10.24425/acs.2025.153962

Archives of Control Sciences
Volume 35(LXXI), 2025
No. 1*, pages 155–199*

A new score function for optimizing transportation problems in complex Pythagorean fuzzy environment

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Transportation problems are a crucial area in logistics and operations research that focuses on effectively transporting products and resources from origin to destination. The complex nature of real-world situations requires the utilization of advanced mathematical models and optimization approaches to formulate effective solutions. Quantifying the accurate supply, demand, and transportation costs of transportation problems is highly challenging due to the market's unpredictable economic and environmental conditions. Several methodologies have been proposed to address the issue of transportation problems with uncertain parameters, utilizing fuzzy theory and its derivatives. Complex Pythagorean fuzzy numbers (CPyFNs) are highly suitable and effective for representing uncertain and ambiguous information in real-world scenarios. This article introduces an innovative analysis of transportation problems in the complex Pythagorean fuzzy information framework. Initially, we establish an improved score function value to evaluate the ordering of CPyFNs more precisely and propose several ranking rules. The following section of the study presents mathematical formulations and optimization models for transportation problems, where the parameters are expressed as CPyFNs. The solution approach is formulated using Vogel's approximation method (VAM) and modified distribution method (MODI) to solve transportation problems in a complex Pythagorean fuzzy setting. An illustrative numerical example is provided to showcase the practicality and effectiveness of the proposed method in

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Rajkumar Verma is grateful for the financial support provided by the Universidad de Talca through FONDO PARA ATRACCIÓN DE INVESTIGADORES POSTDOCTORALES.

Received 14.9.2024.

real-life scenarios. The significance of the research and the extent of future investigation are also emphasized.

Key words: transportation problem, complex Pythagorean fuzzy numbers, minimum transportation cost, Vogel's approximation method, modified distribution method

1. Introduction

The optimization of transportation systems is essential in the fields of logistics and operations research. Its objective is to streamline the movement of goods and resources from their point of origin to their destination. This approach is crucial for reducing expenses, optimizing productivity, and guaranteeing a smooth movement of commodities throughout various supply chain networks. Addressing the escalating complexities of dynamic global circumstances, transportation experts are working towards creating strategies that provide effective, safe, and dependable transportation services while minimizing negative impacts on the environment and local communities. Transportation network management encounters various challenges in real-world scenarios, including insufficient safety records, capacity limitations, technological advancements, productivity enhancements, market fluctuations, unreliability, and safety apprehensions. Overcoming these obstacles is crucial for effectively delivering goods to consumers. The transportation problem is a specific type of linear programming problem that is pivotal in addressing real-world challenges. The main goal of a transportation problem is to maximize the efficiency of moving goods by minimizing transportation costs between different origins and destinations. In the present scenario, the transportation problem has become a standard application for an industrial organization with several manufacturing and distribution centers. Initially, Hitchcock [24] mathematically formulated the transportation problem in 1941. After that, Koopmans [31] provided its modified formulation, referred to as the Hitchcock-Koopmans transportation problem. In 1951, Dantzig [17] developed a simplex transportation scheme for dealing with transportation problems more efficiently. Later, several researchers studied transportation problems and developed different algorithms for finding optimal solutions [7,12,51,62]. In general, it has been considered that the input parameters (like supply, demand, and cost units) are expressed in crisp numbers. However, in real-world situations, the parameters associated with the transportation problem are imprecise and vague due to the involvement of different kinds of uncertainties and lack of sufficient information. Zadeh [67] proposed the theory of fuzzy sets (FSs) for more precisely dealing with vague and imprecise information. FS theory has been extensively applied in different areas to handle many complex problems with uncertain and vague data [8, 10, 55, 58].

A fuzzy transportation problem is a special type of transportation problem that involves input parameters represented as fuzzy numbers. Researchers have endeavored to deal with several categories of transportation problems using imprecise and uncertain data. Chanas et al. [15] introduced a method based on parametric programming to address the challenges associated with fuzzy transportation problem. Geetha and Nair [23] engaged in a discourse over a stochastic iteration of the transportation problem. Chanas and Kuchta [14] proposed the notion of an optimal solution for the transportation problem with fuzzy coefficients, which are expressed as fuzzy numbers. They developed an algorithm for obtaining the optimal solution. Liu and Kao [34] proposed a continuous optimization approach based on the extension principle to address fuzzy transportation problems. Gani and Razak [20] employed trapezoidal fuzzy numbers to depict supply and demand in two-stage fuzzy transportation problems. Pandian and Natarajan [41] devised a solution methodology for the fuzzy transportation problem that incorporates mixed constraints. Kaur and Kumar [29] expanded upon traditional transportation methods to address transportation issues within a fuzzy context. Subsequently, Kaur and Kumar [30] engaged in a discussion regarding unbalanced fuzzy transportation difficulties. Mathur et al. [39] introduced a novel approach for solving transportation problem using trapezoidal fuzzy numbers.

In 1986, Atanassov [9] generalized the idea of fuzzy sets and proposed the theory of intuitionistic fuzzy sets (IFSs) to describe human cognitive information more efficiently. An IFS characterizes an element by a membership degree (MD) and non-membership degree (NMD) satisfying the condition $MD + NMD \leqslant 1$. Since its proposal, many researchers have shown great interest in intuitionistic fuzzy set theory and its applications to various problems related to different application areas [5, 37, 60, 66]. Hussain and Kumar [26] and Gani and Abbas [19] proposed solution methods for transportation problems with intuitionistic fuzzy parameters. [52] developed an efficient approach to handle type-1 intuitionistic fuzzy transportation problems. In addition, Singh and Yadav [53] solved type-2 intuitionistic fuzzy transportation problem using a novel ranking approach. Mahmoodirad et al. [38] formulated a new approach for fully intuitionistic fuzzy transportation problems. Ebrahimnejad and Verdegay [18] suggest an innovative technique to resolve a fully intuitionistic fuzzy transportation problem. Roy et al. [47] developed novel methods for handling transportation problems in an intuitionistic fuzzy framework with single and multi-objectives. Chhibber et al. [16] utilized the TOPSIS approach for solving multiobjective non-linear transportation problems with intuitionistic fuzzy parameters. Beg et al. [11] developed a new technique to solve intuitionistic fuzzy transportation problems.

From these studies, it is clear that the IFS theory has extensive application in several areas. Note that the IFS theory is valid under the condition that the sum of the degrees of membership and non-membership is equal to or less than 1. However, in practical applications, there are many situations where a person gives their assessment information in membership and non-membership degrees towards a particular object as 0.8 and 0.6. Then clearly, this situation cannot be described by using IFS because of $0.8 + 0.6 \nleqslant 1$. To resolve this shortcoming, Yager [64, 65] proposed the notion of the Pythagorean fuzzy set (PyFS) as a new generalization of the IFS theory by relaxing the condition $MD + NMD \leqslant 1$ to $MD^2 + NMD^2 \leqslant 1$. In the last few years, several studies have been reported in the literature, including mathematical results and the applications of PyFSs after the appearance of Yager's work. For instance, Zhang and Xu [69] extended the TOPSIS method in the Pythagorean fuzzy context. Peng and Yang [42] defined subtraction and division operations on PyFSs. Ma and Xu [36] proposed symmetric aggregation operators (AOs) for aggregating Pythagorean fuzzy numbers (PyFNs). Garg [21] introduced correlation coefficients between PyFSs. Zeng et al. [68] defined various distance an similarity measures between PFSs. Peng et al. [43] presented a detailed study on information measures under the Pythagorean fuzzy environment. Verma and Merigó [59] proposed generalized similarity measures with Pythagorean fuzzy information for solving multiple attribute decision-making (MADM) problems. Rani et al. [46] extended the VIKOR approach using entropy and divergence measures in a Pythagorean fuzzy context. Jana et al. [27] proposed the novel Dombi AOs with Pythagorean fuzzy information and discussed their application in MADM. Sarkar and Biswas [49] developed an integrated approach to transportation management using AHP-TOPSIS with Pythagorean fuzzy information. Verma and Mittal [61] defined Pythagorean fuzzy probabilistic ordered weighted cosine similarity measures and utilized them to solve multiple attribute group decision-making (MAGDM) problems. The Pythagorean fuzzy transportation problem was first studied by Kumar et al. [32] in 2019. Umamageswari and Uthra [57] developed a new solution for transportation problems using Pythagorean fuzzy parameters. Nagar et al. [40] proposed a new score function for raking PFNs and used it to solve the Pythagorean fuzzy transportation problem. Sharma et al. [50] utilized a new Fermetean fuzzy score function to solve TPs under the Pythagorean fuzzy environment. Saikia et al. [48] introduced a new similarity measure in Pythagorean fuzzy framework and utilized it to solve transportation optimization models with Pythagorean fuzzy parameters.

As from the above-discussed studies, it has been observed that the FSs, IFSs, and PyFSs have been widely used to solve many complex problems from different application areas. However, these models cannot represent the partial ignorance of the data and its fluctuations at a given time. In complex data sets, uncertainty and vagueness occur concurrently with changes to the phase (periodicity) of the data. To resolve this situation, Ramot et al. [44] proposed the notion of the complex fuzzy set (CFS) in which the membership function of a CFS, restricted to a complex unit circle, consists of two real-valued terms, i.e., amplitude term and phase term. The novelty of CFS is due to the phase term associated with membership, which enables it to handle periodic data. Bi et al. [13] proposed geometric AOs for fusing CFNs. Hu et al. [25] suggested a decision-making approach based on power AOs in a complex fuzzy context. The large-scale learning problems with complex fuzzy data were studied by Sobhi and Dick [54] in 2023. Alkouri and Salleh [6] generalized the theory of CFSs by introducing complex intuitionistic fuzzy sets (CIFSs) and discussed some basic operations associated with them. [22] defined weighted average/weighted geometric AOs for complex intuitionistic fuzzy numbers (CIFNs) and explored their application in decision-making. [45] studied power AOs with CIFNs to handle decision-making issues. [3] discussed the complex intuitionistic fuzzy Hamacher AOs with applications in decision-making scenarios. In 2020, [56] presented the theory of complex Pythagorean fuzzy sets (CPyFSs) as a natural extension of CIFSs. It provides more flexibility to tackle two-dimensional vague information efficiently by relaxing conditions for amplitude and phase terms. [1] defined novel AOs for aggregating complex Pythagorean fuzzy numbers (CPyFNs) based on Dombi operational laws. [2] defined complex Pythagorean fuzzy Yager AOs to deal with MCDM challenges. The VIKOR method was studied by [35] under the complex Pythagorean fuzzy context. [4] proposed an ELECTRE-II decision-making approach for solving the MCGDM problem with complex Pythagorean fuzzy information. [63] defied Hamming and Hausdorff distance measures in the complex Pythagorean fuzzy environment and used them for pattern recognition and medical diagnosis. [28] introduced novel AOs for CPyFNs using Einstein operation laws and discussed their application in the best breed selection of Horsegram. [33] studied new AOs for CPyFNs with confidence levels based on Archimedean operational laws.

All the existing transportation optimization models based on IFS and PFS characterized the uncertainties and vagueness of supply, demand, and transportation cost parameters in terms of MDs and NDs, which may cause the loss of some useful information due to the absence of valuable phase terms. It is significantly necessary to develop optimization models that improve efficiency in solving transportation problems, with parameter information expressed in both amplitude and phase terms. Therefore, this manuscript aims to develop more efficient transportation optimization models under the complex Pythagorean fuzzy information environment to resolve complex and dynamic transportation problems more efficiently.

The notable research contributions of the study are summarized as follows:

- An improved score function value based on the logarithmic function for CPyFNs is proposed to overcome the deficiencies of the existing score function value given in [56].
- We develop transportation optimization models in a complex Pythagorean fuzzy framework to handle transportation problems with parameters expressed in terms of CPyFNs.
- A solution algorithm is designed to solve complex Pythagorean fuzzy transportation problems (CPyFTPs) using the suggested score function value.
- A Rel-life numerical example is also given to illustrate the whole working process of the developed algorithm in practical situations to handle different types of CPyFTPs. Additionally, the superiority of the proposed transportation optimization models is established through the comparative analysis with existing work.

The subsequent sections of the paper are organized in the following manner: Section 2 provides a review of fundamental concepts and definitions related to PyFSs and CPyFSs. Section 3 presents a new score function value for CPyFNs and provides proof of its fundamental properties to establish its validity. The work also includes a comparison with existing score function value to demonstrate the need for and superiority of the suggested score function value. Section 4 proposes mathematical optimization models for different types of transportation problems, where the parameters are specified in terms of CPyFNs. This paper also offers solution algorithm to resolve the formulated transportation models using the suggested score value function. In Section 5 we provide practical numerical examples to demonstrate the operational procedures of the established algorithm and validate their suitability in real-world scenarios. Section 6 comprises concluding thoughts along with future directions.

2. Preliminaries

This section provides a concise overview of fundamental concepts and definitions necessary for further advancement of the study.

2.1. Pythagorean fuzzy set

Definition 1. [64, 65] *A PyFS* $\mathbb{P}$ *defined in a universe of discourse* $\mathscr{G} = \{g_1, g_2, g_n\}$ *is represented by*

$$\mathbb{P} = \Big\{ \langle g, \xi_{\mathbb{P}}(g), \eta_{\mathbb{P}}(g) \rangle \mid g \in \mathscr{G} \Big\}, \tag{1}$$

where $\xi_{\mathcal{P}}: X \longrightarrow [0,1]$ *and* $\eta_{\mathcal{P}}: X \longrightarrow [0,1]$ *symbolize the MemD and NMemD of an element* $g \in \mathcal{G}$ *to the set* $\mathbb{P}$*, respectively, with the condition* $0 \leqslant (\xi_{\mathbb{P}}(g))^2 + (\eta_{\mathbb{P}}(g))^2 \leqslant 1 \; \forall \, g \in \mathcal{G}$.

For any $g \in \mathcal{G}$, the hesitancy degree (HesD) is denoted by

$$\psi_{\mathbb{P}}(g) = \sqrt{1 - (\xi_{\mathbb{P}}(g))^2 - (\eta_{\mathbb{P}}(g))^2}.$$

For a given element $g \in \mathcal{G}$, the pair $\langle \xi_{\mathbb{P}}(g), \eta_{\mathbb{P}}(g) \rangle$ is known as a PyFN and represented by $\aleph = \langle \xi_{\aleph}, \eta_{\aleph} \rangle$ where $\xi_{\aleph}, \; \eta_{\aleph} \in [0,1]$ and $(\xi_{\aleph})^2 + (\eta_{\aleph})^2 \leqslant 1$.

2.2. Complex Pythagorean fuzzy set

Definition 2. [56] *Let* $\mathcal{G} = \{g_1, g_2, \ldots, g_n\}$ *be a universe of discourse. A CPyFS* $\mathbb{CP}$ *defined in* $\mathcal{G}$ *is an object of the form given by*

$$\mathbb{CP} = \left\{ \left\langle g_j, \xi_{\mathbb{CP}}(g_j)\, e^{i2\pi\alpha_{\mathbb{CP}}(g_j)}, \eta_{\mathbb{CP}}(g_j)\, e^{i2\pi\beta_{\mathbb{CP}}(g_j)} \right\rangle | g_j \in \mathcal{G} \right\}, \tag{2}$$

where $\xi_{\mathbb{CP}}(g_j) e^{i2\pi\alpha_{\mathbb{CP}}(g_j)}$ *and* $\eta_{\mathbb{CP}}(g_j)\, e^{i2\pi\beta_{\mathbb{CP}}(g_j)}$ *denote the complex grades of MemD and NMemD of an element* $g \in \mathcal{G}$ *to the set* $\mathbb{CP}$*, respectively, such that* $0 \leqslant (\xi_{\mathbb{CP}}(g_j))^2 + (\eta_{\mathbb{CP}}(g_j))^2 \leqslant 1$ *and* $0 \leqslant (\alpha_{\mathbb{CP}}(g_j))^2 + (\beta_{\mathbb{CP}}(g_j))^2 \leqslant 1$ $\forall \, g_j \in \mathcal{G}$.

The real-valued functions $\xi_{\mathbb{CP}}$ and $\eta_{\mathbb{CP}}$ describe the amplitude terms of MemD and NMemD, respectively. On the other hand, the real-valued functions $\alpha_{\mathbb{CP}}$ and $\beta_{\mathbb{CP}}$ represent the phase terms of MemD and NMemD, respectively. The HesD for a given element $g_j \in \mathcal{G}$ is expressed by $\psi_{\mathbb{CP}}(g_j) e^{i2\pi\gamma_{\mathbb{CP}}(g_j)} = \sqrt{1 - (\xi_{\mathbb{P}}(g_j))^2 - (\eta_{\mathbb{P}}(g_j))^2} e^{2i\pi\sqrt{1-(\alpha_{\mathbb{P}}(g_j))^2-(\beta_{\mathbb{P}}(g_j))^2}}$. For convenience, we consider the pair $\left\langle \xi_{\mathbb{CP}}(g_j) e^{2i\pi\alpha_{\mathbb{CP}}(g_j)}, \eta_{\mathbb{CP}}(g_j) e^{2i\pi\beta_{\mathbb{CP}}(g_j)} \right\rangle$ as a CPyFN and expressed it by $\widetilde{C} = \langle \xi_{\widetilde{C}} e^{i2\pi\alpha_{\widetilde{C}}}, \eta_{\widetilde{C}} e^{i2\pi\beta_{\widetilde{C}}} \rangle$. Let $\Omega_{\widetilde{C}}$ denote the collection of all CyPFNs.

