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Observers for unobservable linear systems

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Observers for unobservable linear systems $\dot{x} = Ax + Bu$, $y = Cx$, $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ are proposed. It is shown that there exist full-order and reduced-order observers for systems satisfying the condition rank $\begin{bmatrix} A \\ C \end{bmatrix} = n$. Procedures for computation of the matrices of the observers are given and illustrated by numerical examples.

Key words: observability, eigenvalues assignment, linear system, feedback, procedure

1. Introduction

The concepts of the controllability and observability introduced by Kalman [9, 10] have been the basic notions of the modern control theory. It well-known that if the linear system is controllable then by the use of state feedbacks it is possible to modify the dynamical properties of the closed-loop systems [1,2,8,11,12,16,19]. If the linear system is observable then it is possible to design an observer which reconstructs the state vector of the system [1, 2, 5, 8, 10–17, 19].

The concept of observers for linear systems has been proposed by David G. Luenberger in [13] and extended to reduced order observers in [13, 14]. In the paper [13] he has shown that an identity observer can be designed to have arbitrary dynamics if the original system is completely observable and in [14, 15] the considerations have been extended to multi-input multi-output linear systems and reduced order observers. Design of a robust finite-time observer for a class of uncertain nonlinear systems has been proposed in [17].

Descriptor systems of integer and fractional order have been analyzed in [5–7, 11, 12]. The stabilization of positive descriptor fractional linear systems with

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two different fractional order by decentralized controller have been investigated in [18].

In this paper procedures for design of full-order and reduced-order observers will be proposed. Some preliminaries on the solution of the matrix algebraic equations and the controllability and observability of linear systems are recalled in Section 2. Two methods for design of the full-order observers for unobservable linear system are proposed in Section 3. In Section 4 a method for designing reduced-order observers is given. Concluding remarks will be given in Section 5.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n \times m$ real matrices, I_n – the $n \times n$ identity matrix.

2. Preliminaries

Consider the matrix equation

$$PX = Q, \tag{1}$$

where $P \in \Re^{n\times m}$, $Q \in \Re^{n\times p}$ are given and $X \in \Re^{m\times p}$ is unknown matrix.

Theorem 1. *The matrix equation (1) has a solution X if and only if*

$$\operatorname{rank}\begin{bmatrix} P & Q \end{bmatrix} = \operatorname{rank} P \tag{2}$$

Proof follows immediately from the Kronecker-Cappelli theorem [3, 16].

Consider the liner continuous-time systems

$$\dot{x} = Ax + Bu, \tag{3a}$$

$$y = Cx, \tag{3b}$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$.

Definition 1 ([1, 2, 4, 9–12, 16, 19]). *The system (3) (the pair (A, B)) is called controllable if there exists an input $u(t) \in \Re^m$, $t \in \begin{bmatrix} 0 & t_f \end{bmatrix}$ which steers the state of the system from initial state $x(0) \in \Re^n$ to the given final state $x_f = x(t_f)$.*

Theorem 2 ([1,2,4,9–12,16,19]). *The system (3) (the pair (A, B)) is controllable if and only if one of the following conditions is satisfied:*

1. Kalman condition

$$\operatorname{rank}\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = n. \tag{4a}$$

2. Hautus condition

$$\operatorname{rank}\begin{bmatrix} I_n s - A & B \end{bmatrix} = n \ \textit{for} \ \ s \in f \ \ (f \textit{ is the field of complex numbers}). \tag{4b}$$

Definition 2 ([1,2,4,9–12,16,19]). *The linear system (3) is called observable if knowing its input* $u(t) \in \Re^m$ *and its output* $y(t) \in \Re^p$ *for* $t \in \begin{bmatrix} 0 & t_f \end{bmatrix}$ *it is possible find its unique initial condition* $x(0) \in \Re^n$.

Theorem 3 ([1,2,4,9–12,16,19]). *The system (3) (the pair* (A, C)*) is observable if and only if one of the following conditions is satisfied:*

1. Kalman condition

$$\operatorname{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n. \tag{5a}$$

2. Hautus condition

$$\operatorname{rank}\begin{bmatrix} I_n s - A \\ C \end{bmatrix} = n \quad \textit{for} \;\; s \in f \quad (f \textit{ is the field of complex numbers}). \tag{5b}$$

3. Full order observers

3.1. Method 1

Consider the liner continuous-time systems (3).

