the number degrees of freedom is $d = -n - 1$, while, as already shown in [12], in the analysis of variance of the median of the population the number $n - 3 = -d - 2$ occurs, i.e., is smaller by 2. These facts can be used to formulate a hypothesis of approximating the PDF $p_\tau(\tau; n)$ of the normalized ratio τ (7) of the population median μ by the distributions $p_u(m; n - 2)$ (3), (4) of the normalized ratio $u = (m - \mu)/\sigma$ of the median estimator m for a number of observations $n - 2$, $i.e.$, smaller by 2. This hypothesis can be easily verified, since the exact expressions for the probability distributions $p_\tau(\tau; n)$ of the normalized median τ (9) at numbers of $n = 2, \ldots, 10$ are

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known [14] and also based on the sample median distributions $p_u(m;n)$ (4), (5). Namely, Fig. 1 shows pairs of population normalized median PDFs $p_\tau(\tau;n)$ [14] for $n = 5,\ldots,10$ and sample normalized median PDFs $p_u(u;n-2)$ for $n-2 = 3,\ldots,8$. From these data, one can see a very good convergence of these PDFs, even practically indistinguishable.

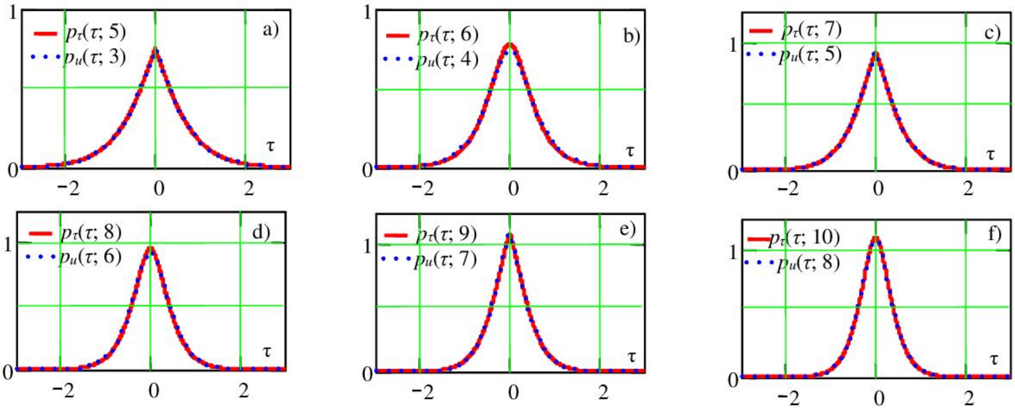

Fig. 1. PDFs of the normalized population median $p_\tau(\tau;n)$ for $n = 5,\ldots,10$ (solid red) and the normalized sample median $p_u(\tau;n-2)$ (dot blue) for $n = 3,\ldots,8$ accordingly: $n = 5$ (a); $n = 6$ (b); $n = 7$ (c), $n = 8$ (d), $n = 9$ (e), $n = 10$ (f).

More informative are the differences between $p_u(\tau;n-2)$ and $p_\tau(\tau;n)$ PDFs $\Delta p(\tau;n) = p_u(\tau;n-2) - p_\tau(\tau;n)$. For odd $n \geq 5$, the difference between these PDFs is less than 0.002, while for even n, although it is slightly larger but still very small. Due to the closeness of the PDFs, the closeness of the standard deviations of the exact $\sigma_\tau(n)$ and approximated $\sigma_u(n-2)$ determined by distributions (5) and (6) is expected. The standard deviations of the normalized sample median can be determined by (5) and (6). Namely, for odd n:

$$\sigma 1_u(n) = \sqrt{var1_u(n)} = \sqrt{\frac{n!}{\left(\left(\frac{n-1}{2}\right)!\right)^2 2^{\frac{n-9}{2}}}\left[\sum_{i=0}^{\frac{n-1}{2}} \frac{(-1)^i C^i_{\frac{n-1}{2}}}{2^i(n+1+2i)^3}\right]}, \tag{13}$$

and for the even n:

$$\sigma 2_u(n) = \sqrt{\frac{(n)!}{\left(\left(\frac{n}{2}-1\right)!\right)^2 2^{n-3}n^3}\left[\sum_{i=0}^{\frac{n}{2}-2}\left[\frac{(-1)^i C^i_{\frac{n}{2}-1}\cdot((n+1+i)^2+\frac{3}{4}n^2)}{2^{i-\frac{n}{2}-2}(n+2+2i)^3}\right]+\frac{1+(-1)^{\frac{n}{2}-1}\cdot 3}{n}\right]}. \tag{14}$$

The exact $\sigma_\tau(n)$ [14] and approximate $\sigma_u(n)$ values (13), (14) of standard deviations of the sample medians for $n = 4,\ldots,10$ are given in Table 1. The analysis shows that the standard deviation of the sample median determined for $n-2$ observations very well approximates the standard deviation of the population median for n observations. The relative differences

$$\delta_\sigma(n) = \left(\frac{\sigma_u(n-2)}{\sigma_\tau(n)} - 1\right)\cdot 100\%$$

between these standard deviations are given in Table 1. This table shows that when the number of observations is $n \geq 6$, the differences between standard approximated and exact standard deviations of the normalized median and also relative differences between the $\sigma_{u_A,\mathrm{mod}}(n)$ (8),