Definition 3. [56] *Let*

$$\mathbb{CP} = \left\{ \left\langle g_j, \xi_{\mathbb{CP}}(g_j) e^{i2\pi\alpha_{\mathbb{CP}}(g_j)}, \eta_{\mathbb{CP}}(g_j) e^{i2\pi\beta_{\mathbb{CP}}(g_j)} \right\rangle | g_j \in \mathcal{G} \right\},$$

$$\mathbb{CP}1 = \left\{ \left\langle g_j, \xi_{\mathbb{CP}1}(g_j) e^{i2\pi\alpha_{\mathbb{CP}1}(g_j)}, \eta_{\mathbb{CP}1}(g_j) e^{i2\pi\beta_{\mathbb{CP}1}(g_j)} \right\rangle | g_j \in \mathcal{G} \right\} \text{ and}$$

$$\mathbb{CP}2 = \left\{ \left\langle g_j, \xi_{\mathbb{CP}2}(g_j) e^{i2\pi\alpha_{\mathbb{CP}2}(g_j)}, \eta_{\mathbb{CP}2}(g_j) e^{i2\pi\beta_{\mathbb{CP}2}(g_j)} \right\rangle | g_j \in \mathcal{G} \right\}$$

be three CPyFSs defined in th universe of discourse $\mathcal{G}$*. Then set operational laws on CPyFSs can be described as follows:*

(i) $\mathbb{CP}^C = \left\{\left\langle g_j, \eta_{\mathbb{CP}}\left(g_j\right)e^{i2\pi\beta_{\mathbb{CP}}\left(g_j\right)}, \xi_{\mathbb{CP}}\left(g_j\right)e^{i2\pi\alpha_{\mathbb{CP}}\left(g_j\right)}\right\rangle | g_j \in \mathcal{G}\right\}$;

(ii) $\mathbb{CP}1 \subseteq \mathbb{CP}2$ *if* $\xi_{\mathbb{CP}1}\left(g_j\right) \leqslant \xi_{\mathbb{CP}2}\left(g_j\right)$, $\eta_{\mathbb{CP}1}\left(g_j\right) \geqslant \eta_{\mathbb{CP}2}\left(g_j\right)$, $\alpha_{\mathbb{CP}1}\left(g_j\right) \leqslant \xi_{\mathbb{CP}2}\left(g_j\right)$, *and* $\beta_{\mathbb{CP}1}\left(g_j\right) \geqslant \beta_{\mathbb{CP}2}\left(g_j\right)$;

(iii) $\mathbb{CP}1 = \mathbb{CP}2$ *if and only if* $\mathbb{CP}1 \subseteq \mathbb{CP}2$ *and* $\mathbb{CP}2 \subseteq \mathbb{CP}1$;

(iv) $$\mathbb{CP}1\cup\mathbb{CP}2 = \left\{\left\langle g_j, \max\left(\xi_{\mathbb{CP}1}\left(g_j\right), \xi_{\mathbb{CP}2}\left(g_j\right)\right) e^{i2\pi\max\left(\alpha_{\mathbb{CP}1}\left(g_j\right),\alpha_{\mathbb{CP}2}\left(g_j\right)\right)},\right.\right.$$
$$\left.\left.\min\left(\eta_{\mathbb{CP}1}\left(g_j\right), \eta_{\mathbb{CP}2}\left(g_j\right)\right) e^{i2\pi\min\left(\beta_{\mathbb{CP}1}\left(g_j\right),\beta_{\mathbb{CP}2}\left(g_j\right)\right)}\right\rangle | g_j \in \mathcal{G}\right\};$$

(v) $$\mathbb{CP}1\cap\mathbb{CP}2 = \left\{\left\langle g_j, \min\left(\xi_{\mathbb{CP}1}\left(g_j\right), \xi_{\mathbb{CP}2}\left(g_j\right)\right) e^{i2\pi\min\left(\alpha_{\mathbb{CP}1}\left(g_j\right),\alpha_{\mathbb{CP}2}\left(g_j\right)\right)},\right.\right.$$
$$\left.\left.\max\left(\eta_{\mathbb{CP}1}\left(g_j\right), \eta_{\mathbb{CP}2}\left(g_j\right)\right) e^{i2\pi\max\left(\beta_{\mathbb{CP}1}\left(g_j\right),\beta_{\mathbb{CP}2}\left(g_j\right)\right)}\right\rangle | g_j \in \mathcal{G}\right\}.$$

Definition 4. [56] *Let* $\widetilde{C} = \langle \xi_{\widetilde{C}}e^{i2\pi\alpha_{\widetilde{C}}}, \eta_{\widetilde{C}}e^{i2\pi\beta_{\widetilde{C}}}\rangle$, $\widetilde{C}1 = \langle \xi_{\widetilde{C}1}e^{i2\pi\alpha_{\widetilde{C}1}}, \eta_{\widetilde{C}1}e^{i2\pi\beta_{\widetilde{C}1}}\rangle$ *and* $\widetilde{C}2 = \langle \xi_{\widetilde{C}2}e^{i2\pi\alpha_{\widetilde{C}2}}, \eta_{\widetilde{C}2}e^{i2\pi\beta_{\widetilde{C}2}}\rangle$ *be three CPyFNs. The operational laws on CPyFNs are defined as follows:*

(i) $\widetilde{C}^C = \langle \eta_{\widetilde{C}}e^{i2\pi\beta_{\widetilde{C}}}, \xi_{\widetilde{C}}e^{i2\pi\alpha_{\widetilde{C}}}\rangle$;

(ii) $\widetilde{C}1 \leqslant \widetilde{C}2$ *if* $\xi_{\widetilde{C}1} \leqslant \xi_{\widetilde{C}2}$, $\eta_{\widetilde{C}1} \geqslant \eta_{\widetilde{C}2}$, $\alpha_{\widetilde{C}1} \leqslant \alpha_{\widetilde{C}2}$, *and* $\beta_{\widetilde{C}1} \geqslant \beta_{\widetilde{C}2}$;

(iii) $\widetilde{C}1 = \widetilde{C}2$ *if and only if* $\widetilde{C}1 \leqslant \widetilde{C}2$ *and* $\widetilde{C}2 \leqslant \widetilde{C}1$;

(iv) $$\widetilde{C}1\cup\widetilde{C}2 = \left\langle \max\left(\xi_{\widetilde{C}1}, \xi_{\widetilde{C}2}\right)e^{i2\pi\max(\alpha_{\widetilde{C}1},\alpha_{\widetilde{C}2})}, \min\left(\eta_{\widetilde{C}1}, \eta_{\widetilde{C}2}\right) e^{i2\pi\min(\beta_{\widetilde{C}1},\beta_{\widetilde{C}2})}\right\rangle;$$

(v) $$\widetilde{C}1\cap\widetilde{C}2 = \left\langle \min\left(\xi_{\widetilde{C}1}, \xi_{\widetilde{C}2}\right)e^{i2\pi\min(\alpha_{\widetilde{C}1},\alpha_{\widetilde{C}2})}, \max\left(\eta_{\widetilde{C}1}, \eta_{\widetilde{C}2}\right) e^{i2\pi\min(\beta_{\widetilde{C}1},\beta_{\widetilde{C}2})}\right\rangle;$$

(vi) $$\widetilde{C}1 \bigoplus \widetilde{C}2 = \left\langle\sqrt{\left(\xi_{\widetilde{C}1}\right)^2+\left(\xi_{\widetilde{C}2}\right)^2-\left(\xi_{\widetilde{C}1}\right)^2\left(\xi_{\widetilde{C}2}\right)^2}e^{i2\pi\sqrt{(\alpha_{\widetilde{C}1})^2+(\alpha_{\widetilde{C}2})^2-(\alpha_{\widetilde{C}1})^2(\alpha_{\widetilde{C}2})^2}},\right.$$
$$\left.\eta_{\widetilde{C}1}\eta_{\widetilde{C}2}e^{i2\pi\beta_{\widetilde{C}1}\beta_{\widetilde{C}2}}\right\rangle;$$

(vii) $$\widetilde{C}1 \bigotimes \widetilde{C}2 = \left\langle \xi_{\widetilde{C}1}\xi_{\widetilde{C}12}e^{i2\pi\alpha_{\widetilde{C}1}\alpha_{\widetilde{C}2}},\right.$$
$$\left.\sqrt{\left(\eta_{\widetilde{C}1}\right)^2+\left(\eta_{\widetilde{C}2}\right)^2-\left(\eta_{\widetilde{C}1}\right)^2\left(\eta_{\widetilde{C}2}\right)^2}e^{i2\pi\sqrt{(\beta_{\widetilde{C}1})^2+(\beta_{\widetilde{C}2})^2-(\beta_{\widetilde{C}1})^2(\beta_{\widetilde{C}2})^2}}\right\rangle;$$

(viii) $$\lambda\widetilde{C} = \left\langle\sqrt{\left(1-\left(1-\left(\xi_{\widetilde{C}}\right)^2\right)^\lambda\right)}e^{i2\pi\sqrt{\left(1-\left(1-(\alpha_{\widetilde{C}})^2\right)^\lambda\right)}}, \left(\eta_{\widetilde{C}}\right)^\lambda e^{i2\pi(\beta_{\widetilde{C}})^\lambda}\right\rangle;$$

(ix) $$\widetilde{C}^\lambda = \left\langle\left(\xi_{\widetilde{C}}\right)^\lambda e^{i2\pi(\alpha_{\widetilde{C}})^\lambda}, \sqrt{\left(1-\left(1-\left(\eta_{\widetilde{C}}\right)^2\right)^\lambda\right)}e^{i2\pi\sqrt{\left(1-\left(1-(\beta_{\widetilde{C}})^2\right)^\lambda\right)}}\right\rangle.$$

Example 1. Let $\widetilde{C}1 = \left\langle 0.70e^{i2\pi 0.60}, 0.50e^{i2\pi 0.80} \right\rangle$ and $\widetilde{C}2 = \left\langle 0.60e^{i2\pi 0.70}, 0.40e^{i2\pi 0.50} \right\rangle$ be two CPyFNs and $\lambda = 4$, then

(i) $$\widetilde{C}1 \cup \widetilde{C}2 = \left\langle \max(0.70, 0.60) e^{i2\pi \max(0.60,0.70)}, \min(0.50, 0.40) e^{i2\pi \min(0.80,0.50)} \right\rangle = \left\langle 0.70e^{i2\pi 0.70},\ 0.50e^{i2\pi 0.80} \right\rangle;$$

(ii) $$\widetilde{C}1 \cap \widetilde{C}2 = \left\langle \min(0.70, 0.60) e^{i2\pi \min(0.60,0.70)}, \max(0.50, 0.40) e^{i2\pi \max(0.80,0.50)} \right\rangle = \left\langle 0.60e^{i2\pi 0.60},\ 0.40e^{i2\pi 0.50} \right\rangle;$$

(iii) $$\widetilde{C}1 \bigoplus \widetilde{C}2 = \left\langle \sqrt{(0.70)^2+(0.60)^2-(0.70)^2(0.60)^2} \times e^{i2\pi\sqrt{(0.60)^2+(0.70)^2-(0.60)^2(0.70)^2}},\ 0.50 \times 0.40 e^{i2\pi 0.80\times 0.50} \right\rangle = \left\langle 0.6736e^{i2\pi 0.6736},\ 0.2000e^{i2\pi 0.4000} \right\rangle;$$

(iv) $$\widetilde{C}1 \bigotimes \widetilde{C}2 = \left\langle 0.70\times 0.60e^{i2\pi 0.60\times 0.70},\ \sqrt{(0.50)^2+(0.40)^2-(0.50)^2(0.40)^2} \times e^{i2\pi\sqrt{(0.80)^2+(0.50)^2-(0.80)^2(0.50)^2}} \right\rangle = \left\langle 0.3700e^{i2\pi 0.7300},\ 0.4200e^{i2\pi 0.4200} \right\rangle;$$

(v) $$4\widetilde{C}1 = \left\langle \sqrt{\left(1-\left(1-(0.70)^2\right)^4\right)} e^{i2\pi\sqrt{\left(1-\left(1-(0.60)^2\right)^4\right)}},\ (0.50)^4 e^{i2\pi(0.80)^4} \right\rangle = \left\langle 0.9656e^{i2\pi 0.9123},\ 0.0625e^{i2\pi 0.4096} \right\rangle;$$

(vi) $$\widetilde{C}1^4 = \left\langle (0.70)^4 e^{i2\pi(0.60)^4},\ \sqrt{\left(1-\left(1-(0.50)^2\right)^4\right)} e^{i2\pi\sqrt{\left(1-\left(1-(0.80)^2\right)^4\right)}} \right\rangle = \left\langle 0.2401e^{i2\pi 0.1296},\ 0.8268e^{i2\pi 0.9916} \right\rangle.$$

Definition 5. [56] *Let* $\widetilde{C} = \left\langle \xi_{\widetilde{C}} e^{2i\pi\alpha_{\widetilde{C}}}, \eta_{\widetilde{C}} e^{i2\pi\beta_{\widetilde{C}}} \right\rangle$ *be a CPyFN, where* $\xi_{\widetilde{C}}, \eta_{\widetilde{C}} \in [0, 1]$, $\alpha_{\widetilde{C}}, \beta_{\widetilde{C}} \in [0, 1]$, $\xi_{\widetilde{C}}^2 + \eta_{\widetilde{C}}^2 \leqslant 1$ *and* $\alpha_{\widetilde{C}}^2 + \beta_{\widetilde{C}}^2 \leqslant 1$. *The score function value of the CPyFN* $\widetilde{C} = \left\langle \xi_{\widetilde{C}} e^{2i\pi\alpha_{\widetilde{C}}}, \eta_{\widetilde{C}} e^{i2\pi\beta_{\widetilde{C}}} \right\rangle$ *is defined as follows:*

$$\mathscr{S}\bar{\mathscr{F}}\mathscr{V}\left(\widetilde{C}\right) = \xi_{\widetilde{C}}^2 - \eta_{\widetilde{C}}^2 + \alpha_{\widetilde{C}}^2 - \beta_{\widetilde{C}}^2, \tag{3}$$

and the accuracy function value of the CPyFN $\widetilde{C} = \left\langle \xi_{\widetilde{C}} e^{2i\pi\alpha_{\widetilde{C}}}, \eta_{\widetilde{C}} e^{i2\pi\beta_{\widetilde{C}}} \right\rangle$ *is given as follows:*

$$\mathscr{A}\bar{\mathscr{F}}\mathscr{V}\left(\widetilde{C}\right) = \xi_{\widetilde{C}}^2 + \eta_{\widetilde{C}}^2 + \alpha_{\widetilde{C}}^2 + \beta_{\widetilde{C}}^2, \tag{4}$$

where $\mathscr{S}\bar{\mathscr{F}}\mathscr{V}\left(\widetilde{C}\right) \in [-2, 2]$ *and* $\mathscr{A}\bar{\mathscr{F}}\mathscr{V}\left(\widetilde{C}\right) \in [0, 2]$.

For any two CPyFNs

$$\widetilde{C}1 = \left\langle \xi_{\widetilde{C}1} e^{i2\pi\alpha_{\widetilde{C}1}}, \eta_{\widetilde{C}1} e^{i2\pi\beta_{\widetilde{C}1}} \right\rangle \quad \text{and} \quad \widetilde{C}2 = \left\langle \xi_{\widetilde{C}2} e^{i2\pi\alpha_{\widetilde{C}2}}, \eta_{\widetilde{C}2} e^{i2\pi\beta_{\widetilde{C}2}} \right\rangle,$$

the following ranking methodology was suggested by [56]:

(i) If $\mathscr{S}\bar{\mathscr{F}}\mathscr{V}\left(\widetilde{C}1\right) > \mathscr{S}\bar{\mathscr{F}}\mathscr{V}\left(\widetilde{C}2\right)$, then $\widetilde{C}1 \succ \widetilde{C}2$;

(ii) If $\mathscr{S}\bar{\mathscr{F}}\mathscr{V}\left(\widetilde{C}1\right) = \mathscr{S}\bar{\mathscr{F}}\mathscr{V}\left(\widetilde{C}2\right)$, then accuracy function is used as follows

(a) $\mathscr{A}\bar{\mathscr{F}}\mathscr{V}\left(\widetilde{C}1\right) > \mathscr{A}\bar{\mathscr{F}}\mathscr{V}\left(\widetilde{C}2\right)$, then $\widetilde{C}1 \succ \widetilde{C}2$;

(b) $\mathscr{A}\bar{\mathscr{F}}\mathscr{V}\left(\widetilde{C}1\right) = \mathscr{A}\bar{\mathscr{F}}\mathscr{V}\left(\widetilde{C}2\right)$, then $\widetilde{C}1 = \widetilde{C}2$.

3. An improved score function value for CPyFNs

This section discusses the shortcomings of the existing score function value of CPyFNs given by [56] with the help of some counter-numerical examples. Then, we define an improved score function value of CPyFNs to overcome the shortcomings of the existing score function value of CPyFNs and to obtain a robust ranking order among CPyFNs in practical scenarios.