It is assumed that the pair (A, C) is not observable but satisfies the condition

$$\operatorname{rank}\begin{bmatrix} A \\ C \end{bmatrix} = n. \tag{6}$$

We are looking for full-order observer of the system (1) in the form

$$\dot{\hat{x}} = F\hat{x} + Gu + Hy, \tag{7}$$

where $\hat{x} = \hat{x}(t) \in \Re^n$, $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ and $F \in \Re^{n\times n}$, $G \in \Re^{n\times m}$, $H \in \Re^{n\times p}$.

Let

$$e = x - \hat{x}. \tag{8}$$

Substituting (3) and (7) into $\dot{e} = \dot{x} - \dot{\hat{x}}$ we obtain

$$\dot{e} = Ax + Bu - F\hat{x} - Gu - HCx = (A - HC)x - F\hat{x} + (B - G)u = Fe \tag{9}$$

if and only if

$$F = A - HC \tag{10a}$$

and

$$G = B. \tag{10b}$$

To design the observer the matrix H should be chosen so that the matrix F is asymptotically stable with desired eigenvalues located in the left-hand side of the complex plane.

Knowing the matrices A, F and C we may find the matrix H from the equation

$$HC = A - F \tag{11}$$

if and only if

$$\operatorname{rank}\begin{bmatrix} A^T - F^T & C^T \end{bmatrix} = \operatorname{rank} C^T. \tag{12}$$

Therefore, the following theorem has been proved.

Theorem 4. *There exists the full-order observer (7) of the system (3) if the conditions (6) and (12) are satisfied.*

If the conditions of Theorem 4 are satisfied then using the following procedure we may find the desired observer.

Procedure 1. ***Step 1***. *Choose the asymptotically stable matrix F with desired eigenvalues $Res_k < 0$, $k = 1, \ldots, n$.*

Step 2. *Check the condition (12) for the chosen matrix F.*

Step 3. *Compute the desired matrix $H \in \Re^{n \times p}$ using (10a).*

Example 1. Consider the system (3) with the matrices

$$A = \begin{bmatrix} -6 & 1 & 0 \\ 2 & -5 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}. \tag{13}$$

The pair (A,C) given by (13) is unobservable

$$\operatorname{rank}\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ -3 & -3 & 1 \end{bmatrix} = 2 \tag{14}$$

but it satisfies the condition (6)

$$\operatorname{rank}\begin{bmatrix} A \\ C \end{bmatrix} = \operatorname{rank}\begin{bmatrix} -6 & 1 & 0 \\ 2 & -5 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = 3. \tag{15}$$

Using Procedure 1 we obtain the following,

Step 1. In this case we choose the asymptotically stable matrix F in the form

$$F = \begin{bmatrix} -6 & 1 & 2 \\ 2 & -5 & 1 \\ 1 & 1 & -6 \end{bmatrix}. \tag{16}$$

Step 2. The condition (12) for the matrices A, C, F given by (13) and (16) is satisfied since

$$\operatorname{rank}\begin{bmatrix}A^T - F^T & C^T\end{bmatrix} = \operatorname{rank}\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -3 & -1 & 7 & 1\end{bmatrix} = \operatorname{rank} C^T = \operatorname{rank}\begin{bmatrix}0\\0\\1\end{bmatrix} = 1. \tag{17}$$

Step 3. Using (10a), (13) and (16) we obtain

$$C^T H^T = \begin{bmatrix}0\\0\\1\end{bmatrix}\begin{bmatrix}h_1 & h_2 & h_3\end{bmatrix}^T = A^T - F^T = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ -3 & -1 & 7\end{bmatrix} \tag{18a}$$

and

$$H^T = \begin{bmatrix}h_1 & h_2 & h_3\end{bmatrix}^T = \begin{bmatrix}-3 & -1 & 7\end{bmatrix}. \tag{18b}$$

Therefore, the desired observer has the form

$$\dot{\hat{x}} = F\hat{x} + Gu + Hy = \begin{bmatrix}-6 & 1 & 2\\ 2 & -5 & 1\\ 1 & 1 & -6\end{bmatrix}\hat{x} + \begin{bmatrix}1\\1\\0\end{bmatrix}u + \begin{bmatrix}-3\\-1\\7\end{bmatrix}y. \tag{19}$$

3.2. Method 2

This method is based on the following assumption:

Assumption 1. *For the given unobservable pair $(A,\ C)$ satisfying the condition (6) there exists nonsingular matrix $M \in \Re^{n\times n}$ such that*

$$\begin{bmatrix}A\\C\end{bmatrix}M = \begin{bmatrix}\bar{A}\\ \bar{C}\end{bmatrix} \tag{20}$$

and the pair $(\bar{A},\ \bar{C})$ is observable.