$\sigma_{u,\mathrm{mod}}(n-2) = \sigma_u(n-2)\cdot\sqrt{n-3}$ are less than 1% (0.56%), i.e., they are negligibly small. Therefore, the standard deviation of the normalized population median can be approximated as:

$$u_A(\tau) = \sigma_\tau(n) \approx \sigma_u(n-2); \quad \sigma_{u,\mathrm{mod}}(n-2) = \sigma_u(n-2)\cdot\sqrt{n-3}, \tag{15}$$

Table 1. Exact $\sigma_\tau(n)$ [14] and approximate $\sigma_u(n-2)$ (15) values of the standard deviation of the normalized population median and also exact $\sigma_{u_A,\mathrm{mod}}(n)$ and approximated $\sigma_{u,\mathrm{mod}}(n-2)$ modified values and relative differences (in %) between them ($n = 4,\dots,10$).

n	4	5	6	7	8	9	10
Exact, $\sigma_\tau(n)$	1.0548	0.8299	0.6518	0.5952	0.5113	0.4845	0.4326
Approx., $\sigma_u(n-2)$	1	0.7993	0.6482	0.5926	0.5108	0.4854	0.4328
Exact, $\sigma_{u_A,\mathrm{mod}}(n)$	1.0548	1.174	1.129	1.191	1.143	1.187	1.145
Approx., $\sigma_{u,\mathrm{mod}}(n-2)$	1	1.130	1.123	1.185	1.142	1.189	1.145
$\delta_\sigma(n)$, %	5.48	3.82	0.56	0.44	0.10	−0.20	−0.05

Similarly, the difference $\Delta F(\tau;n) = F_u(\tau;n-2) - F_\tau(\tau;n)$ between the distribution functions $F_u(\tau;n-2)$ and $F_\tau(\tau;n)$ is very small. Namely, for odd $n \geq 5$ the differences $\Delta F(\tau;n)$ are less than $2\cdot10^{-3}$, (i.e., less than 0.2% of DF maximal value 1) and for even $n \geq 6$ the differences $\Delta F(\tau;n)$ are less than 10^{-2} (i.e., less than 1%). When n =8 and 10 these differences are less than $5\cdot10^{-3}$ (i.e., less than 0.5%).

For accuracy comparison purposes for the $n = 4,\dots,10$ in Table 2, the exact $U_{p,\tau}(\tau;n)$ [14] and approximated $U_{p,u}(u;n-2)$ values of expanded uncertainties

$$U_{p,u}(u;n-2) = \mathrm{solve}\left\{F_u[U_{p,u}(u;n-2)] = \frac{p+1}{2}\right\}, \tag{16}$$

Table 2. Exact $U_{p,\tau}(\tau;n)$ [14], approximate $U_{p,u}(u;n-2)$ expanded uncertainties and exact $k_{Up}(p;n)$, approximated $k_{Up,u}(p;n-2)$ values of coverage factors and also relative difference $\delta_{Up}(n)$ between exact and approximate uncertainties.

n		4	5	6	7	8	9	10
$p = 0.90$	Exact $U_{p,\tau}(\tau;n)$	1.5024	1.3144	1.0367	0.9702	0.8287	0.7948	0.7060
	Approx. $U_{p,u}(u,n-2)$	1.6359	1.3067	1.0608	0.9715	0.8368	0.7971	0.7097
	Exact $k_{Up}(p;n)$	1.4244	1.5839	1.5904	1.6300	1.6209	1.6407	1.6321
	Approx. $k_{Up,u}(p;n-2)$	1.6359	1.6348	1.6366	1.6394	1.6382	1.6420	1.6399
$p = 0.95$	Exact $U_{p,\tau}(\tau;n)$	2.0000	1.6841	1.3226	1.2237	1.0421	0.9945	0.8811
	Approx. $U_{p,u}(u,n-2)$	2.0565	1.6681	1.3267	1.2267	1.0428	0.9994	0.8817
	Exact $k_{Up}(p,n)$	1.8961	2.0294	2.0252	2.0558	2.0382	2.0529	2.0369
	Approx. $k_{Up,u}(p;n-2)$	2.0565	2.0870	2.0468	2.0700	2.0415	2.0588	2.0373
$p = 0.99$	Exact $U_{p,\tau}(\tau;n)$	3.4456	2.6121	2.0252	1.8030	1.5343	1.4364	1.2728
	Approx. $U_{p,u}(u,n-2)$	2.9951	2.4913	1.916	1.7978	1.4959	1.4464	1.2573
	Exact $k_{Up}(p,n)$	3.2667	3.1477	3.1071	3.0290	3.0009	2.9650	2.9423
	Approx. $k_{Up,u}(p;n-2)$	2.9951	3.1169	2.9559	3.0338	2.9287	2.9797	2.9050
$\delta_{Up}(n)$, %	$p = 0.90$	8.9	−0.58	2.33	0.13	0.97	0.28	0.52
	$p = 0.95$	2.8	−0.95	0.31	0.24	0.07	0.49	0.07
	$p = 0.99$	−13	−4.62	−5.4	−0.28	−2.5	0.7	−1.22

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and exact $k_{Up}(p;n)$ (12) and approximated $k_{Up,u}(p,n-2)$,

$$k_{Up,u}(p;n-2) = \frac{U_{p,u}(u;n-2)}{\sigma_u(n-2)} \tag{17}$$

values of coverage factors are presented. The relative approximation errors

$$\delta_{Up}(n) = \left(\frac{U_{p,u}(u;n-2)}{U_{p,\tau}(\tau;n)} - 1\right) \cdot 100\%$$

of these expanded uncertainties are also given in Table 2.