3.1. Counter-numerical examples

Example 2. Let

$$\widetilde{C}1 = \left\langle 0.60e^{i2\pi 0.50}, 0.60e^{i2\pi 0.50} \right\rangle \text{ and } \widetilde{C}2 = \left\langle 0.40e^{i2\pi 0.70}, 0.40e^{i2\pi 0.70} \right\rangle$$

be two CPyFNs. By applying the score function value given in Eq. (3) $\mathscr{S}\bar{\mathscr{F}}\mathscr{V}\left(\widetilde{C}\right)$, we obtain $\mathscr{S}\bar{\mathscr{F}}\mathscr{V}\left(\widetilde{C}1\right) = 0$ and $\mathscr{S}\bar{\mathscr{F}}\mathscr{V}\left(\widetilde{C}2\right) = 0$ which implies that "$\widetilde{C}1 = \widetilde{C}2$". It is obviously not true in this case because $\widetilde{C}1 = \left\langle 0.60e^{i2\pi 0.50}, 0.60e^{i2\pi 0.50} \right\rangle$ and $\widetilde{C}2 = \left\langle 0.40e^{i2\pi 0.70}, 0.40e^{i2\pi 0.70} \right\rangle$ are two different CPyFNs satisfy the condition $\xi_{\widetilde{C}1} \neq \xi_{\widetilde{C}2}, \eta_{\widetilde{C}1} \neq \eta_{\widetilde{C}2}, \alpha_{\widetilde{C}1} \neq \alpha_{\widetilde{C}2}$, and $\beta_{\widetilde{C}1} \neq \beta_{\widetilde{C}2}$. Hence the score function value $\mathscr{S}\bar{\mathscr{F}}\mathscr{V}\left(\widetilde{C}\right)$ given in Eq. (3) produces the counter-intuitive ranking order for the CPyFNs $\widetilde{C}1 = \left\langle 0.60e^{i2\pi 0.50}, 0.60e^{i2\pi 0.50} \right\rangle$ and $\widetilde{C}2 = \left\langle 0.40e^{i2\pi 0.70}, 0.40e^{i2\pi 0.70} \right\rangle$.

Example 3. Let

$$\widetilde{C}3 = \left\langle \sqrt{0.20}e^{i2\pi 0.30}, 0.60e^{i2\pi 0.50} \right\rangle \text{ and } \widetilde{C}4 = \left\langle 0.30e^{i2\pi\sqrt{0.20}}, 0.50e^{i2\pi 0.60} \right\rangle$$

be two CPyFNs. By applying the score value function given in Eq. (3) $\mathscr{S}\bar{\mathscr{F}}\mathscr{V}\left(\widetilde{C}\right)$, we obtain $\mathscr{S}\bar{\mathscr{F}}\mathscr{V}\left(\widetilde{C}3\right) = -0.3200$ and $\mathscr{S}\bar{\mathscr{F}}\mathscr{V}\left(\widetilde{C}4\right) = -0.3200$

which implies that "$\widetilde{C}3 = \widetilde{C}4$". It is obviously not true in this case because $\widetilde{C}3 = \left\langle \sqrt{0.20}e^{i2\pi 0.30}, 0.60e^{i2\pi 0.50} \right\rangle$ and $\widetilde{C}4 = \left\langle 0.30e^{i2\pi\sqrt{0.20}}, 0.50e^{i2\pi 0.60} \right\rangle$ are two different CPyFNs satisfy the condition $\xi_{\widetilde{C}3} \neq \xi_{\widetilde{C}4}$, $\eta_{\widetilde{C}3} \neq \eta_{\widetilde{C}4}$, $\alpha_{\widetilde{C}3} \neq \alpha_{\widetilde{C}4}$, and $\beta_{\widetilde{C}3} \neq \beta_{\widetilde{C}4}$. Hence the score function value $\bar{\mathcal{SFV}}\left(\widetilde{C}\right)$ given in Eq. (3) produces the counter-intuitive ranking order for the CPyFNs $\widetilde{C}3 = \left\langle \sqrt{0.20}e^{i2\pi 0.30}, 0.60e^{i2\pi 0.50} \right\rangle$ and $\widetilde{C}4 = \left\langle 0.30e^{i2\pi\sqrt{0.20}}, 0.50e^{i2\pi 0.60} \right\rangle$.

Example 4. Let

$$\widetilde{C}5 = \left\langle 0.60e^{i2\pi 0.45}, 0.40e^{i2\pi 0.45} \right\rangle \quad \text{and} \quad \widetilde{C}6 = \left\langle 0.70e^{i2\pi 0.30}, \sqrt{0.29}e^{i2\pi 0.30} \right\rangle$$

be two CPyFNs.By applying the score function value given in Eq. (3) $\bar{\mathcal{SFV}}\left(\widetilde{C}\right)$, we obtain $\bar{\mathcal{SFV}}\left(\widetilde{C}5\right) = 0.2000$ and $\bar{\mathcal{SFV}}\left(\widetilde{C}6\right) = 0.2000$ which implies that $\widetilde{C}5 = \widetilde{C}6$. It is obviously not true in this case because $\widetilde{C}5 = \left\langle 0.60e^{i2\pi 0.45}, 0.40e^{i2\pi 0.45} \right\rangle$ and $\widetilde{C}6 = \left\langle 0.70e^{i2\pi 0.30}, \sqrt{0.29}e^{i2\pi 0.30} \right\rangle$ are two different CPyFNs hold the condition $\xi_{\widetilde{C}5} \neq \xi_{\widetilde{C}6}, \eta_{\widetilde{C}5} \neq \eta_{\widetilde{C}6}, \alpha_{\widetilde{C}5} \neq \alpha_{\widetilde{C}6}$, and $\beta_{\widetilde{C}5} \neq \beta_{\widetilde{C}6}$. Hence the score function value $\bar{\mathcal{SFV}}\left(\widetilde{C}\right)$ given in Eq. (3) produces the counter-intuitive ranking order for the CPyFNs $\widetilde{C}5 = \left\langle 0.60e^{i2\pi 0.45}, 0.40e^{i2\pi 0.45} \right\rangle$ and $\widetilde{C}6 = \left\langle 0.70e^{i2\pi 0.30}, \sqrt{0.29}e^{i2\pi 0.30} \right\rangle$.

In the following, we define an improved score value function $\widehat{\mathcal{SFV}}_{IMP}\left(\widetilde{C}\right)$ for CyPFNs in order to overcome the above discussed shortcomings of the score function value given in Eq. (3).

3.2. An improved score function

First, let us consider the following function

$$h(m,n) = \left[\frac{\left(1 + m^2 - n^2\right) - \ln\left(1 + \left(1 - m^2 - n^2\right)\right)}{\left(1 + m^2 + n^2\right)} \right], \tag{5}$$

where $m, n \in [0,\ 1]$ and $0 \leqslant m^2 + n^2 \leqslant 1$.

Theorem 1. *The function $h(m,n)$ holds the following properties:*

(i) $h(m,n)$ is increases monotonically with respect to m and decreases monotonically with respect to n,

(ii) $h(m,n) = 0$ *if and only if* $m = 0$ *and* $n = 1$,

(iii) $h(m,n) = 1$ *if and only if* $m = 1$ *and* $n = 0$,

(iv) $0 \leqslant h(m,n) \leqslant 1$.

Proof. (i) Computing the first-order partial derivatives of Eq. (5) with respect to m and n, we get

$$\frac{\partial h(m,n)}{\partial m} = \left\{ \frac{(1+m^2+n^2)\left(\frac{6m-2mn^2-2m^3}{2-m^2-n^2}\right) - 2m\left(1+m^2-n^2-\ln\left(2-m^2-n^2\right)\right)}{\left(1+m^2+n^2\right)^2} \right\};$$

$$\frac{\partial h(m,n)}{\partial n} = \left\{ \frac{(1+m^2+n^2)\left(\frac{2n^3+2m^2n-2n}{2-m^2-n^2}\right) - 2n\left(1+m^2-n^2-\ln\left(2-m^2-n^2\right)\right)}{\left(1+m^2+n^2\right)^2} \right\}.$$

Since $0 \leqslant m \leqslant 1$, $0 \leqslant vn \leqslant 1$ and $0 \leqslant m^2 + v^2 \leqslant 1$, then we have $\dfrac{\partial h(m,n)}{\partial m} \geqslant 0$ and $\dfrac{\partial h(m,n)}{\partial n} \leqslant 0$.

Thus, we can deduce that the function $h(m,n)$ exhibits a monotonic increase with respect to m and a monotonic decrease with respect to n.

(ii) & (iii) According to the part (i), the function $h(m,n)$ will get the minimum and the maximum values when $m = 0, n = 1$ and $m = 1, n = 0$, respectively. On the other hand, when $m = 0, n = 1$, then

$$h(0,1) = \left[\frac{\left(1+0^2-1^2\right) - \ln\left(1+\left(1-0^2-1^2\right)\right)}{\left(1+0^2+1^2\right)} \right] = 0,$$

and when $m = 1$, $n = 0$, then

$$h(u,v) = \left[\frac{\left(1+1^2-0^2\right) - \ln\left(1+\left(1-1^2-0^2\right)\right)}{\left(1+1^2+0^2\right)} \right] = 1.$$

(iv) Based on the previous findings, it is evident that the function $h(m,n)$ produces a spectrum of values. Specifically, the minimum achievable value of $h(m,n)$ is 0, and the maximum attainable value is 1. Hence, $0 \leqslant h(m,n) \leqslant 1$.

This validates the theorem. □

Figure 1 graphically illustrates the function $h(m,n)$ to better enhance clarity about its characteristics.

To compare the CyPFNs, we define an improved score function value of a CyPFN $\widetilde{C} = \left\langle \xi_{\widetilde{C}} e^{2i\pi\alpha_{\widetilde{C}}}, \eta_{\widetilde{C}} e^{i2\pi\beta_{\widetilde{C}}} \right\rangle$ as follows:

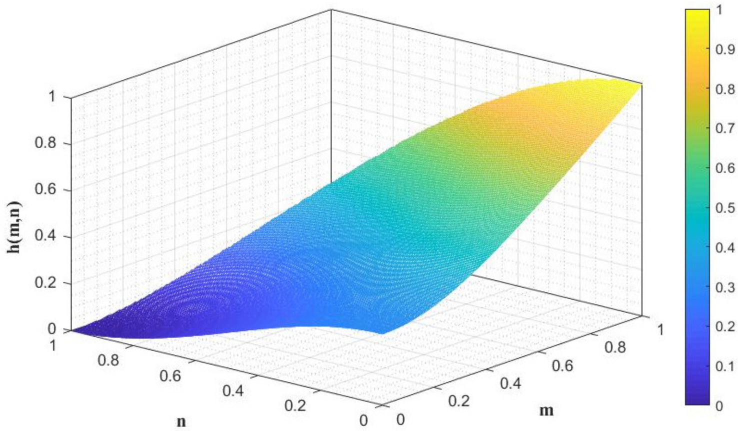

Figure 1: Function $h(m,n)$

Definition 6. *Suppose that* $\widetilde{C} = \langle \xi_{\widetilde{C}} e^{2i\pi\alpha_{\widetilde{C}}}, \eta_{\widetilde{C}} e^{i2\pi\beta_{\widetilde{C}}} \rangle$ *is a CPyFN, where* $\xi_{\widetilde{C}}, \eta_{\widetilde{C}} \in [0,1]$, $\alpha_{\widetilde{C}}, \beta_{\widetilde{C}} \in [0,1]$, $\xi^2_{\widetilde{C}} + \eta^2_{\widetilde{C}} \leqslant 1$ *and* $\alpha^2_{\widetilde{C}} + \beta^2_{\widetilde{C}} \leqslant 1$. *Using function* $h(m,n)$, *we define an improved score function value for CPyFN* $\widetilde{C} = \langle \xi_{\widetilde{C}} e^{2i\pi\alpha_{\widetilde{C}}}, \eta_{\widetilde{C}} e^{i2\pi\beta_{\widetilde{C}}} \rangle$ *as follows:*

$$\underset{IMP}{\widetilde{\mathcal{SFV}}}\left(\widetilde{C}\right) = \frac{1}{(1+2\pi)}\left[\left\{\frac{1+\xi^2_{\widetilde{C}} - \eta^2_{\widetilde{C}} - \ln\left(1+\left(1-\xi^2_{\widetilde{C}} - \eta^2_{\widetilde{C}}\right)\right)}{\left(1+\xi^2_{\widetilde{C}} + \eta^2_{\widetilde{C}}\right)}\right\} + 2\pi\left\{\frac{1+\alpha^2_{\widetilde{C}} - \beta^2_{\widetilde{C}} - \ln\left(1+\left(1-\alpha^2_{\widetilde{C}} - \beta^2_{\widetilde{C}}\right)\right)}{\left(1+\alpha^2_{\widetilde{C}} + \beta^2_{\widetilde{C}}\right)}\right\}\right]$$
$$= \frac{1}{(1+2\pi)}\left[\left\{\frac{1+\xi^2_{\widetilde{C}} - \eta^2_{\widetilde{C}} - \ln\left(2-\xi^2_{\widetilde{C}} - \eta^2_{\widetilde{C}}\right)}{\left(1+\xi^2_{\widetilde{C}} + \eta^2_{\widetilde{C}}\right)}\right\} + 2\pi\left\{\frac{1+\alpha^2_{\widetilde{C}} - \beta^2_{\widetilde{C}} - \ln\left(2-\alpha^2_{\widetilde{C}} - \beta^2_{\widetilde{C}}\right)}{\left(1+\alpha^2_{\widetilde{C}} + \beta^2_{\widetilde{C}}\right)}\right\}\right]. \tag{6}$$

Theorem 2. *For a given CPyFN* $\widetilde{C} = \langle \xi_{\widetilde{C}} e^{2i\pi\alpha_{\widetilde{C}}}, \eta_{\widetilde{C}} e^{i2\pi\beta_{\widetilde{C}}} \rangle$, *where* $\xi_{\widetilde{C}}, \eta_{\widetilde{C}} \in [0,1]$, $\alpha_{\widetilde{C}}, \beta_{\widetilde{C}} \in [0,1]$, $\xi^2_{\widetilde{C}} + \eta^2_{\widetilde{C}} \leqslant 1$ *and* $\alpha^2_{\widetilde{C}} + \beta^2_{\widetilde{C}} \leqslant 1$, *the score function* $\underset{IMP}{\widetilde{\mathcal{SFV}}}\left(\widetilde{C}\right)$ *defined in Eq. (6) satisfies the following properties:*

(i) When $\alpha_{\widetilde{C}}$ *and* $\beta_{\widetilde{C}}$ *are fixed, then* $\underset{IMP}{\widetilde{\mathcal{SFV}}}\left(\widetilde{C}\right)$ *increases monotonically with respect to* $\xi_{\widetilde{C}}$ *and decreases monotonically with respect to* $\eta_{\widetilde{C}}$.

(ii) When $\xi_{\widetilde{C}}$ *and* $\eta_{\widetilde{C}}$ *are fixed, then* $\underset{IMP}{\widetilde{\mathcal{SFV}}}\left(\widetilde{C}\right)$ *increases monotonically with respect to* $\alpha_{\widetilde{C}}$ *and decreases monotonically with respect to* $\beta_{\widetilde{C}}$.

(iii) $\underset{IMP}{\widetilde{\mathcal{SFV}}}\left(\widetilde{C}\right) = 0$ *if and only if* $\xi_{\widetilde{C}} = 0$, $\eta_{\widetilde{C}} = 1$, $\alpha_{\widetilde{C}} = 0$, *and* $\beta_{\widetilde{C}} = 1$.

(iv) $\underset{IMP}{\widetilde{\mathcal{SFV}}}\left(\widetilde{C}\right) = 1$ *if and only if* $\xi_{\widetilde{C}} = 1$, $\eta_{\widetilde{C}} = 0$, $\alpha_{\widetilde{C}} = 1$, *and* $\beta_{\widetilde{C}} = 0$.

(v) $0 \leqslant \underset{IMP}{\widetilde{\mathcal{SFV}}}\left(\widetilde{C}\right) \leqslant 1$.

Proof. Since $\widetilde{C} = \left\langle \xi_{\widetilde{C}} e^{2i\pi\alpha_{\widetilde{C}}}, \eta_{\widetilde{C}} e^{i2\pi\beta_{\widetilde{C}}} \right\rangle$ is a CPyFN, then according to the Definition 2, we know that $\xi_{\widetilde{C}}, \eta_{\widetilde{C}} \in [0, 1]$, $\alpha_{\widetilde{C}}, \beta_{\widetilde{C}} \in [0, 1]$, $\xi^2_{\widetilde{C}} + \eta^2_{\widetilde{C}} \leqslant 1$ and $\alpha^2_{\widetilde{C}} + \beta^2_{\widetilde{C}} \leqslant 1$.