Note that for large class of unobservable pairs $(A,\ C)$ satisfying (6) there exists a nonsingular permutation M such that (20) holds.

To design the desired full-order observer the classical method can be applied [1, 5, 8, 13, 15].

Example 2. Design the observer for uncontrollable and unobservable system (3) with the matrices

$$A = \begin{bmatrix}1 & 0\\ 1 & 2\end{bmatrix}, \qquad B = \begin{bmatrix}0\\1\end{bmatrix}, \qquad C = \begin{bmatrix}1 & 0\end{bmatrix} \tag{21}$$

Note that the pair $(A,\ C)$ given by (21) satisfies the condition (6) since

$$\operatorname{rank}\begin{bmatrix} A \\ C \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 0 \end{bmatrix} = 2. \tag{22}$$

Let

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \qquad \bar{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}. \tag{23}$$

In this case the desired nonsingular matrix M has the form

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{24}$$

and

$$\begin{bmatrix} \bar{A} \\ \bar{C} \end{bmatrix} = \begin{bmatrix} A \\ C \end{bmatrix} M = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}. \tag{25}$$

In this case the pair (23) is observable and using the well known method we may design for this pair the full-order observer.

4. Reduced order observers

Consider the linear system (3) with the matrix $C \in \Re^{p\times n}$ satisfying the assumption

$$\operatorname{rank} C = \operatorname{rank} C_1 = p, \tag{26}$$

where $C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$, $C_1 \in \Re^{p\times p}$.

If $\det C_1 = 0$ and $\operatorname{rank} C = p$ then by permutation of the columns we may obtain $\det C_1 \neq 0$.

If the condition (26) is satisfied then the matrix

$$V = \begin{bmatrix} C_1^{-1} & -C_1^{-1}C_2 \\ 0 & I_{n-p} \end{bmatrix} \in \Re^{n\times n} \tag{27}$$

is nonsingular and

$$\bar{C} = CV = \begin{bmatrix} C_1 & C_2 \end{bmatrix}\begin{bmatrix} C_1^{-1} & -C_1^{-1}C_2 \\ 0 & I_{n-p} \end{bmatrix} = \begin{bmatrix} I_n & 0 \end{bmatrix}. \tag{28}$$

We define the new state vector

$$\bar{x} = V^{-1}x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad x_1 \in \Re^{p}, \quad x_2 \in \Re^{n-p} \tag{29}$$

and applying the linear transformation (27) to the system (6) we obtain

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u, \tag{30a}$$

$$y = \bar{C}\bar{x} = x_1\,, \tag{30b}$$

where

$$\begin{aligned} \bar{A} &= V^{-1}AV = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{11} \in \Re^{p\times p}, \quad A_{22} \in \Re^{(n-p)\times(n-p)}, \\ \bar{B} &= V^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad B_1 \in \Re^{p\times m}, \quad B_2 \in \Re^{(n-p)\times m}. \end{aligned} \tag{30c}$$

From (30c) it follows that the vector $y = x_1$ is known and the reduced order observer should reconstruct only the sub-vector $x_2 \in \Re^{n-p}$.

From (30) we have

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1u, \tag{31a}$$

$$\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2u. \tag{31b}$$

Substituting $y = x_1$ to (31b) we obtain

$$\dot{x}_2 = A_{22}x_2 + \bar{u}, \tag{32a}$$

$$\bar{y} = A_{12}x_2\,, \tag{32b}$$

where

$$\bar{u} = B_2u + A_{21}y, \qquad \bar{y} = \dot{y} - A_{11}y - B_1u. \tag{32c}$$

Therefore, the reconstruction of the vector $\bar{x}$ has been reduced to finding the sub-vector x_2 by the reduced order observer described by the equations (32).

Lemma 1. *The pair* $(A_{22},\ A_{12})$ *satisfies the condition*

$$\operatorname{rank}\begin{bmatrix} A_{22} \\ A_{12} \end{bmatrix} = p \tag{33}$$

if and only if

$$\operatorname{rank}\begin{bmatrix} A \\ C \end{bmatrix} = n. \tag{34}$$

Proof. From (28) and (32) we have

$$\begin{aligned} \operatorname{rank}\begin{bmatrix} A \\ C \end{bmatrix} &= \operatorname{rank}\left\{\begin{bmatrix} V^{-1} & 0 \\ 0 & I_{n-p} \end{bmatrix}\begin{bmatrix} A \\ C \end{bmatrix}\begin{bmatrix} V & 0 \\ 0 & I_{n-p} \end{bmatrix}\right\} = \operatorname{rank}\begin{bmatrix} \bar{A} \\ \bar{C} \end{bmatrix} \\ &= \operatorname{rank}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ I_p & 0 \end{bmatrix} = p + \operatorname{rank}\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}. \end{aligned} \tag{35}$$