From data presented in Table 2, it can be seen that when $n \geq 7$ and $p = 0.90, 0.95$ the approximated value of expanded uncertainty differs from the exact value by less than 1%, and for $p = 0.99$ is less than 2,5%. It is generally accepted [1] that the uncertainty is represented by no more than two significant figures, which corresponds to approximately 5% accuracy. Therefore, from the point of view of standard and expanded uncertainty, the accuracy of the proposed approximation meets the requirements, i.e., it is sufficient.

It follows from the results above that the basic parameters i.e. standard uncertainty and expanded uncertainty (confidence interval) of the population median can be determined with sufficient precision in a very simple way based on the estimation from sample of n size value of the absolute median deviation s and uses of the values of the corresponding coefficients relating to the standard and expanded uncertainties for the normalized sample median for $n-2$ (Fig. 2).

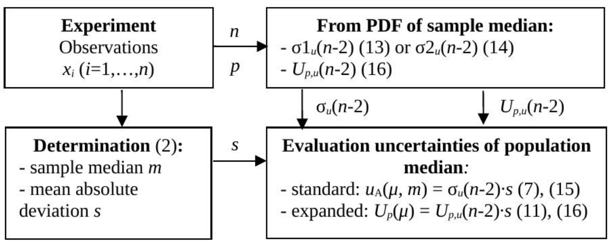

Fig. 2. Algorithm of uncertainty evaluation for the Laplace population median.

3. Investigation with a Monte Carlo method

The effectiveness of the proposed approximation of probability distribution of the population median and the parameters determined on its basis, mainly the standard and expanded uncertainties, was tested for the sample size $n = 11, \ldots, 70$, using the MCM [22].

3.1. Description of the investigation algorithm

The number of simulations was $M = 10^5$. During the tests the normalized DE population probability distribution is used: $\mathrm{DE}(\mu 0, \sigma 0) = \mathrm{DE}(0, 1)$, i.e., the value of the population median is taken as $\mu 0 = 0$ and the value of population median absolute deviation is $\sigma 0 = 1$.

For $n = 11, \ldots, 60$ the M ($j = 1, \ldots, M$) random samples with $i = 1, \ldots, n$ values $x_{j,i}$ were generated by the formula:

$$x_{j,i} = \begin{cases} \mu 0 + \sigma 0 \cdot \ln(2 \cdot z_{j,i}), & \text{if } z_{j,i} \leq 0.5, \\ \mu 0 - \sigma 0 \cdot \ln(2 \cdot (1 - z_{j,i})), & \text{if } 0.5 < z_{j,i} < 1, \end{cases} \tag{18}$$
$$z_{j,i} = \mathrm{rnd}(1); \; i = 1, \ldots, n; \; j = 1, \ldots, M.$$

For each number of observations n, there were determined:

1) the sample median m_j (2) and arithmetic mean $\overline{x}_j$ and also sample median absolute deviation s_j (2) and sample standard deviation $S_{j;(\overline{x})}$:

$$s_j = \frac{1}{n}\sum_{i=1}^{n}|x_{j,i} - m_j|, \qquad S_{j;(\overline{x})} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_{j,i} - \overline{x}_j)}; \tag{19}$$

2) the mean $\overline{\tau}$ and standard deviation $s_{MC}(n)$ of this estimate:

$$\overline{\tau} = \frac{1}{M}\sum_{j=1}^{M}\tau_j; \qquad \tau_j = \frac{\mu 0 - m_j}{s_j} = -\frac{m_j}{s_j}; \qquad s_{MC}(n) = \sqrt{\frac{1}{M-1}\sum_{j=1}^{M}(\tau_j - \overline{\tau})^2}; \tag{20}$$

3) the modified value of normalized standard uncertainty (8):

$$\sigma_{u_A,\mathrm{mod},MC}(n) = s_{MC}(n)\cdot\sqrt{n-3}; \tag{21}$$

4) the standard uncertainties when the measurement result is sample median m and mean $\overline{x}$:

$$u_A(\mu, m) = \sigma_{u_A,\mathrm{mod},MC}(n)\cdot\frac{s}{\sqrt{n-3}}, \qquad u_A(\mu,\overline{x}) = \frac{S_{(\overline{x})}}{\sqrt{n-3}}; \tag{22}$$

5) the estimates of expanded uncertainty of normalized values of ratio (5) for the confidence levels p =0.90, 0.95 and 0.99:

$$U_{p,MC}(\tau) = \frac{U_{p,R} - U_{p,L}}{2}; \tag{23}$$

where $U_{p,L} = \tau^{(s)}_{[M\cdot\frac{p}{2}]}$, $U_{p,R} = \tau^{(s)}_{M-[M\cdot\frac{p}{2}]}$, - are the left and right estimate values of expanded uncertainties determined by sorted values $\tau^{(s)}_j$;