(i) By differentiating Eq. (6) with respect to $\xi_{\widetilde{C}}$ and $\eta_{\widetilde{C}}$ using first order partial derivatives, we derive the following:

$$\frac{\partial \underset{IMP}{\widetilde{\mathcal{SFV}}}\left(\widetilde{C}\right)}{\partial \xi_{\widetilde{C}}} = \frac{\partial\left(\frac{1}{(1+2\pi)}\left[\left\{\frac{1+\xi^2_{\widetilde{C}}-\eta^2_{\widetilde{C}}-\ln\left(2-\xi^2_{\widetilde{C}}-\eta^2_{\widetilde{C}}\right)}{\left(1+\xi^2_{\widetilde{C}}+\eta^2_{\widetilde{C}}\right)}\right\}+2\pi\left\{\frac{1+\alpha^2_{\widetilde{C}}-\beta^2_{\widetilde{C}}-\ln\left(2-\alpha^2_{\widetilde{C}}-\beta^2_{\widetilde{C}}\right)}{\left(1+\alpha^2_{\widetilde{C}}+\beta^2_{\widetilde{C}}\right)}\right\}\right]\right)}{\partial \xi_{\widetilde{C}}}$$

$$= \frac{1}{(1+2\pi)}\left[\frac{\partial\left\{\frac{1+\xi^2_{\widetilde{C}}-\eta^2_{\widetilde{C}}-\ln\left(2-\xi^2_{\widetilde{C}}-\eta^2_{\widetilde{C}}\right)}{\left(1+\xi^2_{\widetilde{C}}+\eta^2_{\widetilde{C}}\right)}\right\}}{\partial \xi_{\widetilde{C}}} + 2\pi\frac{\partial\left\{\frac{1+\alpha^2_{\widetilde{C}}-\beta^2_{\widetilde{C}}-\ln\left(2-\alpha^2_{\widetilde{C}}-\beta^2_{\widetilde{C}}\right)}{\left(1+\alpha^2_{\widetilde{C}}+\beta^2_{\widetilde{C}}\right)}\right\}}{\partial \xi_{\widetilde{C}}}\right]$$

$$= \frac{1}{(1+2\pi)}\left[\left\{\frac{\frac{\left(6\xi_{\widetilde{C}}-2\xi^3_{\widetilde{C}}-2\xi_{\widetilde{C}}\eta^2_{\widetilde{C}}\right)}{\left(2-\xi^2_{\widetilde{C}}-\eta^2_{\widetilde{C}}\right)}\left(1+\xi^2_{\widetilde{C}}+\eta^2_{\widetilde{C}}\right)-2\xi_{\widetilde{C}}\left(1+\xi^2_{\widetilde{C}}-\eta^2_{\widetilde{C}}-\ln\left(2-\xi^2_{\widetilde{C}}-\eta^2_{\widetilde{C}}\right)\right)}{\left(1+\xi^2_{\widetilde{C}}+\eta^2_{\widetilde{C}}\right)^2}\right\}\right]$$

$$\geqslant 0. \tag{7}$$

and

$$\frac{\partial \widehat{\mathcal{SFV}}_{IMP}\left(\widetilde{C}\right)}{\partial \eta_{\widetilde{C}}} = \frac{\partial\left(\frac{1}{(1+2\pi)}\left[\left\{\frac{1+\xi_{\widetilde{C}}^2-\eta_{\widetilde{C}}^2-\ln\left(2-\xi_{\widetilde{C}}^2-\eta_{\widetilde{C}}^2\right)}{\left(1+\xi_{\widetilde{C}}^2+\eta_{\widetilde{C}}^2\right)}\right\}+2\pi\left\{\frac{1+\alpha_{\widetilde{C}}^2-\beta_{\widetilde{C}}^2-\ln\left(2-\alpha_{\widetilde{C}}^2-\beta_{\widetilde{C}}^2\right)}{\left(1+\alpha_{\widetilde{C}}^2+\beta_{\widetilde{C}}^2\right)}\right\}\right]\right)}{\partial \eta_{\widetilde{C}}}$$

$$= \frac{1}{(1+2\pi)}\left[\frac{\partial\left\{\frac{1+\xi_{\widetilde{C}}^2-\eta_{\widetilde{C}}^2-\ln\left(2-\xi_{\widetilde{C}}^2-\eta_{\widetilde{C}}^2\right)}{\left(1+\xi_{\widetilde{C}}^2+\eta_{\widetilde{C}}^2\right)}\right\}}{\partial \eta_{\widetilde{C}}}+2\pi\frac{\partial\left\{\frac{1+\alpha_{\widetilde{C}}^2-\beta_{\widetilde{C}}^2-\ln\left(2-\alpha_{\widetilde{C}}^2-\beta_{\widetilde{C}}^2\right)}{\left(1+\alpha_{\widetilde{C}}^2+\beta_{\widetilde{C}}^2\right)}\right\}}{\partial \eta_{\widetilde{C}}}\right]$$

$$= \frac{1}{(1+2\pi)}\left[\left\{\frac{\frac{\left(2\eta_{\widetilde{C}}^3+2\xi_{\widetilde{C}}^2\eta_{\widetilde{C}}-2\eta_{\widetilde{C}}\right)}{\left(2-\xi_{\widetilde{C}}^2-\eta_{\widetilde{C}}^2\right)}\left(1+\xi_{\widetilde{C}}^2+\eta_{\widetilde{C}}^2\right)-2\eta_{\widetilde{C}}\left(1+\xi_{\widetilde{C}}^2-\eta_{\widetilde{C}}^2-\ln\left(2-\xi_{\widetilde{C}}^2-\eta_{\widetilde{C}}^2\right)\right)}{\left(1+\xi_{\widetilde{C}}^2+\eta_{\widetilde{C}}^2\right)^2}\right\}\right]$$

$$\leqslant 0. \tag{8}$$

The results are obtained in Eqs. (7) and (8) show that $\widehat{\mathcal{SFV}}_{IMP}\left(\widetilde{C}\right)$ increases monotonically with respect to $\xi_{\widetilde{C}}$ and decreases monotonically with respect to $\eta_{\widetilde{C}}$.

(ii) By differentiating Eq. (6) with respect to $\alpha_{\widetilde{C}}$ and $\beta_{\widetilde{C}}$ using first order partial derivatives, we obtain the following:

$$\frac{\partial \widehat{\mathcal{SFV}}_{IMP}\left(\widetilde{C}\right)}{\partial \alpha_{\widetilde{C}}} = \frac{\partial\left(\frac{1}{(1+2\pi)}\left[\left\{\frac{1+\xi_{\widetilde{C}}^2-\eta_{\widetilde{C}}^2-\ln\left(2-\xi_{\widetilde{C}}^2-\eta_{\widetilde{C}}^2\right)}{\left(1+\xi_{\widetilde{C}}^2+\eta_{\widetilde{C}}^2\right)}\right\}+2\pi\left\{\frac{1+\alpha_{\widetilde{C}}^2-\beta_{\widetilde{C}}^2-\ln\left(2-\alpha_{\widetilde{C}}^2-\beta_{\widetilde{C}}^2\right)}{\left(1+\alpha_{\widetilde{C}}^2+\beta_{\widetilde{C}}^2\right)}\right\}\right]\right)}{\partial \alpha_{\widetilde{C}}}$$

$$= \frac{1}{(1+2\pi)}\left[\frac{\partial\left\{\frac{1+\xi_{\widetilde{C}}^2-\eta_{\widetilde{C}}^2-\ln\left(2-\xi_{\widetilde{C}}^2-\eta_{\widetilde{C}}^2\right)}{\left(1+\xi_{\widetilde{C}}^2+\eta_{\widetilde{C}}^2\right)}\right\}}{\partial \alpha_{\widetilde{C}}}+2\pi\frac{\partial\left\{\frac{1+\alpha_{\widetilde{C}}^2-\beta_{\widetilde{C}}^2-\ln\left(2-\alpha_{\widetilde{C}}^2-\beta_{\widetilde{C}}^2\right)}{\left(1+\alpha_{\widetilde{C}}^2+\beta_{\widetilde{C}}^2\right)}\right\}}{\partial \alpha_{\widetilde{C}}}\right]$$

$$= \frac{2\pi}{(1+2\pi)}\left[\frac{\frac{\left(6\alpha_{\widetilde{C}}-2\alpha_{\widetilde{C}}^3-2\alpha_{\widetilde{C}}\beta_{\widetilde{C}}^2\right)}{\left(2-\alpha_{\widetilde{C}}^2-\beta_{\widetilde{C}}^2\right)}\left(1+\alpha_{\widetilde{C}}^2+\beta_{\widetilde{C}}^2\right)-2\alpha_{\widetilde{C}}\left(1+\alpha_{\widetilde{C}}^2-\beta_{\widetilde{C}}^2-\ln\left(2-\alpha_{\widetilde{C}}^2-\beta_{\widetilde{C}}^2\right)\right)}{\left(1+\alpha_{\widetilde{C}}^2+\beta_{\widetilde{C}}^2\right)^2}\right]$$

$$\geqslant 0. \tag{9}$$

and

$$\frac{\partial \widetilde{\underset{IMP}{\mathcal{SFV}}}\left(\widetilde{C}\right)}{\partial \beta_{\widetilde{C}}} = \frac{\partial \left(\frac{1}{(1+2\pi)}\left[\left\{\frac{1+\xi_{\widetilde{C}}^2-\eta_{\widetilde{C}}^2-\ln\left(2-\xi_{\widetilde{C}}^2-\eta_{\widetilde{C}}^2\right)}{\left(1+\xi_{\widetilde{C}}^2+\eta_{\widetilde{C}}^2\right)}\right\}+2\pi\left\{\frac{1+\alpha_{\widetilde{C}}^2-\beta_{\widetilde{C}}^2-\ln\left(2-\alpha_{\widetilde{C}}^2-\beta_{\widetilde{C}}^2\right)}{\left(1+\alpha_{\widetilde{C}}^2+\beta_{\widetilde{C}}^2\right)}\right\}\right]\right)}{\partial \eta_{\widetilde{C}}}$$

$$= \frac{1}{(1+2\pi)}\left[\frac{\partial\left\{\frac{1+\xi_{\widetilde{C}}^2-\eta_{\widetilde{C}}^2-\ln\left(2-\xi_{\widetilde{C}}^2-\eta_{\widetilde{C}}^2\right)}{\left(1+\xi_{\widetilde{C}}^2+\eta_{\widetilde{C}}^2\right)}\right\}}{\partial \beta_{\widetilde{C}}} + 2\pi\frac{\partial\left\{\frac{1+\alpha_{\widetilde{C}}^2-\beta_{\widetilde{C}}^2-\ln\left(2-\alpha_{\widetilde{C}}^2-\beta_{\widetilde{C}}^2\right)}{\left(1+\alpha_{\widetilde{C}}^2+\beta_{\widetilde{C}}^2\right)}\right\}}{\partial \beta_{\widetilde{C}}}\right]$$

$$= \frac{2\pi}{(1+2\pi)}\left[\left\{\frac{\frac{\left(2\beta_{\widetilde{C}}^3+2\alpha_{\widetilde{C}}^2\beta_{\widetilde{C}}-2\beta_{\widetilde{C}}\right)}{\left(2-\alpha_{\widetilde{C}}^2-\beta_{\widetilde{C}}^2\right)}\left(1+\alpha_{\widetilde{C}}^2+\beta_{\widetilde{C}}^2\right)-2\beta_{\widetilde{C}}\left(1+\alpha_{\widetilde{C}}^2-\beta_{\widetilde{C}}^2-\ln\left(2-\alpha_{\widetilde{C}}^2-\beta_{\widetilde{C}}^2\right)\right)}{\left(1+\alpha_{\widetilde{C}}^2+\beta_{\widetilde{C}}^2\right)^2}\right\}\right]$$

$$\leqslant 0. \tag{10}$$

The results are shown in Eqs. (9) and (10) demonstrate that the value of $\widetilde{\underset{IMP}{\mathcal{SFV}}}\left(\widetilde{C}\right)$ monotonically increases as $\alpha_{\widetilde{C}}$ increases, and monotonically decreases as $\beta_{\widetilde{C}}$ decreases.

(iii) & (iv) According to the part (i) and (ii), the score function $\widetilde{\underset{IMP}{\mathcal{SFV}}}\left(\widetilde{C}\right)$ will achieve the minimum and the maximum values when $\xi_{\widetilde{C}} = 0$, $\eta_{\widetilde{C}} = 1$, $\alpha_{\widetilde{C}} = 0$, & $\beta_{\widetilde{C}} = 1$ and $\xi_{\widetilde{C}} = 1$, $\eta_{\widetilde{C}} = 0$, $\alpha_{\widetilde{C}} = 1$, & $\beta_{\widetilde{C}} = 0$, respectively. On the counter side, when $\xi_{\widetilde{C}} = 0$, $\eta_{\widetilde{C}} = 1$, $\alpha_{\widetilde{C}} = 0$, & $\beta_{\widetilde{C}} = 1$, then

$$\widetilde{\underset{IMP}{\mathcal{SFV}}}\left(\widetilde{C}\right) = \frac{1}{(1+2\pi)}\left[\left\{\frac{1+0^2-1^2-\ln\left(2-0^2-1^2\right)}{\left(1+0^2+1^2\right)}\right\} + 2\pi\left\{\frac{1+0^2-1^2-\ln\left(2-0^2-1^2\right)}{\left(1+0^2+1^2\right)}\right\}\right]$$

$$= \frac{1}{(1+2\pi)}\left[\left\{\frac{0}{2}\right\}+2\pi\left\{\frac{0}{2}\right\}\right] = 0,$$

and when $\xi_{\widetilde{C}} = 1, \eta_{\widetilde{C}} = 0, \alpha_{\widetilde{C}} = 1, \; \& \; \beta_{\widetilde{C}} = 0$, then

$$\overbrace{\mathcal{SFV}}_{IMP}\left(\widetilde{C}\right) = \frac{1}{(1+2\pi)}\left[\left\{\frac{1+1^2-0^2-\ln\left(2-1^2-0^2\right)}{\left(1+1^2+0^2\right)}\right\} + 2\pi\left\{\frac{1+1^2-0^2-\ln\left(2-1^2-0^2\right)}{\left(1+1^2+0^2\right)}\right\}\right]$$

$$= \frac{1}{(1+2\pi)}\left[\left\{\frac{2}{2}\right\} + 2\pi\left\{\frac{2}{2}\right\}\right] = 1.$$

(iv) Because $\widetilde{C} = \left\langle \xi_{\widetilde{C}} e^{2i\pi\alpha_{\widetilde{C}}}, \eta_{\widetilde{C}} e^{i2\pi\beta_{\widetilde{C}}} \right\rangle$ is a CyPFN, where $\xi_{\widetilde{C}}, \eta_{\widetilde{C}} \in [0, 1]$, $\alpha_{\widetilde{C}}, \beta_{\widetilde{C}} \in [0, 1]$, $\xi_{\widetilde{C}}^2 + \eta_{\widetilde{C}}^2 \leqslant 1$ and $\alpha_{\widetilde{C}}^2 + \beta_{\widetilde{C}}^2 \leqslant 1$, it can be seen that

$$1+\xi_{\widetilde{C}}^2-\eta_{\widetilde{C}}^2-\ln\left(2-\xi_{\widetilde{C}}^2-\eta_{\widetilde{C}}^2\right) \in [0, 2], \quad \left(1+\xi_{\widetilde{C}}^2+\eta_{\widetilde{C}}^2\right) \in [1, 2], \tag{11}$$

and

$$1+\alpha_{\widetilde{C}}^2-\beta_{\widetilde{C}}^2-\ln\left(2-\alpha_{\widetilde{C}}^2-\beta_{\widetilde{C}}^2\right) \in [0, 2], \quad \left(1+\alpha_{\widetilde{C}}^2+\beta_{\widetilde{C}}^2\right) \in [1, 2], \tag{12}$$

which imply that

$$\left\{\frac{1+\xi_{\widetilde{C}}^2-\eta_{\widetilde{C}}^2-\ln\left(2-\xi_{\widetilde{C}}^2-\eta_{\widetilde{C}}^2\right)}{\left(1+\xi_{\widetilde{C}}^2+\eta_{\widetilde{C}}^2\right)}\right\} \in [0, 1] \quad \text{and} \quad \left\{\frac{1+\alpha_{\widetilde{C}}^2-\beta_{\widetilde{C}}^2-\ln\left(2-\alpha_{\widetilde{C}}^2-\beta_{\widetilde{C}}^2\right)}{\left(1+\alpha_{\widetilde{C}}^2+\beta_{\widetilde{C}}^2\right)}\right\} \in [0, 1]. \tag{13}$$

and

$$\left[\left\{\frac{1+\xi_{\widetilde{C}}^2-\eta_{\widetilde{C}}^2-\ln\left(2-\xi_{\widetilde{C}}^2-\eta_{\widetilde{C}}^2\right)}{\left(1+\xi_{\widetilde{C}}^2+\eta_{\widetilde{C}}^2\right)}\right\} + 2\pi\left\{\frac{1+\alpha_{\widetilde{C}}^2-\beta_{\widetilde{C}}^2-\ln\left(2-\alpha_{\widetilde{C}}^2-\beta_{\widetilde{C}}^2\right)}{\left(1+\alpha_{\widetilde{C}}^2+\beta_{\widetilde{C}}^2\right)}\right\}\right] \in [0, 1+2\pi]. \tag{14}$$

Based on the Eqs. (11)–(14), we obtain $\overbrace{\mathcal{SFV}}_{IMP}\left(\widetilde{C}\right) \in [0, 1]$.