Therefore, the pair $(A_{22},\ A_{12})$ satisfies the condition (33) if and only if the pair $(A,\ C)$ satisfies the condition (34). □

Remark 1. *If the pair $(A,\ C)$ satisfies the condition (6) then the pair $(A_{22},\ A_{12})$ can be observable.*

The matrix H is chosen so that that the eigenvalue of the matrix

$$F = A_{22} - HA_{12} \tag{36}$$

are in the left hand side of the complex plane, enough far from the origin of the complex plane.

To design the reduced-order observer the following procedure can be used.

Procedure 2. ***Step 1***. *Using the matrix (27) transform the matrix C to the form (28) and compute the matrices (30c).*

Step 2. *Choose the desired eigenvalues $s_1, s_2, \ldots, s_{n-p}$ in left-hand side of the complex plane and compute the matrix H satisfying the equation*

$$\det\left[I_{n-p}s - A_{22} + HA_{12}\right] = p(s) = (s - s_1)(s - s_2)\ldots(s - s_{n-p}). \tag{37}$$

Step 3. *Knowing the matrix H find the desired equation of the reduce-order observer.*

Example 3. Find the reduce-order observer for the system (3) with the matrices (13).

Using Procedure 2 we obtain the following.

Step 1. In this case $p = 1$ and taking into account the form of matrix $C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ we choose $\bar{x}_1 = x_3$.

In this case we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = A_{22}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + B_2 u, \tag{38a}$$

$$\bar{y} = C_2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{38b}$$

where

$$A_{22} = \begin{bmatrix} -6 & 1 \\ 1 & -5 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \qquad \bar{y} = \dot{y} - y. \tag{38c}$$

Note that the pair $(A_{22},\ B_2)$ s controllable and the pair $(A_{22},\ C_2)$ is observable since

$$\begin{aligned}\operatorname{rank}\begin{bmatrix}B_2 & A_{22}B_2\end{bmatrix} &= \operatorname{rank}\begin{bmatrix}1 & -5\\ 1 & -4\end{bmatrix} = 2,\\ \operatorname{rank}\begin{bmatrix}C_2\\ C_2A_{22}\end{bmatrix} &= \operatorname{rank}\begin{bmatrix}1 & 1\\ -5 & -4\end{bmatrix} = 2.\end{aligned} \tag{39}$$

Step 2. We choose the eigenvalues of the observer $s_1 = s_2 = -10$ and using (37) we obtain

$$\begin{aligned}\det\begin{bmatrix}I_2s - A_{22} + HC_2\end{bmatrix} &= \begin{vmatrix}s+6+h_1 & h_1-1\\ h_2-1 & s+5+h_2\end{vmatrix}\\ &= s^2 + (11+h_1+h_2)s + 6h_1 + 7h_2 + 29\\ &= (s+10)^2 = s^2 + 20s + 100.\end{aligned} \tag{40}$$

From comparison of the coefficients we have

$$\begin{bmatrix}1 & 1\\ 6 & 7\end{bmatrix}\begin{bmatrix}h_1\\ h_2\end{bmatrix} = \begin{bmatrix}9\\ 71\end{bmatrix} \tag{41}$$

and

$$H = \begin{bmatrix}h_1\\ h_2\end{bmatrix} = \begin{bmatrix}1 & 1\\ 6 & 7\end{bmatrix}^{-1}\begin{bmatrix}9\\ 71\end{bmatrix} = \begin{bmatrix}-8\\ 17\end{bmatrix}. \tag{42}$$

Step 3. The desired observer has the form

$$\begin{aligned}\dot{\hat{x}}_2 &= (A_{22} - HC_2)\hat{x} + B_2u + H\dot{y}\\ &= \begin{bmatrix}2 & 9\\ -16 & -22\end{bmatrix}\hat{x} + \begin{bmatrix}1\\ 1\end{bmatrix}u + \begin{bmatrix}-8\\ 17\end{bmatrix}\dot{y}.\end{aligned} \tag{43}$$

5. Concluding remarks

Procedures for design full-order and reduced-order observers for unobservable linear continuous-time systems have been proposed. Two Procedures for designing of the full-order observers for unobservable linear system have been proposed. A method for designing reduced-order observers has been also given. The presented methods can be extended to discrete-time linear system and to descriptor linear continuous-time and discrete-time systems. An open problem is an extension of the methods to fractional order linear systems.

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