6) the estimates of expanded uncertainty of population median, when the measurement result is sample median and sample arithmetic mean:

$$U_{p,MC}(\mu, m) = U_{p,MC}(\tau)\cdot s_{MC}(n), \qquad U_{p,MC}(\mu,\overline{x}) = t_p(n-1)\cdot\sqrt{\frac{n}{n-1}}\cdot S_{(\overline{x})}; \tag{24}$$

7) the value of the coverage factor:

$$k_{Up,MC}(n) = \frac{U_{p,MC}(\tau_m)}{s_{MC}(n)}; \tag{25}$$

8) the relative (in %) differences between $\sigma_u(n-2)$ (15) and $\sigma_{u_A,\mathrm{mod},MC}(n)$ (21) and also between $k_{Up,u}(n-2)$ (17) and $k_{Up,MC}(n)$ (25):

$$\delta_{u_A}(n) = \left(\frac{\sigma_{u,\mathrm{mod}}(n-2)}{\sigma_{u_A,\mathrm{mod},MC}(n)} - 1\right)\cdot 100\%, \qquad \delta_{Up}(n) = \left(\frac{k_{Up,u}(n-2)}{k_{Up,MC}(n)} - 1\right)\cdot 100\%. \tag{26}$$

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3.2. Results of the Monte Carlo investigation for $n = 11, \ldots, 70$

The values of modified standard deviation $\sigma_{u,\text{mod}}(n-2)$ (15) determined by approximation and $\sigma_{u_A,\text{mod},MC}(n)$ (25) determined by MCM are shown in Fig. 3a and given in Table 3. The relative differences $\delta_{u_A}(n)$ between $\sigma_{u,\text{mod}}(n-2)$ and $\sigma_{u_A,\text{mod},MC}(n)$ are shown in Fig. 3b.

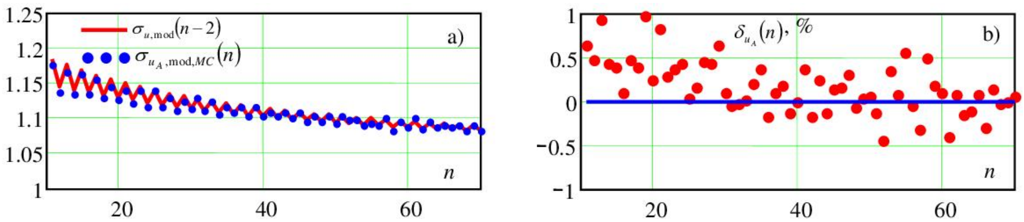

Fig. 3. Dependences of modified normalized standard deviations $\sigma_{u,\text{mod}}(n-2)$ and $\sigma_{u_A,\text{mod},MC}(n)$ from the number of observations (a) relative differences (in %) between the modified values of standard deviations $\sigma_{u,\text{mod}}(n-2)$ determined by approximation and (b) $\sigma_{u_A,\text{mod},MC}(n)$ determined by MCM.

From data presented in Table 3 and shown in Fig. 3b, it can be seen that differences between approximated values and determined by MCM do not exceed 1%. So, it can be concluded that when the number of observations increased $n > 10$, from the point of view of evaluation of standard uncertainty the proposed approximation also ensures sufficient accuracy.

Table 3. The values of modified standard deviations $\sigma_{u,\text{mod}}(n-2)$ and $\sigma_{u_A,\text{mod},MC}(n)$.

n	11	12	13	14	15	16	17	18	19	20
$\sigma_{u,\text{mod}}(n-2)$	1.183	1.143	1.176	1.14	1.168	1.136	1.161	1.132	1.155	1.129
$\sigma_{u_A,\text{mod},MC}(n)$	1.176	1.138	1.165	1.135	1.164	1.135	1.156	1.128	1.144	1.126
n	21	22	23	24	25	26	27	28	29	30
$\sigma_{u,\text{mod}}(n-2)$	1.149	1.125	1.144	1.122	1.139	1.119	1.135	1.116	1.131	1.113
$\sigma_{u_A,\text{mod},MC}(n)$	1.140	1.122	1.140	1.117	1.139	1.117	1.13	1.111	1.124	1.112
n	31	32	33	34	35	36	37	38	39	40
$\sigma_{u,\text{mod}}(n-2)$	1.127	1.111	1.124	1.108	1.121	1.106	1.118	1.104	1.115	1.102
$\sigma_{u_A,\text{mod},MC}(n)$	1.128	1.111	1.124	1.106	1.117	1.108	1.117	1.102	1.117	1.102
n	41	42	43	44	45	46	47	48	49	50
$\sigma_{u,\text{mod}}(n-2)$	1.113	1.100	1.111	1.098	1.108	1.097	1.106	1.095	1.104	1.094
$\sigma_{u_A,\text{mod},MC}(n)$	1.109	1.102	1.108	1.100	1.107	1.095	1.103	1.096	1.104	1.093
n	51	52	53	54	55	56	57	58	59	60
$\sigma_{u,\text{mod}}(n-2)$	1.103	1.092	1.101	1.091	1.099	1.090	1.098	1.088	1.096	1.087
$\sigma_{u_A,\text{mod},MC}(n)$	1.104	1.097	1.097	1.09	1.093	1.09	1.101	1.083	1.094	1.086
n	61	62	63	64	65	66	67	68	69	70
$\sigma_{u,\text{mod}}(n-2)$	1.095	1.086	1.093	1.085	1.092	1.084	1.090	1.083	1.089	1.081
$\sigma_{u_A,\text{mod},MC}(n)$	1.099	1.085	1.095	1.086	1.091	1.087	1.089	1.083	1.089	1.081