This proves the theorem. □

Theorem 3. *Let*

$$\widetilde{C}1 = \left\langle \xi_{\widetilde{C}1} e^{i2\pi\alpha_{\widetilde{C}1}}, \eta_{\widetilde{C}1} e^{i2\pi\beta_{\widetilde{C}1}} \right\rangle \quad \text{and} \quad \widetilde{C}2 = \left\langle \xi_{\widetilde{C}2} e^{i2\pi\alpha_{\widetilde{C}2}}, \eta_{\widetilde{C}2} e^{i2\pi\beta_{\widetilde{C}2}} \right\rangle$$

be two CPyFNs, where $\xi_{\widetilde{C}i}, \eta_{\widetilde{C}i} \in [0, 1]$, $\alpha_{\widetilde{C}i}, \beta_{\widetilde{C}i} \in [0, 1]$, $\xi^2_{\widetilde{C}i} + \eta^2_{\widetilde{C}i} \leqslant 1$ *and* $\alpha^2_{\widetilde{C}i} + \beta^2_{\widetilde{C}i} \leqslant 1$ $(i = 1, 2)$. *If* $\widetilde{C}1 \leqslant \widetilde{C}2$ *i.e.,* $\xi_{\widetilde{C}1} \leqslant \xi_{\widetilde{C}2}$, $\eta_{\widetilde{C}1} \geqslant \eta_{\widetilde{C}2}$, $\alpha_{\widetilde{C}1} \leqslant \alpha_{\widetilde{C}2}$, *and* $\beta_{\widetilde{C}1} \geqslant \beta_{\widetilde{C}2}$, *then* $\underset{IMP}{\widehat{\mathcal{SFV}}}\left(\widetilde{C}1\right) \leqslant \underset{IMP}{\widehat{\mathcal{SFV}}}\left(\widetilde{C}2\right)$.

Proof. From Theorem 2, we have the following facts (i) When $\alpha_{\widetilde{C}}$ and $\beta_{\widetilde{C}}$ are fixed, the score function value $\underset{IMP}{\widehat{\mathcal{SFV}}}\left(\widetilde{C}\right)$ monotonically increases and decreases with respect to $\xi_{\widetilde{C}}$ and $\eta_{\widetilde{C}}$, respectively. (ii) When $\xi_{\widetilde{C}}$ and $\eta_{\widetilde{C}}$ are fixed, then $\underset{IMP}{\widehat{\mathcal{SFV}}}\left(\widetilde{C}\right)$ monotonically increases and decreases with respect to $\alpha_{\widetilde{C}_\wp}$ and $\beta_{\widetilde{C}}$, respectively. Therefore when $\widetilde{C}1 \leqslant \widetilde{C}2$ i.e., $\xi_{\widetilde{C}1} \leqslant \xi_{\widetilde{C}2}$, $\eta_{\widetilde{C}1} \geqslant \eta_{\widetilde{C}2}$, $\alpha_{\widetilde{C}1} \leqslant \alpha_{\widetilde{C}2}$, and $\beta_{\widetilde{C}1} \geqslant \beta_{\widetilde{C}2}$, then we get $\underset{IMP}{\widehat{\mathcal{SFV}}}\left(\widetilde{C}1\right) \leqslant \underset{IMP}{\widehat{\mathcal{SFV}}}\left(\widetilde{C}2\right)$. □

Definition 7. *Let*

$$\widetilde{C}1 = \left\langle \xi_{\widetilde{C}1} e^{i2\pi\alpha_{\widetilde{C}1}},\ \eta_{\widetilde{C}1} e^{i2\pi\beta_{\widetilde{C}1}} \right\rangle \quad \text{and} \quad \widetilde{C}2 = \left\langle \xi_{\widetilde{C}2} e^{i2\pi\alpha_{\widetilde{C}2}},\ \eta_{\widetilde{C}2} e^{i2\pi\beta_{\widetilde{C}2}} \right\rangle$$

be two CPyFNs. We have established the ranking rules based on the values of the improved score function, denoted as $\underset{IMP}{\widehat{\mathcal{SFV}}}\left(\widetilde{C}\right)$. *The rules are listed as follows:*

1. *If* $\underset{IMP}{\widehat{\mathcal{SFV}}}\left(\widetilde{C}1\right) > \underset{IMP}{\widehat{\mathcal{SFV}}}\left(\widetilde{C}2\right)$, *then* $\widetilde{C}1 > \widetilde{C}2$;
2. *If* $\underset{IMP}{\widehat{\mathcal{SFV}}}\left(\widetilde{C}1\right) < \underset{IMP}{\widehat{\mathcal{SFV}}}\left(\widetilde{C}2\right)$, *then* $\widetilde{C}1 < \widetilde{C}2$;
3. *If* $\underset{IMP}{\widehat{\mathcal{SFV}}}\left(\widetilde{C}1\right) = \underset{IMP}{\widehat{\mathcal{SFV}}}\left(\widetilde{C}2\right)$, *then* $\widetilde{C}1 = \widetilde{C}2$.

To demonstrate the practicality of the suggested score function value $\underset{IMP}{\widehat{\mathcal{SFV}}}\left(\widetilde{C}\right)$ for ranking CPyFNs, we solved Examples 2–4 using our proposed score function value $\underset{IMP}{\widehat{\mathcal{SFV}}}\left(\widetilde{C}\right)$. The resulting outcomes are recorded in Table 1.

The findings presented in Table 1 clearly demonstrate that the suggested score function value, denoted as $\underset{IMP}{\widehat{\mathcal{SFV}}}\left(\widetilde{C}\right)$, is capable of ranking the CPyFNs with

Table 1: The comparison results

Examples	$\mathcal{SFV}(\widetilde{C})$	Ranking result	$\underset{IMP}{\widehat{\mathcal{SFV}}}(\widetilde{C})$	Ranking result
$\widetilde{C}1=\langle 0.60e^{i2\pi 0.50}, 0.60e^{i2\pi 0.50}\rangle$ $\widetilde{C}2=\langle 0.40e^{i2\pi 0.70}, 0.40e^{i2\pi 0.70}\rangle$	$\mathcal{SFV}(\widetilde{C}1)=0$ $\mathcal{SFV}(\widetilde{C}2)=0$	$\widetilde{C}1=\widetilde{C}2$	$\underset{IMP}{\widehat{\mathcal{SFV}}}(\widetilde{C}1)=0.4021$ $\underset{IMP}{\widehat{\mathcal{SFV}}}(\widetilde{C}2)=0.4771$	$\widetilde{C}1<\widetilde{C}2$
$\widetilde{C}3=\left\langle\sqrt{0.20}e^{i2\pi 0.30}, 0.60e^{i2\pi 0.50}\right\rangle$ $\widetilde{C}4=\left\langle 0.30e^{i2\pi\sqrt{0.20}}, 0.50e^{i2\pi 0.60}\right\rangle$	$\mathcal{SFV}(\widetilde{C}3)=-0.3200$ $\mathcal{SFV}(\widetilde{C}4)=-0.3200$	$\widetilde{C}3=\widetilde{C}4$	$\underset{IMP}{\widehat{\mathcal{SFV}}}(\widetilde{C}3)=0.2563$ $\underset{IMP}{\widehat{\mathcal{SFV}}}(\widetilde{C}4)=0.2970$	$\widetilde{C}3<\widetilde{C}4$
$\widetilde{C}5=\langle 0.60e^{i2\pi 0.45}, 0.40e^{i2\pi 0.45}\rangle$ $\widetilde{C}6=\left\langle 0.70e^{i2\pi 0.30}, \sqrt{0.29}e^{i2\pi 0.30}\right\rangle$	$\mathcal{SFV}(\widetilde{C}5)=0.2000$ $\mathcal{SFV}(\widetilde{C}6)=0.2000$	$\widetilde{C}5=\widetilde{C}6$	$\underset{IMP}{\widehat{\mathcal{SFV}}}(\widetilde{C}5)=0.4003$ $\underset{IMP}{\widehat{\mathcal{SFV}}}(\widetilde{C}6)=0.3705$	$\widetilde{C}5>\widetilde{C}6$

greater efficiency and effectively addresses all the limitations of the current score function value given in Eq. (3). Therefore, we may deduce that the performance of the suggested score function value $\underset{IMP}{\widehat{\mathcal{SFV}}}(\widetilde{C})$, surpasses that of the existing score function value to ranking CPyFNs in real-life scenarios.

4. Transportation problem under complex Pythagorean fuzzy framework

The transportation problem is a linear programming problem (LPP) that focuses on finding the most efficient way to distribute items from multiple suppliers to several consumers. The objective is to minimize the overall transportation expenses while ensuring that the supply and demand constraints are fulfilled. Here, we present a fundamental explanation of the mathematical model for the transportation problem in a crisp environment. Table 2 presents a classical transportation problem.

Table 2: A classical transportation problem

Source/Destination	D_1	D_2	$\cdots$	D_n	Supply
S_1	$\mathfrak{I}_{11}$	$\mathfrak{I}_{12}$	$\cdots$	$\mathfrak{I}_{1n}$	p_1
S_2	$\mathfrak{I}_{21}$	$\mathfrak{I}_{22}$	$\cdots$	$\mathfrak{I}_{2n}$	p_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
S_m	$\mathfrak{I}_{m1}$	$\mathfrak{I}_{m2}$	$\cdots$	$\mathfrak{I}_{mn}$	p_m
Demand	q_1	q_2	$\cdots$	q_n	

where:

(1) m – number of sources (suppliers);

(2) n – number of destinations (consumers);

(3) p_i – supply available at the source S_i, $i = 1, 2, \ldots, m$;

(4) q_j – demand at the destination D_j, $j = 1, 2, \ldots, n$;

(5) $\Im_{ij}$ – the transportation cost of transporting one unit from the i-th source to the j-th destination for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$;

(6) k_{ij} – quantity of items/products transported from the S_i source to the D_j destination.

Let us assume that k_{ij} quantity of items/products transported from the i-th source to the j-th destination. The objective is to minimize the overall transportation cost, which can be formulated mathematically as follows:

$$\begin{aligned} &\text{minimize } Z = \sum_{i=1}^{m}\sum_{j=1}^{n} \Im_{ij} k_{ij} \\ &\text{subject to } \begin{cases} \sum_{j=1}^{n} k_{ij} = p_i, & \text{for } i = 1, 2, \ldots, m, \\ \sum_{i=1}^{m} k_{ij} = q_j, & \text{for } j = 1, 2, \ldots, n, \\ k_{ij} \geqslant 0, & \text{for } i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n. \end{cases} \end{aligned} \tag{15}$$

Additionally, the problem is known as balance transportation problem if the total supply equals the total demand, i.e.,

$$\sum_{i=1}^{m} p_i = \sum_{j=1}^{n} q_j\,. \tag{16}$$

In classical transportation issues, the transportation parameters such as cost $\Im_{ij}$, supply p_i, and demand q_j are typically assumed to have deterministic values, which means they are precise and not subject to uncertainty. Nevertheless, in practical scenarios, the exact values of transportation parameters may be uncertain due to various factors like fluctuating supply capacities, variable demand, unpredictable transportation costs, and unforeseen disruptions that affect the reliability of the transportation network. The CPyFNs have a robust modeling capacity to depict ambiguous information data. Thus, CPyFNs are employed to represent the ambiguous variables of the transportation problem being discussed in these situations. The problem in discussion is commonly referred to as the complex

Pythagorean fuzzy transportation problem. As a result, we obtain the following transportation problems with parameters represented in the form of CPyFNs.

4.1. Type1-CPyFTP: When the transportation cost is given in the form of CPyFNs

Here, we assume that the decision experts have uncertainty on the transportation cost. To characterize this uncertainty, we utilize CPyFNs $\mathfrak{I}_{ij}^{(CPyFN)}$. A tabular representation of Type1-CPyFTP is shown in Table 3. Let us assume that k_{ij} quantity of items/products transported from the i-th source to the j-th destination. The optimization model for Type1-CPyFTP is as follows:

$$\begin{aligned} &\text{minimize } \bar{Z} = \sum_{i=1}^{m} \sum_{j=1}^{n} \mathfrak{I}_{ij}^{(CPyFN)} k_{ij} \\ &\text{subject to } \begin{cases} \sum_{j=1}^{n} k_{ij} = p_i, & \text{for } i = 1,2,\ldots,m, \\ \sum_{i=1}^{m} k_{ij} = q_j, & \text{for } j = 1,2,\ldots,n, \\ k_{ij} \geqslant 0, & \text{for } i = 1,2,\ldots,m; \ j = 1,2,\ldots,n. \end{cases} \end{aligned} \tag{17}$$

where:

(1) m denotes the number of sources (suppliers);

(2) n represents the number of destinations (consumers);

(3) p_i is the supply available at the source S_i, $i = 1,2,\ldots,m$;

(4) q_j is the demand at the destination D_j, $j = 1,2,\ldots,n$;

Table 3: Tabular representation of a Type1-CPyFTP

Source/Destination	D_1	D_2	$\cdots$	D_n	Supply
S_1	$\mathfrak{I}_{11}^{(CPyFN)}$	$\mathfrak{I}_{12}^{(CPyFN)}$	$\cdots$	$\mathfrak{I}_{1n}^{(CPyFN)}$	p_1
S_2	$\mathfrak{I}_{21}^{(CPyFN)}$	$\mathfrak{I}_{22}^{(CPyFN)}$	$\cdots$	$\mathfrak{I}_{2n}^{(CPyFN)}$	p_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
S_m	$\mathfrak{I}_{m1}^{(CPyFN)}$	$\mathfrak{I}_{m2}^{(CPyFN)}$	$\cdots$	$\mathfrak{I}_{mn}^{(CPyFN)}$	p_m
Demand	q_1	q_2	$\cdots$	q_n	

(5) $\mathfrak{I}_{ij}^{(CPyFN)}$ is the complex Pythagorean fuzzy transportation cost of transporting one unit from the i-th source to the j-th destination for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$;

(6) k_{ij} – quantity of items/products transported from the S_i source to the D_j destination;

and $\mathfrak{I}_{ij}^{(CPyFN)} = \left\langle \xi_{\mathfrak{I}_{ij}} e^{i2\pi\alpha_{\mathfrak{I}_{ij}}}, \eta_{\mathfrak{I}_{ij}} e^{i2\pi\beta_{\mathfrak{I}_{ij}}} \right\rangle$ satisfying $\xi_{\mathfrak{I}_{ij}}, \eta_{\mathfrak{I}_{ij}} \in [0, 1]$, $\alpha_{\mathfrak{I}_{ij}}, \beta_{\mathfrak{I}_{ij}} \in [0, 1]$, $0 \leqslant \xi_{\mathfrak{I}_{ij}}^2 + \eta_{\mathfrak{I}_{ij}}^2 \leqslant 1$, $0 \leqslant \alpha_{\mathfrak{I}_{ij}}^2 + \beta_{\mathfrak{I}_{ij}}^2 \leqslant 1$.

4.2. Type2-CPyFTP: When the supply and demand are represented in the form of CPyFNs

In this scenario, we assume that the decision experts have uncertainty over the units of supply and demand. To express these values, we utilize CPyFNs $p_i^{(CPyFN)}$ and $q_i^{(CPyFN)}$. A tabular representation of Type2-CPyFTP is presented in Table 4. Let us assume that k_{ij} quantity of items/products transported from the i-th source to the j-th destination. The optimization model for Type2-CPyFTP is presented below:

$$\text{minimize } \bar{Z} = \sum_{i=1}^{m} \sum_{j=1}^{n} \mathfrak{I}_{ij} k_{ij}$$

$$\text{subject to } \begin{cases} \sum_{j=1}^{n} k_{ij} = p_i^{(CPyFN)}, & \text{for } i = 1, 2, \ldots, m, \\ \sum_{i=1}^{m} k_{ij} = q_j^{(CPyFN)}, & \text{for } j = 1, 2, \ldots, n, \\ k_{ij} \geqslant 0, & \text{for } i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n. \end{cases} \tag{18}$$

where:

(1) m denotes the number of sources (suppliers);

(2) n represents the number of destinations (consumers);

(3) $p_i^{(CPyFN)}$ is the complex Pythagorean fuzzy supply available at the source S_i, $i = 1, 2, \ldots, m$;

(4) $q_j^{(CPyFN)}$ is the complex Pythagorean fuzzy demand at the destination D_j, $j = 1, 2, \ldots, n$;

(5) $\mathfrak{I}_{ij}$ is the transportation cost of transporting one unit from the i-th source to the j-th destination for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$;

(6) k_{ij} – quantity of items/products transported from the S_i source to the D_j destination;

and $p_i^{(CPyFN)} = \left\langle \xi_{p_i} e^{i2\pi\alpha_{p_i}}, \eta_{p_i} e^{i2\pi\beta_{p_i}} \right\rangle$ and $q_j^{(CPyFN)} = \left\langle \xi_{q_j} e^{i2\pi\alpha_{q_j}}, \eta_{q_j} e^{i2\pi\beta_{q_j}} \right\rangle$ with hold the conditions $\xi_{p_i}, \eta_{p_i}, \xi_{q_j}, \eta_{q_{ij}} \in [0,\ 1]$, $\alpha_{p_i}, \beta_{p_i}, \alpha_{q_j}, \beta_{q_j} \in [0,\ 1]$, $0 \leqslant \xi_{p_i}^2 + \eta_{p_i}^2 \leqslant 1$, $0 \leqslant \xi_{q_j}^2 + \eta_{q_j}^2 \leqslant 1$, $0 \leqslant \alpha_{p_i}^2 + \beta_{p_i}^2 \leqslant 1$ and $0 \leqslant \alpha_{q_j}^2 + \beta_{q_j}^2 \leqslant 1$.

Table 4: Tabular representation of a Type2-CPyFTP

Source/Destination	D_1	D_2	$\cdots$	D_n	Supply
S_1	$\Im_{11}$	$\Im_{12}$	$\cdots$	$\Im_{1n}$	$p_1^{(CPyFN)}$
S_2	$\Im_{21}$	$\Im_{22}$	$\cdots$	$\Im_{2n}$	$p_2^{(CPyFN)}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
S_m	$\Im_{m1}$	$\Im_{m2}$	$\cdots$	$\Im_{mn}$	$p_m^{(CPyFN)}$
Demand	$q_1^{(CPyFN)}$	$q_2^{(CPyFN)}$	$\cdots$	$q_n^{(CPyFN)}$	

4.3. Type3-CPyFTP: When all the transportation parameters are expressed in the form of CPyFNs

In this scenario, we assume that the decision experts have uncertainty regarding various transportation parameters, including transportation cost, supply, and demand. To represent these values, we utilize CPyFNs $\Im_{ij}^{(CPyFN)}$, $p_i^{(CPyFN)}$, and $q_i^{(CPyFN)}$. A tabular representation of Type3-CPyFTP is shown in Table 5. Let us assume that k_{ij} quantity of items/products transported from the i-th source to the j-th destination. The optimization model for Type3-CPyFTP is presented below:

$$\text{minimize} \ \ \bar{Z} = \sum_{i=1}^{m}\sum_{j=1}^{n} \Im_{ij}^{(CPyFN)} k_{ij}$$

$$\text{subject to} \ \begin{cases} \displaystyle\sum_{j=1}^{n} k_{ij} = p_i^{(CPyFN)}, & \text{for } \ i = 1,2,\ldots,m, \\ \displaystyle\sum_{i=1}^{m} k_{ij} = q_j^{(CPyFN)}, & \text{for } \ j = 1,2,\ldots,n, \\ k_{ij} \geqslant 0, & \text{for } \ i = 1,2,\ldots,m; \ \ j = 1,2,\ldots,n. \end{cases} \tag{19}$$

where:

Table 5: Tabular representation of a Type3-CPyFTP

Source/Destination	D_1	D_2	$\cdots$	D_n	Supply
S_1	$\mathfrak{J}_{11}^{(CPyFN)}$	$\mathfrak{J}_{12}^{(CPyFN)}$	$\cdots$	$\mathfrak{J}_{1n}^{(CPyFN)}$	$p_1^{(CPyFN)}$
S_2	$\mathfrak{J}_{21}^{(CPyFN)}$	$\mathfrak{J}_{22}^{(CPyFN)}$	$\cdots$	$\mathfrak{J}_{2n}^{(CPyFN)}$	$p_2^{(CPyFN)}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
S_m	$\mathfrak{J}_{m1}^{(CPyFN)}$	$\mathfrak{J}_{m2}^{(CPyFN)}$	$\cdots$	$\mathfrak{J}_{mn}^{(CPyFN)}$	$p_m^{(CPyFN)}$
Demand	$q_1^{(CPyFN)}$	$q_2^{(CPyFN)}$	$\cdots$	$q_n^{(CPyFN)}$	

(1) m denotes the number of sources (suppliers);

(2) n represents the number of destinations (consumers);

(3) $p_i^{(CPyFN)}$ is the complex Pythagorean fuzzy supply available at the source S_i, $i = 1, 2, \ldots, m$;

(4) $q_j^{(CPyFN)}$ is the complex Pythagorean fuzzy demand at the destination D_j, $j = 1, 2, \ldots, n$;

(5) $\mathfrak{J}_{ij}^{(CPyFN)}$ is the complex Pythagorean fuzzy transportation cost of transporting one unit from the i-th source to the j-th destination for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$;

(6) k_{ij} = quantity of items/products transported from the S_i source to the D_j destination,

where $\mathfrak{J}_{ij}^{(CPyFN)} = \left\langle \xi_{\mathfrak{J}_{ij}} e^{i2\pi\alpha_{\mathfrak{J}_{ij}}}, \eta_{\mathfrak{J}_{ij}} e^{i2\pi\beta_{\mathfrak{J}_{ij}}} \right\rangle$ satisfying $\xi_{\mathfrak{J}_{ij}}, \eta_{\mathfrak{J}_{ij}} \in [0,\ 1]$, $\alpha_{\mathfrak{J}_{ij}}, \beta_{\mathfrak{J}_{ij}} \in [0,\ 1]$, $0 \leqslant \xi_{\mathfrak{J}_{ij}}^2 + \eta_{\mathfrak{J}_{ij}}^2 \leqslant 1$, $0 \leqslant \alpha_{\mathfrak{J}_{ij}}^2 + \beta_{\mathfrak{J}_{ij}}^2 \leqslant 1$; $p_i^{(CPyFN)} = \left\langle \xi_{p_i} e^{i2\pi\alpha_{p_i}}, \eta_{p_i} e^{i2\pi\beta_{p_i}} \right\rangle$ and $q_j^{(CPyFN)} = \left\langle \xi_{q_j} e^{i2\pi\alpha_{q_j}}, \eta_{q_j} e^{i2\pi\beta_{q_j}} \right\rangle$ with hold the conditions $\xi_{p_i}, \eta_{p_i}, \xi_{q_j}, \eta_{q_{ij}} \in [0,\ 1], \alpha_{p_i}, \beta_{p_i}, \alpha_{q_j}, \beta_{q_j} \in [0,\ 1], 0 \leqslant \xi_{p_i}^2 + \eta_{p_i}^2 \leqslant 1$, $0 \leqslant \xi_{q_j}^2 + \eta_{q_j}^2 \leqslant 1$, $0 \leqslant \alpha_{p_i}^2 + \beta_{p_i}^2 \leqslant 1$ and $0 \leqslant \alpha_{q_j}^2 + \beta_{q_j}^2 \leqslant 1$.

5. Algorithm for solution of CPyFTPs

This section presents a methodological approach to addressing Type1-CPyFTP, Type2-CPyFTP, and Type3-CPyFTP problems utilizing the newly proposed score function value. Initially, we calculate the initial basic feasible solution (IBFS) for the given transportation problem in a complex Pythagorean fuzzy framework using the suggested steps. In addition, we employ the widely recognized modified distribution (MODI) method to determine the optimal solution for the particular transportation problem.

Step 1: Create a tabular representation of the CPyFTP. This phase involves meticulously examining the data associated with the transportation problem to ascertain the suitable categorization for the presented CPyFTP.

Step 2: Transforming CPyFTPs into an equivalent crisp TP.

This step is divided into three sub-steps, as detailed below.

Step 2a: *TypeI-CPyFTP* – Compute the score function value of each CPyFN cost by using advanced score function value $\underset{IMP}{\overbrace{\mathcal{SFV}}}$ given in Eq. (6). Replace all the CPyFN cost with their corresponding score function values to obtain an equivalent crisp transportation problem.

Or

Step 2b: *Type2-CPyFTP* – Compute the score function value of each CPyFN supply and CPyFN demand unit by using advanced score function value $\underset{IMP}{\overbrace{\mathcal{SFV}}}$ given in Eq. (6). Replace all the CPyFN supply and CPyFN demand units with their corresponding score function values to obtain an equivalent crisp transportation problem.

Or

Step 2c: *Type3-CPyFTP* – Compute the score function value of each CPyFN cost, CPyFN supply and CPyFN demand unit by using advanced score function value $\underset{IMP}{\overbrace{\mathcal{SFV}}}$ given in Eq. (6). Replace all the CPyFN costs, CPyFN supply and CPyFN demand units with their corresponding score function values to obtain an equivalent crisp transportation problem.

Step 3: Check whether the given transportation problem is balanced according to the following mathematical expressions:

(i) *TypeI-CPyFTP* – $\sum_{i=1}^{m} p_i = \sum_{j=1}^{n} q_j$

(ii) *Type2-CPyFTP* – $\sum_{i=1}^{m} \underset{IMP}{\overbrace{\mathcal{SFV}}} \left(p_i^{(CPyFN)}\right) = \sum_{j=1}^{n} \underset{IMP}{\overbrace{\mathcal{SFV}}} \left(q_j^{(CPyFN)}\right)$

(iii) *Type3-CPyFTP* – $\sum_{i=1}^{m} \underset{IMP}{\overbrace{\mathcal{SFV}}} \left(p_i^{(CPyFN)}\right) = \sum_{j=1}^{n} \underset{IMP}{\overbrace{\mathcal{SFV}}} \left(q_j^{(CPyFN)}\right)$. If the transportation problem is balanced, proceed directly to Step 4. If the TP is unbalanced, implying that the demand does not equal the supply, then insert a dummy variable on the Demand/Supply to make it balanced and proceed to Step 4.

Step 4: Now, find the initial basic feasible solution for the equivalent crisp transportation problem using the method provided as follows:

Step 4a: Examine every row in the transportation table and determine the lowest and second lowest costs. Compute the difference between these costs for each row and exhibit the outcomes alongside the transportation table enclosing them in parenthesis corresponding to the respective rows. These differences are called the penalties for the rows.

Step 4b: Examine every column in the transportation table and determine the lowest and second lowest costs. Compute the difference between these costs for each column and exhibit the outcomes alongside the transportation table, enclosing them in parenthesis corresponding to the respective columns. These differences are called the penalties for the columns.

Step 4c: Determine the row or column that has the highest penalty among all rows and columns. In case of a tie, employ any arbitrary tie-breaking choice. Let the highest penalty correspond to the i-th row and find the lowest cost in that row. Assign the maximum feasible amount $k_{ij} = \min(p_i, q_j)$ in the (i, j)-th cell and mark both the i-th row and the j-th column in the customary fashion, enclosing the allocation with brackets.

Step 4d: Recompute the penalties for the columns and rows in the reduced transportation table, continuing with Step 4c. Proceed iterating through the procedure until all the requirements are fulfilled.

Step 5: Check the optimality of the solution in a transportation problem by the MODI method.

6. Numerical examples of CPyFTP

Example 5. Consider a scenario in which a local blood bank in Noida, Uttar Pradesh, India, encounters difficulties in effectively delivering blood supplies to different hospitals in the nearby cities. The timely and cost-effective delivery of blood is vital for patient care, emergency response, and overall healthcare efficiency in the surrounding cities. Assume that a blood bank based in Noida supplies to four hospitals located in different nearby cities represented by Hospital-1 – ($\mathscr{H}1$), Hospital-2 – ($\mathscr{H}2$), Hospital-3 – ($\mathscr{H}3$), and Hospital-4 – ($\mathscr{H}4$). The blood bank operates three collection locations, specifically designated as Collection-Center-1 ($\mathbb{CC}1$), Collection-Center-2 ($\mathbb{CC}2$), and Collection-Center-3 ($\mathbb{CC}3$), for the purpose of receiving blood donations. The objective is to determine the most efficient transportation strategy that minimizes the overall cost of transportation while also guaranteeing that the blood demand at each hospital is satisfied. Now, we will examine three specific scenarios to showcase the effectiveness of our algorithm in solving this problem.

Case-1: When the transportation cost is uncertain and represented in the form of CPyFNs

Step 1: The problem Type1-CPyFTP is represented in a tabular format in Table 6.

Step 2: The improved score function value given in Eq. (6) is used to transform the complex Pythagorean fuzzy costs into crisp costs as follows:

$$\widetilde{\mathcal{SFV}}_{IMP}\left(\mathfrak{I}_{11}^{(CPyFN)}\right) = 0.3220, \quad \widetilde{\mathcal{SFV}}_{IMP}\left(\mathfrak{I}_{12}^{(CPyFN)}\right) = 0.4035,$$

$$\widetilde{\mathcal{SFV}}_{IMP}\left(\mathfrak{I}_{13}^{(CPyFN)}\right) = 0.2544, \quad \widetilde{\mathcal{SFV}}_{IMP}\left(\mathfrak{I}_{14}^{(CPyFN)}\right) = 0.3789,$$

$$\widetilde{\mathcal{SFV}}_{IMP}\left(\mathfrak{I}_{21}^{(CPyFN)}\right) = 0.2210, \quad \widetilde{\mathcal{SFV}}_{IMP}\left(\mathfrak{I}_{22}^{(CPyFN)}\right) = 0.2552,$$

$$\widetilde{\mathcal{SFV}}_{IMP}\left(\mathfrak{I}_{23}^{(CPyFN)}\right) = 0.5316, \quad \widetilde{\mathcal{SFV}}_{IMP}\left(\mathfrak{I}_{24}^{(CPyFN)}\right) = 0.2817,$$

$$\widetilde{\mathcal{SFV}}_{IMP}\left(\mathfrak{I}_{31}^{(CPyFN)}\right) = 0.5064, \quad \widetilde{\mathcal{SFV}}_{IMP}\left(\mathfrak{I}_{32}^{(CPyFN)}\right) = 0.6438,$$

$$\widetilde{\mathcal{SFV}}_{IMP}\left(\mathfrak{I}_{33}^{(CPyFN)}\right) = 0.3587, \quad \widetilde{\mathcal{SFV}}_{IMP}\left(\mathfrak{I}_{34}^{(CPyFN)}\right) = 0.7334.$$

The corresponding equivalent crisp transportation problem is displayed in Table 7, derived from the calculated score function values.

Step 3: Here $\sum_{i=1}^{3} p_i = 350$ and $\sum_{j=1}^{4} q_j = 350$ which implies $\sum_{i=1}^{3} p_i = \sum_{j=1}^{4} q_j = 350$. This shows that the above transportation problem is balanced.

Step 4: Now, we compute IBFS by following the subsequent steps.

Steps 4a–4d: All these steps are shown in Tables 8–14.

Hence, we get the IBFS as follows

$$(\mathbb{CC}1, \mathscr{H}1) = k_{11} = 20, \quad (\mathbb{CC}1, \mathscr{H}3) = k_{13} = 50, \quad (\mathbb{CC}1, \mathscr{H}4) = k_{14} = 50,$$

$$(\mathbb{CC}2, \mathscr{H}1) = k_{21} = 50, \quad (\mathbb{CC}2, \mathscr{H}2) = k_{22} = 100, \quad (\mathbb{CC}3, \mathscr{H}3) = k_{33} = 80.$$

The minimum transportation cost corresponding to IBFS is obtained as: $0.3220 \times 20 + 0.2554 \times 50 + 0.3789 \times 50 + 0.2210 \times 50 + 0.2552 \times 100 + 0.3587 \times 80 = 103.42$.

Table 6: Type1-CPyFTP of Ex. 6

$\mathbb{CC}/\mathscr{H}$	$\mathscr{H}1$	$\mathscr{H}2$	$\mathscr{H}3$	$\mathscr{H}4$	Supply
$\mathbb{CC}1$	$\langle 0.7e^{i2\pi 0.5}, 0.3e^{i2\pi 0.8}\rangle$	$\langle 0.4e^{i2\pi 0.6}, 0.8e^{i2\pi 0.6}\rangle$	$\langle 0.5e^{i2\pi 0.4}, 0.4e^{i2\pi 0.7}\rangle$	$\langle 0.9e^{i2\pi 0.3}, 0.1e^{i2\pi 0.4}\rangle$	120
$\mathbb{CC}2$	$\langle 0.3e^{i2\pi 0.4}, 0.6e^{i2\pi 0.7}\rangle$	$\langle 0.7e^{i2\pi 0.2}, 0.4e^{i2\pi 0.5}\rangle$	$\langle 0.5e^{i2\pi 0.6}, 0.8e^{i2\pi 0.3}\rangle$	$\langle 0.8e^{i2\pi 0.3}, 0.2e^{i2\pi 0.6}\rangle$	150
$\mathbb{CC}3$	$\langle 0.5e^{i2\pi 0.6}, 0.6e^{i2\pi 0.4}\rangle$	$\langle 0.4e^{i2\pi 0.7}, 0.5e^{i2\pi 0.2}\rangle$	$\langle 0.3e^{i2\pi 0.5}, 0.8e^{i2\pi 0.5}\rangle$	$\langle 0.7e^{i2\pi 0.8}, 0.5e^{i2\pi 0.3}\rangle$	80
Demand	70	100	130	50	

* Note: Here, the demand and supply values are given in units.