The values of the coverage factors $k_{Up,u}(p; n-2)$ determined by approximation and $k_{Up,MC}(p; n)$ determined by MCM for confidence levels $p = 0.90$, 0.95 and 0.99 when $n = 11, \ldots, 70$ are given in Table 4 and shown in Fig. 4a. To evaluate the accuracy, the relative deviations (in %) between coverage factors $k_{Up,u}(p;n-2)$ determined by approximation and $k_{Up,MC}(p;n)$ determined by MCM, which are calculated by (26), are presented in Fig. 4b.

The results obtained by MCM show that the proposed method for determining both the standard and the expanded uncertainties of the population median is very accurate. Namely, the relative deviations of the coefficients, calculated according to approximate dependences, from the coefficients determined by MCM do not exceed about ±1% (Fig. 4b). From the point of view of uncertainty of measurement, this is a very high accuracy of its evaluation [1].

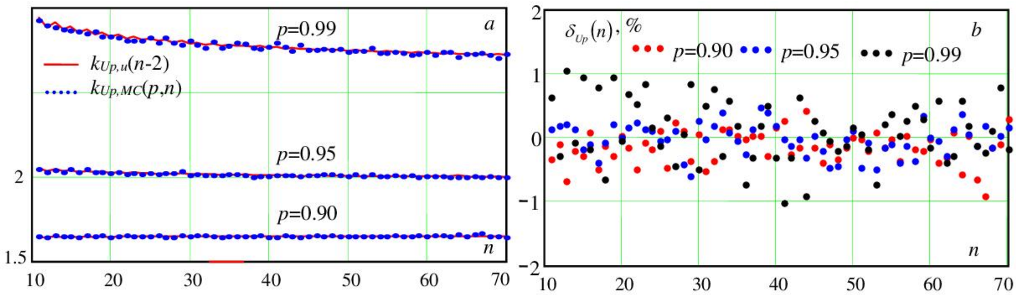

Fig. 4. Dependences of coverage factors $k_{Up,u}(p;n-2)$ determined by approximation and $k_{Up,MC}(p;n)$ determined by MCM (a) relative differences between values of the coverage factors $k_{Up,u}(p;n-2)$ determined by approximation and $k_{Up,MC}(p;n)$ determined by MCM, (b) for confidence levels: 0.90, 0.95 and 0.99.

Table 4. The values of coverage factors $k_{Up,u}(p;n-2)$ determined by approximation and $k_{Up,MC}(p;n)$ by MCM.

	$p = 0.90$		$p = 0.95$		$p = 0.99$	
n	$k_{Up,MC}(p;n)$	$k_{Up,u}(p;n-2)$	$k_{Up,MC}(p;n)$	$k_{Up,u}(p;n-2)$	$k_{Up,MC}(p;n)$	$k_{Up,u}(p;n-2)$
11	1.649	1.644	2.048	2.051	2.923	2.941
12	1.643	1.641	2.030	2.034	2.893	2.885
13	1.654	1.645	2.040	2.044	2.882	2.912
14	1.646	1.642	2.028	2.031	2.870	2.867
15	1.650	1.645	2.043	2.039	2.862	2.889
16	1.642	1.643	2.030	2.028	2.858	2.853
17	1.654	1.646	2.043	2.035	2.848	2.870
18	1.646	1.644	2.027	2.025	2.859	2.840
19	1.651	1.646	2.027	2.031	2.828	2.854
20	1.644	1.644	2.023	2.023	2.830	2.829
21	1.649	1.646	2.025	2.028	2.822	2.841
22	1.653	1.645	2.016	2.021	2.804	2.819
23	1.648	1.647	2.023	2.025	2.806	2.829
24	1.648	1.645	2.017	2.019	2.805	2.810
25	1.645	1.647	2.024	2.023	2.823	2.819
26	1.653	1.645	2.018	2.017	2.793	2.802
27	1.643	1.647	2.019	2.021	2.823	2.810
28	1.644	1.646	2.015	2.016	2.793	2.795
29	1.653	1.647	2.021	2.019	2.779	2.802
30	1.645	1.646	2.009	2.014	2.802	2.788

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Tabke 4. [cont.]