Table 7: Crisp transportation problem corresponding Type1-CPyFTP for Ex. 6

$\mathbb{CC}\mathscr{H}$	$\mathscr{H}1$	$\mathscr{H}2$	$\mathscr{H}3$	$\mathscr{H}4$	Supply
$\mathbb{CC}1$	0.3220	0.4035	0.2544	0.3789	120
$\mathbb{CC}2$	0.2210	0.2552	0.5316	0.2817	150
$\mathbb{CC}3$	0.5064	0.6438	0.3587	0.7334	80
Demand	70	100	130	50	

Table 8: First allocation

$\mathbb{CC}/\mathscr{H}$	$\mathscr{H}1$	$\mathscr{H}2$	$\mathscr{H}3$	$\mathscr{H}4$	Supply	Penalties
$\mathbb{CC}1$	0.3220	0.4035	0.2544	0.3789	120	(0.0676)
$\mathbb{CC}2$	0.2210	0.2552 (**100**)	0.5316	0.2817	50	(0.0342)
$\mathbb{CC}3$	0.5064	0.6438	0.3587	0.7334	80	(0.1477)
Demand	70	0	130	50		
Penalties	(0.1010)	(0.1483)	(0.1043)	(0.0972)		

Table 9: Second allocation

$\mathbb{CC}/\mathscr{H}$	$\mathscr{H}1$	$\mathscr{H}2$	$\mathscr{H}3$	$\mathscr{H}4$	Supply	Penalties
$\mathbb{CC}1$	0.3220	0.4035	0.2544	0.3789	120	(0.0676)
$\mathbb{CC}2$	0.2210	0.2552 (**100**)	0.5316	0.2817	50	(0.0607)
$\mathbb{CC}3$	0.5064	0.6438	0.3587 (**80**)	0.7334	0	(0.1477)
Demand	70	0	50	50		
Penalties	(0.1010)	–	(0.1043)	(0.0972)		

Table 10: Third allocation

$\mathbb{CC}/\mathscr{H}$	$\mathscr{H}1$	$\mathscr{H}2$	$\mathscr{H}3$	$\mathscr{H}4$	Supply	Penalties
$\mathbb{CC}1$	0.3220	0.4035	0.2544 (**50**)	0.3789	70	(0.0676)
$\mathbb{CC}2$	0.2210	0.2552 (**100**)	0.5316	0.2817	50	(0.0342)
$\mathbb{CC}3$	0.5064	0.6438	0.3587 (**80**)	0.7334	0	–
Demand	70	0	0	50		
Penalties	(0.1010)	–	(0.2772)	(0.0972)		

Table 11: Forth allocation

$\mathbb{CC}/\mathscr{H}$	$\mathscr{H}1$	$\mathscr{H}2$	$\mathscr{H}3$	$\mathscr{H}4$	Supply	Penalties
$\mathbb{CC}1$	0.3220	0.4035	0.2544 (**50**)	0.3789	70	(0.0569)
$\mathbb{CC}2$	0.2210 (**50**)	0.2552 (**100**)	0.5316	0.2817	0	(0.0607)
$\mathbb{CC}3$	0.5064	0.6438	0.3587 (**80**)	0.7334	0	–
Demand	20	0	0	50		
Penalties	(0.1010)	–	–	(0.0972)		

Table 12: Fifth allocation

$\mathbb{CC}/\mathscr{H}$	$\mathscr{H}1$	$\mathscr{H}2$	$\mathscr{H}3$	$\mathscr{H}4$	Supply	Penalties
$\mathbb{CC}1$	0.3220	0.4035	0.2544 (**50**)	0.3789 (**50**)	20	(0.0676)
$\mathbb{CC}2$	0.2210 (**50**)	0.2552 (**100**)	0.5316	0.2817	0	–
$\mathbb{CC}3$	0.5064	0.6438	0.3587 (**80**)	0.7334	0	–
Demand	20	0	0	0		
Penalties	(0.3220)	–	–	(0.3789)		

Table 13: Sixth allocation

$\mathbb{CC}/\mathscr{H}$	$\mathscr{H}1$	$\mathscr{H}2$	$\mathscr{H}3$	$\mathscr{H}4$	Supply	Penalties
$\mathbb{CC}1$	0.3220 (**20**)	0.4035	0.2544 (**50**)	0.3789 (**50**)	0	(0.0676)
$\mathbb{CC}2$	0.2210 (**50**)	0.2552 (**100**)	0.5316	0.2817	0	–
$\mathbb{CC}3$	0.5064	0.6438	0.3587 (**80**)	0.7334	0	–
Demand	70	0	0	0		
Penalties	(0.3220)	–	–	–		

Table 14: Final allocations for Type1-CPyFTP of Ex. 6

$\mathbb{CC}/\mathscr{H}$	$\mathscr{H}1$	$\mathscr{H}2$	$\mathscr{H}3$	$\mathscr{H}4$	Supply
$\mathbb{CC}1$	0.3220 (**20**)	0.4035	0.2544 (**50**)	0.3789 (**50**)	120
$\mathbb{CC}2$	0.2210 (**50**)	0.2552 (**100**)	0.5316	0.2817	150
$\mathbb{CC}3$	0.5064	0.6438	0.3587 (**80**)	0.7334	80
Demand	70	100	130	50	

Steps 5: Now, we examine the optimality of the transportation problem. Here, the number of allocated cells = 6 is equal to $m + n - 1 = 3 + 4 - 1 = 6$, indicating a non-degenerate solution. Following that, the MODI method is employed in order to assess the optimality of the IBFS for the designated Type1-CPyFTP. The outcomes obtained are summarized as follows:

$$(\mathbb{CC}1, \mathscr{H}1) = k_{11} = 20, \quad (\mathbb{CC}1, \mathscr{H}3) = k_{13} = 50, \quad (\mathbb{CC}1, \mathscr{H}4) = k_{14} = 50,$$
$$(\mathbb{CC}2, \mathscr{H}1) = k_{21} = 50, \quad (\mathbb{CC}2, \mathscr{H}2) = k_{22} = 100, \quad (\mathbb{CC}3, \mathscr{H}3) = k_{33} = 80.$$

The minimum total transportation cost is obtained as: $0.3220 \times 20 + 0.2554 \times 50 + 0.3789 \times 50 + 0.2210 \times 50 + 0.2552 \times 100 + 0.3587 \times 80 = 103.42$.

Case-2: When the supply and demand are represented in the form of CPyFNs

Step 1: The problem Type2-CPyFTP is displayed in a tabular fashion in Table 15.

Table 15: Type2-CPyFTP of Ex. 6

$\mathbb{CC}/\mathscr{H}$	$\mathscr{H}1$	$\mathscr{H}2$	$\mathscr{H}3$	$\mathscr{H}4$	Supply
$\mathbb{CC}1$	1500	1250	2000	2500	$\langle 0.9e^{i2\pi 0.7}, 0.4e^{i2\pi 0.6} \rangle$
$\mathbb{CC}2$	1800	2200	2800	2000	$\langle 0.7e^{i2\pi 0.5}, 0.6e^{i2\pi 0.8} \rangle$
$\mathbb{CC}3$	2100	2700	3000	2300	$\langle 0.8e^{i2\pi 0.5}, 0.6e^{i2\pi 0.4} \rangle$
Demand	$\langle 0.5e^{i2\pi 0.4}, 0.7e^{i2\pi 0.3} \rangle$	$\langle 0.8e^{i2\pi 0.5}, 0.4e^{i2\pi 0.6} \rangle$	$\langle 0.7e^{i2\pi 0.5}, 0.5e^{i2\pi 0.4} \rangle$	$\langle 0.9e^{i2\pi 0.5}, 0.2e^{i2\pi 0.7} \rangle$	

Note: Here, the cost values are given in rupees/unit.

Step 2: The improved score function value given in Eq. (6) is used to transform the complex Pythagorean fuzzy supplies and demands into crisp values as follows:

$$\widetilde{\mathcal{SFV}}_{IMP}\left(p_1^{(CPyFN)}\right) = 0.5747, \qquad \widetilde{\mathcal{SFV}}_{IMP}\left(p_2^{(CPyFN)}\right) = 0.3043,$$
$$\widetilde{\mathcal{SFV}}_{IMP}\left(p_3^{(CPyFN)}\right) = 0.4710, \qquad \widetilde{\mathcal{SFV}}_{IMP}\left(q_1^{(CPyFN)}\right) = 0.3940,$$
$$\widetilde{\mathcal{SFV}}_{IMP}\left(q_2^{(CPyFN)}\right) = 0.3994, \qquad \widetilde{\mathcal{SFV}}_{IMP}\left(q_3^{(CPyFN)}\right) = 0.4628,$$
$$\widetilde{\mathcal{SFV}}_{IMP}\left(q_4^{(CPyFN)}\right) = 0.3834.$$

The corresponding equivalent crisp transportation problem is displayed in Table 16, derived from the calculated score function values.

Table 16: Crisp transportation problem corresponding Type2-CPyFTP for Ex. 6

$\mathbb{CC}/\mathscr{H}$	$\mathscr{H}1$	$\mathscr{H}2$	$\mathscr{H}3$	$\mathscr{H}4$	Supply
$\mathbb{CC}1$	1500	1250	2000	2500	0.5747
$\mathbb{CC}2$	1800	2200	2800	2000	0.3043
$\mathbb{CC}3$	2100	2700	3000	2300	0.4710
Demand	0.3940	0.3994	0.4628	0.3834	

Step 3: Here $\sum_{i=1}^{3} p_i = 1.3500$ and $\sum_{j=1}^{4} q_j = 1.6396$, which implies $\sum_{i=1}^{3} p_i = 1.3500 < \sum_{j=1}^{4} q_j = 1.6396$. This shows that the above transportation problem is unbalanced. To balance it, we add a dummy collection center denoted by $\overset{dummy}{\mathbb{CC}4}$ having supply $\sum_{j=1}^{4} q_j - \sum_{i=1}^{3} p_i = 1.6386 - 1.3500 = 0.2896$ and cost values as zero. The corresponding balanced transportation problem is shown in Table 17.

Table 17: Balanced crisp transportation problem corresponding Type2-CPyFTP of Ex. 6

$\mathbb{CC}/\mathscr{H}$	$\mathscr{H}1$	$\mathscr{H}2$	$\mathscr{H}3$	$\mathscr{H}4$	Supply
$\mathbb{CC}1$	1500	1250	2000	2500	0.5747
$\mathbb{CC}2$	1800	2200	2800	2000	0.3043
$\mathbb{CC}3$	2100	2700	3000	2300	0.4710
$\overset{dummy}{\mathbb{CC}4}$	0	0	0	0	0.2896
Demand	0.3940	0.3994	0.4628	0.3834	

Step 4: Now, we compute IBFS by following the subsequent steps.

Steps 4a–4d: After following all these steps, the final allocations are shown in Table 18.

Hence, we get the IBFS as follows

$$(\mathbb{CC}1, \mathscr{H}1) = k_{11} = 0.0021, \quad (\mathbb{CC}1, \mathscr{H}2) = k_{12} = 0.3994,$$
$$(\mathbb{CC}1, \mathscr{H}3) = k_{13} = 0.1732, \quad (\mathbb{CC}2, \mathscr{H}1) = k_{21} = 0.3043,$$
$$(\mathbb{CC}3, \mathscr{H}1) = k_{31} = 0.0876, \quad (\mathbb{CC}3, \mathscr{H}4) = k_{34} = 0.3834,$$

Table 18: Final allocation for Type2-CPyFTP of Ex. 6

$\mathbb{CC}/\mathscr{H}$	$\mathscr{H}1$	$\mathscr{H}2$	$\mathscr{H}3$	$\mathscr{H}4$	Supply
$\mathbb{CC}1$	1500 (**0.0021**)	1250 (**0.3994**)	2000 (**0.1732**)	2500	0
$\mathbb{CC}2$	1800 (**0.3043**)	2200	2800	2000	0
$\mathbb{CC}3$	2100 (**0.0876**)	2700	3000	2300 (**0.3834**)	0
$\overset{dummy}{\mathbb{CC}4}$	0	0	0 (**0.2896**)	0	0
Demand	0	0	0	0	

$$\left(\overset{dummy}{\mathbb{CC}4}, \mathscr{H}3\right) = k_{43} = 0.2896.$$

The minimum transportation cost corresponding to IBFS is obtained as: $0.0021 \times 1500 + 0.3994 \times 1250 + 0.1732 \times 2000 + 0.3043 \times 1800 + 0.0876 \times 2100 + 0.3834 \times 2300 + 0.2896 \times 0 = 2462.40$.

Steps 5: Following that, the MODI approach is employed in order to assess the optimality of the IBFS for the designated Type2-CPyFTP. The outcomes obtained are summarized as follows:

$$\begin{aligned}
(\mathbb{CC}1, \mathscr{H}1) &= k_{11} = 0.0021, \quad & (\mathbb{CC}1, \mathscr{H}2) &= k_{12} = 0.3994,\\
(\mathbb{CC}1, \mathscr{H}3) &= k_{13} = 0.1732, \quad & (\mathbb{CC}2, \mathscr{H}1) &= k_{21} = 0.3043,\\
(\mathbb{CC}3, \mathscr{H}1) &= k_{31} = 0.0876, \quad & (\mathbb{CC}3, \mathscr{H}4) &= k_{34} = 0.3834,\\
\left(\overset{dummy}{\mathbb{CC}4}, \mathscr{H}3\right) &= k_{43} = 0.2896.
\end{aligned}$$

The minimum total transportation cost is obtained as: $0.0021 \times 1500 + 0.3994 \times 1250 + 0.1732 \times 2000 + 0.3043 \times 1800 + 0.0876 \times 2100 + 0.3834 \times 2300 + 0.2896 \times 0 = 2462.40$.

Case-3: When all the transportation parameters such as cost, supply and demand are represented in the form of CPyFNs

Step 1: The problem Type3-CPyFTP is displayed in a tabular fashion in Table 19.

Step 2: The improved score function value given in Eq. (6) is used to transform the complex Pythagorean fuzzy costs, supplies and demands into crisp values as follows:

$$\underset{IMP}{\widetilde{\mathscr{SFV}}}\left(\mathfrak{I}_{11}^{(CPyFN)}\right) = 0.5732, \qquad \underset{IMP}{\widetilde{\mathscr{SFV}}}\left(\mathfrak{I}_{12}^{(CPyFN)}\right) = 0.6131,$$

Table 19: Type3-CPyFTP of Ex. 6

$\mathbb{CC}/\mathscr{H}$	$\mathscr{H}1$	$\mathscr{H}2$	$\mathscr{H}3$	$\mathscr{H}4$	Supply
$\mathbb{CC}1$	$\langle 0.6e^{i2\pi 0.7}, 0.4e^{i2\pi 0.5}\rangle$	$\langle 0.5e^{i2\pi 0.8}, 0.4e^{i2\pi 0.6}\rangle$	$\langle 0.7e^{i2\pi 0.4}, 0.7e^{i2\pi 0.8}\rangle$	$\langle 0.9e^{i2\pi 0.6}, 0.2e^{i2\pi 0.3}\rangle$	$\langle 0.8e^{i2\pi 0.4}, 0.4e^{i2\pi 0.7}\rangle$
$\mathbb{CC}2$	$\langle 0.9e^{i2\pi 0.5}, 0.1e^{i2\pi 0.6}\rangle$	$\langle 0.6e^{i2\pi 0.8}, 0.7e^{i2\pi 0.2}\rangle$	$\langle 0.5e^{i2\pi 0.5}, 0.6e^{i2\pi 0.6}\rangle$	$\langle 0.7e^{i2\pi 0.4}, 0.3e^{i2\pi 0.5}\rangle$	$\langle 0.9e^{i2\pi 0.3}, 0.2e^{i2\pi 0.8}\rangle$
$\mathbb{CC}3$	$\langle 0.7e^{i2\pi 0.5}, 0.2e^{i2\pi 0.4}\rangle$	$\langle 0.5e^{i2\pi 0.7}, 0.8e^{i2\pi 0.6}\rangle$	$\langle 0.8e^{i2\pi 0.6}, 0.4e^{i2\pi 0.7}\rangle$	$\langle 0.6e^{i2\pi 0.8}, 0.2e^{i2\pi 0.4}\rangle$	$\langle 0.8e^{i2\pi 0.5}, 0.3e^{i2\pi 0.5}\rangle$
Demand	$\langle 0.7e^{i2\pi 0.6}, 0.5e^{i2\pi 0.3}\rangle$	$\langle 0.9e^{i2\pi 0.5}, 0.2e^{i2\pi 0.7}\rangle$	$\langle 0.8e^{i2\pi 0.7}, 0.5e^{i2\pi 0.5}\rangle$	$\langle 0.6e^{i2\pi 0.4}, 0.6e^{i2\pi 0.7}\rangle$	

$$\underset{IMP}{\widetilde{\mathscr{SFV}}}\left(\Im_{13}^{(CPyFN)}\right) = 0.2298, \quad \underset{IMP}{\widetilde{\mathscr{SFV}}}\left(\Im_{14}^{(CPyFN)}\right) = 0.6159,$$

$$\underset{IMP}{\widetilde{\mathscr{SFV}}}\left(\Im_{21}^{(CPyFN)}\right) = 0.4237, \quad \underset{IMP}{\widetilde{\mathscr{SFV}}}\left(\Im_{22}^{(CPyFN)}\right) = 0.7332,$$

$$\underset{IMP}{\widetilde{\mathscr{SFV}}}\left(\Im_{23}^{(CPyFN)}\right) = 0.3483, \quad \underset{IMP}{\widetilde{\mathscr{SFV}}}\left(\Im_{24}^{(CPyFN)}\right) = 0.3642,$$

$$\underset{IMP}{\widetilde{\mathscr{SFV}}}\left(\Im_{31}^{(CPyFN)}\right) = 0.4787, \quad \underset{IMP}{\widetilde{\mathscr{SFV}}}\left(\Im_{32}^{(CPyFN)}\right) = 0.4985,$$

$$\underset{IMP}{\widetilde{\mathscr{SFV}}}\left(\Im_{33}^{(CPyFN)}\right) = 0.4395, \quad \underset{IMP}{\widetilde{\mathscr{SFV}}}\left(\Im_{34}^{(CPyFN)}\right) = 0.7053,$$

$$\underset{IMP}{\widetilde{\mathscr{SFV}}}\left(p_{1}^{(CPyFN)}\right) = 0.2924, \quad \underset{IMP}{\widetilde{\mathscr{SFV}}}\left(p_{2}^{(CPyFN)}\right) = 0.2262,$$

$$\underset{IMP}{\widetilde{\mathscr{SFV}}}\left(p_{3}^{(CPyFN)}\right) = 0.4460, \quad \underset{IMP}{\widetilde{\mathscr{SFV}}}\left(q_{1}^{(CPyFN)}\right) = 0.5745,$$

$$\underset{IMP}{\widetilde{\mathscr{SFV}}}\left(q_{2}^{(CPyFN)}\right) = 0.3832, \quad \underset{IMP}{\widetilde{\mathscr{SFV}}}\left(q_{3}^{(CPyFN)}\right) = 0.5936,$$

$$\underset{IMP}{\widetilde{\mathscr{SFV}}}\left(q_{4}^{(CPyFN)}\right) = 0.2535.$$

The corresponding equivalent crisp transportation problem is displayed in Table 20, derived from the calculated score function values.