	$p = 0.90$		$p = 0.95$		$p = 0.99$	
n	$k_{Up,MC}(p;n)$	$k_{Up,u}(p;n-2)$	$k_{Up,MC}(p;n)$	$k_{Up,u}(p;n-2)$	$k_{Up,MC}(p;n)$	$k_{Up,u}(p;n-2)$
31	1.656	1.647	2.018	2.017	2.781	2.795
32	1.652	1.646	2.009	2.013	2.761	2.782
33	1.645	1.647	2.008	2.016	2.796	2.788
34	1.644	1.646	2.010	2.012	2.761	2.777
35	1.647	1.647	2.015	2.014	2.777	2.782
36	1.647	1.646	2.016	2.011	2.792	2.772
37	1.647	1.647	2.010	2.013	2.785	2.777
38	1.646	1.646	2.000	2.009	2.762	2.767
39	1.652	1.647	2.004	2.012	2.758	2.771
40	1.644	1.647	2.005	2.008	2.771	2.763
41	1.643	1.647	2.011	2.010	2.795	2.767
42	1.651	1.647	2.010	2.008	2.767	2.758
43	1.650	1.647	2.010	2.009	2.745	2.762
44	1.640	1.647	2.003	2.007	2.780	2.755
45	1.650	1.647	2.008	2.008	2.751	2.758
46	1.653	1.647	2.010	2.006	2.749	2.751
47	1.649	1.647	2.017	2.007	2.756	2.754
48	1.652	1.647	2.014	2.005	2.7539	2.747
49	1.650	1.647	2.007	2.006	2.754	2.751
50	1.647	1.647	2.002	2.004	2.737	2.741
51	1.648	1.647	2.015	2.006	2.746	2.747
52	1.650	1.647	2.005	2.004	2.746	2.741
53	1.646	1.647	2.015	2.005	2.764	2.744
54	1.649	1.647	2.006	2.003	2.733	2.739
55	1.648	1.647	2.006	2.004	2.735	2.741
56	1.653	1.647	2.010	2.002	2.751	2.736
57	1.647	1.647	2.006	2.003	2.731	2.738
58	1.650	1.647	2.009	2.002	2.720	2.733
59	1.651	1.647	1.996	2.003	2.728	2.736
60	1.647	1.647	2.001	2.001	2.735	2.731
61	1.654	1.647	2.003	2.002	2.717	2.733
62	1.652	1.647	2.006	2.001	2.739	2.728
63	1.646	1.647	1.999	2.001	2.738	2.730
64	1.657	1.647	2.000	2.000	2.710	2.726
65	1.647	1.647	2.000	2.001	2.723	2.728
66	1.658	1.647	2.002	1.999	2.727	2.724
67	1.663	1.647	1.997	2.000	2.733	2.726
68	1.650	1.647	2.002	1.999	2.719	2.722
69	1.650	1.647	2.000	2.000	2.704	2.725
70	1.643	1.647	1.996	1.999	2.725	2.720

As was mentioned above, the normalized sample median has an asymptotically normal distribution. Due to this, for the large n as approximated values of coverage factor $k_{Up}(n)$, which determine the expanded uncertainty, the corresponding percentiles of Student t-distribution [1] can be used. For example, when $n = 70$ (number degrees of freedom $d = 70 - 1 = 69$) for $p = 0.90$, 0.95 and 0.99 coverage factors from t-distribution are: 1.667, 1.995, 2.678. After comparing the values of coefficients $k_{Up,u}(n-2)$: 1.647 (1.2%), 1.999 (0.2%) and 2.720 (1.5%) in the last line in Table 4 for $n = 70$. It can be seen that approximated values of $k_{Up,u}(n-2)$ are much closer to limit values, the differences are less than 1.5%, $i.e.$, they are negligible.

3.3. Comparison of the uncertainties evaluated by proposed and standard procedure according to GUM [1]

In the case of the Laplace population (1), the parameter μ is both the median and the expected value of the population. Therefore, in accordance with the standard procedure [1], the mean value $\bar{x}$ can be assumed as the estimator of μ, i.e., as the best measurement result. Thus, using standard uncertainty $u_A(\mu|\overline{x}.)$ (26), it is possible to answer the question: how does uncertainty $u_A(\mu|\overline{x})$ differ from the uncertainty $u_A(\mu|m)$ (m = med) when the measurement result is the sample median. Or in other words, how much will be the measurement uncertainty lower if the median is be used as the result instead of the mean value ? The relative deviations of these uncertainties, expressed in percentages, can be used to answer this question:

$$R_{u_A}(\mu, n) = \left(\frac{u_A(\mu, \overline{x})}{u_A(\mu, m)} - 1\right) \cdot 100\%. \tag{27}$$

These deviations present increasing standard uncertainty of measurement with multiple observations obtained from the Laplace population when the arithmetic mean instead of the sample median is used as the measurement result. The deviations $R_{u_A}(\mu, n)$ (27), depending on the number of observations, are shown in Fig. 5.

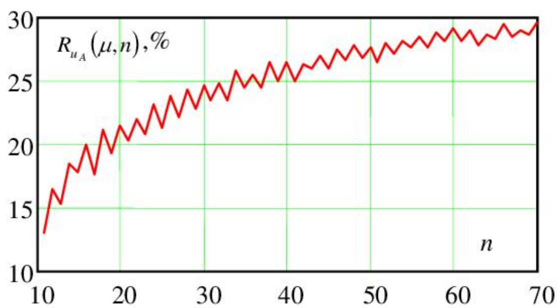

Fig. 5. Increase in standard measurement uncertainties when the arithmetic mean value is used instead of the sample median as the measurement result.

This figure shows that when $n \geq 12$, using the sample median as the measurement result reduces the standard uncertainty by more than 15% compared to using the result as the arithmetic mean, and when $n \approx\geq 35$, using the sample median provides a reduction in the standard uncertainty by more than 25%. To obtain the same level of uncertainty when the mean value is used as the measurement result, the number of observations would be approximately 55 instead of 35 when determining the median. With an increase in the number of observations, the effectiveness of the median increases but this increase is very slow, namely at $n = 70$ its value is about 29%. But here,

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the same level of uncertainty when using the mean value as the measurement result, the number of observations would be approximately 116 instead of 70 when determining the median. Only with very large numbers of observations (theoretically at $n \to \infty$) can it reach its maximum value of about 41%. Similar results are obtained when determining the expanded uncertainty.

4. Conclusions

The article proposes and investigates simple and accurate approximations for the evaluation of the standard and expanded uncertainties of the population median based on the processing of a random sample of size $n \geq 11$ drawn from a Laplace-distributed population.

The approximation of the distribution of the population median is based on well-known and studied distributions of the sample median for the number of observations $n - 2$. For the practical application of the obtained results, appropriate values of the coefficients necessary for the evaluation of the standard and expanded uncertainties for the number of observations $n = 11, \ldots, 70$ were determined and given in the corresponding tables. For a large number of observations ($n > 70, \to \infty$), the approximate values of the coverage factor from Student's t-distribution can be used to calculate the expanded uncertainty.

The algorithm of the uncertainty evaluation is very simple and consists of four steps:

1. determining the sample median m by (2);
2. determining the sample absolute median mean deviation s by (2);
3. determining the standard uncertainty $u_A(\mu)$ by (7) and (15) or by modified standard deviations $\sigma_{u,\text{mod}}(n-2)$ from Table 3;
4. for a given confidence level p, determining the expanded uncertainty $U_p(\mu)$ by (11) and (16) or by coverage factor $k_{Up,u}(p; n-2)$ (17) from Table 4 and standard uncertainty.

The accuracy of the proposed method was tested by comparing the results of approximate values of standard and expanded uncertainties with the results obtained using the exact population median distributions for $n \leq 10$, and by the Monte Carlo simulation, with the number of trials $M = 10^5$ for a number of observations $n = 11, \ldots, 70$.

Based on comparisons with exact results ($n \leq 10$), it has been shown that:

(i) when $n \geq 5$ the differences between approximated and exact standard deviations of the normalized median are less than 1%, i.e., negligibly small;

(ii) when $n \geq 7$ and $p = 0.90$ and 0.95, the approximated value of expanded uncertainty differs from the exact value by less than 1%, and for $p = 0.99$ difference does not exceed $\approx 2.5\%$.

The results obtained by Monte Carlo simulation ($n = 11, \ldots, 70$) have shown that the proposed approximated method for determining both standard and expanded uncertainty of the population median is very accurate. Namely, relative deviations of the standard deviation and expanded uncertainties ($p = 0.90$, 0.95, 0,99), determined according to proposed approximate dependences, from the values determined by the Monte Carlo simulation do not exceed about 1%. From the point of view of uncertainty of measurement, this is a very high accuracy of uncertainty evaluation [1].

Using the arithmetic mean as the measurement result, i.e., the estimate of the Laplace population location parameter, instead of the sample median, will generally result in an increased uncertainty of up to about 25–40%, or would require an increase in the number of observations about 1.5–2 times to obtain the same uncertainty.

A comparison of the obtained results with the results given in literature sources has shown that the proposed approximation is more accurate (please see also the Appendix) and is easier to use.

5. Appendix

The efficiency of the proposed approximation is also checked by comparison with the data given in [13] namely related to the determination of cumulative probability

$$P\left[n^{\frac{3}{2}} W_n = \sqrt{n} \cdot \tau < z\right] \quad (z = \sqrt{n} \cdot \tau).$$

In Table 5 the relative errors δ_{appr} (in %) between approximated [13] and exact values or determined by MCM values are given also. Analysis of data presented in Table 5 shows that the proposed approximation in comparison with the approximation given in [13] is much more accurate, especially for large values of $|z|$.

Table 5. Comparison of exact and approximate cumulative probabilities P given in [13] and determined by proposed method for selected sample sizes.

n	z	Exact	MCM	Approximated					
				Median/χ2 [13]	δ_{appr},%	Norm./Norm. [13]	δ_{appr}, %	Proposed, $(n-2)$	δ_{appr}, %
$n = 5$	−4.072	0.0194	0.0197	0.034	75.3	0.025	28.9	0.0186	−4.1
	−2.429	0.0766	0.0766	0.087	13.6	0.050	−34.7	0.0758	−1.0
	−1.565	0.1552	0.1545	0.155	−0.1	0.100	−3.4	0.1544	−0.5
$n = 9$	-3.691	0.0107	0.0108	0.020	1.5	0.010	−49.2	0.0110	2.8
	−2.589	0.0396	0.0394	0.050	26.3	0.025	−36.9	0.0400	1.0
	−1.967	0.0795	0.0791	0.087	9.4	0.050	−37.1	0.0797	0.25
	−1.418	0.1416	0.1409	0.143	1.0	0.100	−29.4	0.1417	0.07
$n = 15$	−2.913		0.0170	0.024	41.2	0.010	−41.2	0.0176	3.5
	−2.273		0.0413	0.049	18.6	0.025	−39.5	0.0425	2.9
	−1.817		0.0752	0.081	7.7	0.050	−33.5	0.0765	1.7
	−1.359		0.1323	0.134	1.3	0.100	−24.4	0.1329	0.45
$n = 33$	-2.544		0.0184	0.022	19.6	0.010	−45.7	0.0189	2.7
	−2.085		0.0391	0.043	10	0.025	−36.1	0.0400	2.3
	-1.717		0.0689	0.073	6	0.050	−27.4	0.0701	1.7
	−1.315		0.1213	0.124	67.2	0.100	−17.6	0.1229	1.3

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References

[1] Joint Committee for Guides in Metrology. (2008). *Evaluation of measurement data – Guide to the expression of uncertainty in measurement* (JCGM 100:2008). http://www.bipm.org/utils/common/documents/jcgm/JCGM_100_2008_E.pdf

[2] Kotz, S., Kozubowski, T.J., & Podgórski, K. (2001). Introduction. In *The Laplace Distribution and Generalizations* (pp. 229–230). Birkhäuser Boston. https://doi.org/10.1007/978-1-4612-0173-1_5

[3] Johnson, N.L., Kotz, S., & Balakrishnan, N. (1995). *Continuous univariate distributions, Volume 2*. John Wiley & Sons. https://doi.org/10.2307/2340444

[4] Geraci, M. & Cortina Borja, M. (2018). The Laplace distribution. *Significance*, *15*(4), 10–11.

[5] Johannesson, P., Podgórski, K., & Rychlik, I. (2017). Laplace distribution models for road topography and roughness. *International Journal of Vehicle Performance*, *3*(2), 224. https://doi.org/10.1504/ijvp.2017.085032

[6] Duong, N.C., Speyer, J.L., & Idan, M. (2022). Laplace estimation for scalar linear systems. *Automatica*, *144*, 110301. https://doi.org/10.1016/j.automatica.2022.110301

[7] Al Hayek, N. (2021). Parameter Estimation for Discrete Laplace Distribution. *Lobachevskii Journal of Mathematics*, *42*(1), 368–373. https://doi.org/10.1134/s1995080221020116

[8] Lawrence, J. (2013). Distribution of the Median in Samples from the Laplace Distribution. *Open Journal of Statistics*, *03*(06), 422–426. https://doi.org/10.4236/ojs.2013.36050

[9] Fischer, A., Gaunt, R.E., & Sarantsev, A. (2024). Modified method of moments for generalized Laplace distributions. *Communications in Statistics - Simulation and Computation*, 1–18. https://doi.org/10.1080/03610918.2024.2307463

[10] Afful, R.B. (2020). *Statistical Inference for the Discrete Laplace Distribution* [Master's theses, The University of Regina]. https://ourspace.uregina.ca/handle/10294/9338

[11] Fisz, M. (1967). *Probability Theory and Mathematical Statistics* (3rd ed.). John Wiley & Sons, Inc.

[12] Dorozhovets, M. (2020). Forward and inverse problems of Type A uncertainty evaluation. *Measurement*, *165*, 108072. https://doi.org/10.1016/j.measurement.2020.108072

[13] Bain, L.J., & Engelhardt, M. (1973). Interval Estimation for the Two-parameter Double Exponential Distribution. *Technometrics*, *15*(3), 875–887. https://doi.org/10.1080/00401706.1973.10489120

[14] Dorozhovets, M. (2021). Exact distributions and interval estimation of the parameters of double exponential (Laplace) population for a small number of observations. *Measurement*, *182*, 108857. https://doi.org/10.1016/j.measurement.2020.108857

[15] Kappenman, R.F. (1975). Conditional Confidence Intervals for Double Exponential Distribution Parameters. *Technometrics*, *17*(1), 233–235. https://doi.org/10.2307/1268356

[16] Alrasheedi, M.A. (2012). Confidence Intervals for Double Exponential Distribution: A Simulation Approach. *World Academy of Science, Engineering and Technology, International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering*, *6*(1), 84–88.

[17] Iliopoulos, G., & MirMostafaee, S.M.T.K. (2013). Exact prediction intervals for order statistics from the Laplace distribution based on the maximum-likelihood estimators. *Statistics*, *48*(2), 575–592. https://doi.org/10.1080/02331888.2013.766795

[18] Kang, S.B., Cho, Y.S., & Han, J.T. (2005). Estimation for the double exponential distribution based on Type-II censored samples. *Journal of the Korean Data and Information Science Society*, *16*(1), 115–126.

[19] Childs, A., & Balakrishnan, N. (2000). Conditional inference procedures for the Laplace distribution when the observed samples are progressively censored. *Metrika*, *52*(2), 253–265. https://doi.org/10.1007/s001840000092

[20] Iliopoulos, G., & Balakrishnan, N. (2011). Exact likelihood inference for Laplace distribution based on Type-II censored samples. *Journal of Statistical Planning and Inference*, *141*(2), 1224–1239. https://doi.org/10.1016/j.jspi.2010.09.024

[21] Tafiadi, M., & Iliopoulos, G. (2017). Exact inference for the difference of Laplace location parameters. *Metrika*, *80*(6–8), 829–861. https://doi.org/10.1007/s00184-017-0630-3

[22] Joint Committee for Guides in Metrology. (2008). *Evaluation of measurement data – Supplement 1 to the Guide to the Expression of Uncertainty in Measurement' – propagation of distributions using a Monte Carlo method* (JCGM 101:2008).

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