Table 20: Crisp transportation problem corresponding Type3-CPyFTP of Ex. 6

$\mathbb{CC}/\mathscr{H}$	$\mathscr{H}1$	$\mathscr{H}2$	$\mathscr{H}3$	$\mathscr{H}4$	Supply
$\mathbb{CC}1$	0.5732	0.6131	0.2298	0.6159	0.2924
$\mathbb{CC}2$	0.4237	0.7332	0.3483	0.3642	0.2262
$\mathbb{CC}3$	0.4787	0.4985	0.4395	0.7053	0.4460
Demand	0.5745	0.3832	0.5936	0.2535	

Step 3: Here $\sum_{i=1}^{3} p_i = 0.9646$ and $\sum_{j=1}^{4} q_j = 1.8048$, which implies $\sum_{i=1}^{3} p_i = 0.9646 < \sum_{j=1}^{4} q_j = 1.7298$. This shows that the above transportation problem is unbalanced. To balance it, we add a dummy collection center denoted by $\overset{dummy}{\mathbb{CC}4}$

having supply $\sum_{j=1}^{4} q_j - \sum_{i=1}^{3} p_i = 1.7298 - 0.9946 = 0.7352$ and cost values as zero. The corresponding balanced transportation problem is shown in Table 21.

Table 21: Balanced crisp transportation problem corresponding Type3-CPyFTP of Ex. 6

$\mathbb{CC}/\mathscr{H}$	$\mathscr{H}1$	$\mathscr{H}2$	$\mathscr{H}3$	$\mathscr{H}4$	Supply
$\mathbb{CC}1$	0.5732	0.6131	0.2298	0.6159	0.2924
$\mathbb{CC}2$	0.4237	0.7332	0.3483	0.3642	0.2262
$\mathbb{CC}3$	0.4787	0.4985	0.4395	0.7053	0.4460
$\overset{dummy}{\mathbb{CC}4}$	0	0	0	0	0.8402
Demand	0.5745	0.3832	0.5936	0.2535	

Step 4: Now, we compute IBFS by following the subsequent steps.

Steps 4a–4d: After following all these steps, the final allocations are shown in Table 22.

Table 22: Final allocation for Type3-CPyFTP of Ex. 6

$\mathbb{CC}/\mathscr{H}$	$\mathscr{H}1$	$\mathscr{H}2$	$\mathscr{H}3$	$\mathscr{H}4$	Supply
$\mathbb{CC}1$	0.5732	0.6131	0.2298(**0.2924**)	0.6159	0.2924
$\mathbb{CC}2$	0.4237	0.7332	0.3483	0.3642(**0.2262**)	0.2262
$\mathbb{CC}3$	0.4787(**0.1175**)	0.4985	0.4395(**0.3012**)	0.7053(**0.0273**)	0.4460
$\overset{dummy}{\mathbb{CC}4}$	0(**0.4570**)	0(**0.3832**)	0	0	0.8402
Demand	0.5745	0.3832	0.5936	0.2535	

Hence, we get the IBFS as follows

$$(\mathbb{CC}1, \mathscr{H}3) = k_{13} = 0.2924, \quad (\mathbb{CC}2, \mathscr{H}4) = k_{24} = 0.2262,$$
$$(\mathbb{CC}3, \mathscr{H}1) = k_{31} = 0.1175, \quad (\mathbb{CC}3, \mathscr{H}3) = k_{33} = 0.3012,$$
$$(\mathbb{CC}3, \mathscr{H}4) = k_{34} = 0.0273, \quad (\mathbb{CC}4, \mathscr{H}1) = k_{41} = 0.4570,$$
$$\left(\overset{dummy}{\mathbb{CC}4}, \mathscr{H}2\right) = k_{42} = 0.3832.$$

The minimum transportation cost corresponding to IBFS is obtained as: $0.2924 \times 0.2298 + 0.2262 \times 0.3642 + 0.1175 \times 0.4787 + 0.3012 \times 0.4395 + 0.0273 \times 0.7053 + 0 \times 0.4570 + 0 \times 0.3832 = 0.3575$.

Steps 5: Following that, the MODI approach is employed in order to assess the optimality of the IBFS for the designated Type3-CPyFTP. The outcomes obtained are summarized as follows:

$$(\mathbb{CC}1, \mathscr{H}3) = k_{13} = 0.2924, \quad (\mathbb{CC}2, \mathscr{H}4) = k_{24} = 0.2262,$$
$$(\mathbb{CC}3, \mathscr{H}1) = k_{31} = 0.1448, \quad (\mathbb{CC}3, \mathscr{H}3) = k_{33} = 0.3012,$$
$$(\mathbb{CC}4, \mathscr{H}1) = k_{41} = 0.4297, \quad \left(\overset{dummy}{\mathbb{CC}4}, \mathscr{H}2\right) = k_{42} = 0.3832,$$
$$\left(\overset{dummy}{\mathbb{CC}4}, \mathscr{H}4\right) = k_{44} = 0.0273.$$

The minimum total transportation cost is obtained as: $0.2924 \times 0.2298 + 0.2262 \times 0.3642 + 0.1448 \times 0.4787 + 0.3012 \times 0.4395 + 0.4297 \times 0 + 0.3832 \times 0 + 0.0273 \times 0 = 0.3513$.

Furthermore, the aforementioned problems are solved using some conventional methods such as the North-west corner (NWC), Least-cost method (LCM), Row-minima method (RMM), and Column-minima method (CMM). The findings are displayed in Tables 23–25.

Table 23: Comparative results for Type1-CPyFTP of Ex. 6 with the conventional methods

Methods	Optimal allocations	Total transportation costs
NWC	$(\mathbb{CC}1, \mathscr{H}1) = k_{11} = 20$, $(\mathbb{CC}1, \mathscr{H}3) = k_{13} = 50$, $(\mathbb{CC}1, \mathscr{H}4) = k_{14} = 50$, $(\mathbb{CC}2, \mathscr{H}1) = k_{21} = 50$, $(\mathbb{CC}2, \mathscr{H}2) = k_{22} = 100$, $(\mathbb{CC}3, \mathscr{H}3) = k_{33} = 80$	103.42
LCM	$(\mathbb{CC}1, \mathscr{H}1) = k_{11} = 20$, $(\mathbb{CC}1, \mathscr{H}3) = k_{13} = 50$, $(\mathbb{CC}1, \mathscr{H}4) = k_{14} = 50$, $(\mathbb{CC}2, \mathscr{H}1) = k_{21} = 50$, $(\mathbb{CC}2, \mathscr{H}2) = k_{22} = 100$, $(\mathbb{CC}3, \mathscr{H}3) = k_{33} = 80$	103.42
RMM	$(\mathbb{CC}1, \mathscr{H}1) = k_{11} = 20$, $(\mathbb{CC}1, \mathscr{H}3) = k_{13} = 50$, $(\mathbb{CC}1, \mathscr{H}4) = k_{14} = 50$, $(\mathbb{CC}2, \mathscr{H}1) = k_{21} = 50$, $(\mathbb{CC}2, \mathscr{H}2) = k_{22} = 100$, $(\mathbb{CC}3, \mathscr{H}3) = k_{33} = 80$	103.42
CMM	$(\mathbb{CC}1, \mathscr{H}1) = k_{11} = 20$, $(\mathbb{CC}1, \mathscr{H}3) = k_{13} = 50$, $(\mathbb{CC}1, \mathscr{H}4) = k_{14} = 50$, $(\mathbb{CC}2, \mathscr{H}1) = k_{21} = 50$, $(\mathbb{CC}2, \mathscr{H}2) = k_{22} = 100$, $(\mathbb{CC}3, \mathscr{H}3) = k_{33} = 80$	103.42
Proposed method	$(\mathbb{CC}1, \mathscr{H}1) = k_{11} = 20$, $(\mathbb{CC}1, \mathscr{H}3) = k_{13} = 50$, $(\mathbb{CC}1, \mathscr{H}4) = k_{14} = 50$, $(\mathbb{CC}2, \mathscr{H}1) = k_{21} = 50$, $(\mathbb{CC}2, \mathscr{H}2) = k_{22} = 100$, $(\mathbb{CC}3, \mathscr{H}3) = k_{33} = 80$	103.42

The results shown in Tables 23–25 indicate that our approach consistently produces similar optimal allocations and minimal transportation costs across all analyzed methods. This alignment strengthens the effectiveness of our proposed strategy in dealing with transportation problem within the complex Pythagorean fuzzy framework. The results highlight the strength and dependability of our

Table 24: Comparative results for Type2-CPyFTP of Ex. 6 with the conventional methods

Methods	Optimal allocations	Total transportation costs
NWC	$(\mathbb{CC}1, \mathscr{H}1) = k_{11} = 0.0021$, $(\mathbb{CC}1, \mathscr{H}2) = k_{12} = 0.3994$, $(\mathbb{CC}1, \mathscr{H}3) = k_{13} = 0.1732$, $(\mathbb{CC}2, \mathscr{H}1) = k_{21} = 0.3043$, $(\mathbb{CC}3, \mathscr{H}1) = k_{31} = 0.0876$, $(\mathbb{CC}3, \mathscr{H}4) = k_{34} = 0.3834$, $\left(\overset{dummy}{\mathbb{CC}4}, \mathscr{H}3\right) = k_{43} = 0.2896$	2462.40
LCM	$(\mathbb{CC}1, \mathscr{H}1) = k_{11} = 0.0021$, $(\mathbb{CC}1, \mathscr{H}2) = k_{12} = 0.3994$, $(\mathbb{CC}1, \mathscr{H}3) = k_{13} = 0.1732$, $(\mathbb{CC}2, \mathscr{H}1) = k_{21} = 0.3043$, $(\mathbb{CC}3, \mathscr{H}1) = k_{31} = 0.0876$, $(\mathbb{CC}3, \mathscr{H}4) = k_{34} = 0.3834$, $\left(\overset{dummy}{\mathbb{CC}4}, \mathscr{H}3\right) = k_{43} = 0.2896$	2462.40
RMM	$(\mathbb{CC}1, \mathscr{H}1) = k_{11} = 0.0021$, $(\mathbb{CC}1, \mathscr{H}2) = k_{12} = 0.3994$, $(\mathbb{CC}1, \mathscr{H}3) = k_{13} = 0.1732$, $(\mathbb{CC}2, \mathscr{H}1) = k_{21} = 0.3043$, $(\mathbb{CC}3, \mathscr{H}1) = k_{31} = 0.0876$, $(\mathbb{CC}3, \mathscr{H}4) = k_{34} = 0.3834$, $\left(\overset{dummy}{\mathbb{CC}4}, \mathscr{H}3\right) = k_{43} = 0.2896$	2462.40
CMM	$(\mathbb{CC}1, \mathscr{H}1) = k_{11} = 0.0021$, $(\mathbb{CC}1, \mathscr{H}2) = k_{12} = 0.3994$, $(\mathbb{CC}1, \mathscr{H}3) = k_{13} = 0.1732$, $(\mathbb{CC}2, \mathscr{H}1) = k_{21} = 0.3043$, $(\mathbb{CC}3, \mathscr{H}1) = k_{31} = 0.0876$, $(\mathbb{CC}3, \mathscr{H}4) = k_{34} = 0.3834$, $\left(\overset{dummy}{\mathbb{CC}4}, \mathscr{H}3\right) = k_{43} = 0.2896$	2462.40
Proposed method	$(\mathbb{CC}1, \mathscr{H}1) = k_{11} = 0.0021$, $(\mathbb{CC}1, \mathscr{H}2) = k_{12} = 0.3994$, $(\mathbb{CC}1, \mathscr{H}3) = k_{13} = 0.1732$, $(\mathbb{CC}2, \mathscr{H}1) = k_{21} = 0.3043$, $(\mathbb{CC}3, \mathscr{H}1) = k_{31} = 0.0876$, $(\mathbb{CC}3, \mathscr{H}4) = k_{34} = 0.3834$, $\left(\overset{dummy}{\mathbb{CC}4}, \mathscr{H}3\right) = k_{43} = 0.2896$	2462.40

Table 25: Comparative results for Type3-CPyFTP of Ex. 6 with the conventional methods

Methods	Optimal allocations	Total transportation costs
NWC	$(\mathbb{CC}1, \mathscr{H}3) = k_{113} = 0.2924$, $(\mathbb{CC}2, \mathscr{H}4) = k_{24} = 0.2262$, $(\mathbb{CC}3, \mathscr{H}1) = k_{31} = 0.1448$, $(\mathbb{CC}3, \mathscr{H}3) = k_{33} = 0.3012$, $(\mathbb{CC}4, \mathscr{H}1) = k_{41} = 0.4297$, $\left(\overset{dummy}{\mathbb{CC}4}, \mathscr{H}2\right) = k_{42} = 0.3832$, $\left(\overset{dummy}{\mathbb{CC}4}, \mathscr{H}4\right) = k_{44} = 0.0273$	0.3513
LCM	$(\mathbb{CC}1, \mathscr{H}3) = k_{13} = 0.2924$, $(\mathbb{CC}2, \mathscr{H}4) = k_{24} = 0.2262$, $(\mathbb{CC}3, \mathscr{H}1) = k_{31} = 0.1448$, $(\mathbb{CC}3, \mathscr{H}3) = k_{33} = 0.3012$, $(\mathbb{CC}4, \mathscr{H}1) = k_{41} = 0.4297$, $\left(\overset{dummy}{\mathbb{CC}4}, \mathscr{H}2\right) = k_{42} = 0.3832$, $\left(\overset{dummy}{\mathbb{CC}4}, \mathscr{H}4\right) = k_{44} = 0.0273$	0.3513
RMM	$(\mathbb{CC}1, \mathscr{H}3) = k_{13} = 0.2924$, $(\mathbb{CC}2, \mathscr{H}4) = k_{24} = 0.2262$, $(\mathbb{CC}3, \mathscr{H}1) = k_{31} = 0.1448$, $(\mathbb{CC}3, \mathscr{H}3) = k_{33} = 0.3012$, $(\mathbb{CC}4, \mathscr{H}1) = k_{41} = 0.4297$, $\left(\overset{dummy}{\mathbb{CC}4}, \mathscr{H}2\right) = k_{42} = 0.3832$, $\left(\overset{dummy}{\mathbb{CC}4}, \mathscr{H}4\right) = k_{44} = 0.0273$	0.3513
CMM	$(\mathbb{CC}1, \mathscr{H}3) = k_{13} = 0.2924$, $(\mathbb{CC}2, \mathscr{H}4) = k_{24} = 0.2262$, $(\mathbb{CC}3, \mathscr{H}1) = k_{31} = 0.1448$, $(\mathbb{CC}3, \mathscr{H}3) = k_{33} = 0.3012$, $(\mathbb{CC}4, \mathscr{H}1) = k_{41} = 0.4297$, $\left(\overset{dummy}{\mathbb{CC}4}, \mathscr{H}2\right) = k_{42} = 0.3832$, $\left(\overset{dummy}{\mathbb{CC}4}, \mathscr{H}4\right) = k_{44} = 0.0273$	0.3513
Proposed method	$(\mathbb{CC}1, \mathscr{H}3) = k_{13} = 0.2924$, $(\mathbb{CC}2, \mathscr{H}4) = k_{24} = 0.2262$, $(\mathbb{CC}3, \mathscr{H}1) = k_{31} = 0.1448$, $(\mathbb{CC}3, \mathscr{H}3) = k_{33} = 0.3012$, $(\mathbb{CC}4, \mathscr{H}1) = k_{41} = 0.4297$, $\left(\overset{dummy}{\mathbb{CC}4}, \mathscr{H}2\right) = k_{42} = 0.3832$, $\left(\overset{dummy}{\mathbb{CC}4}, \mathscr{H}4\right) = k_{44} = 0.0273$	0.3513

technique, confirming its capacity to handle the complexities inherent in the real-world transportation problem with uncertain and vague information.

The comprehensive examination outlined in Table 26 provides a detailed comparison between the proposed work and the existing research found in the literature.

Table 26: Comparison of the proposed work with existing work in the literature

Existing models	Whether consider MemD	Whether consider NMemD	Whether consider MemD more flexibly	Whether consider NMemD more flexibly	Whether consider phase term of MemD	Whether consider phase term of NMemD
[14]	✓	×	✓	×	×	×
[20]	✓	×	✓	×	×	×
[52]	✓	✓	✓	✓	×	×
[47]	✓	✓	✓	✓	×	×
[16]	✓	✓	✓	✓	×	×
[11]	✓	✓	✓	✓	×	×
[32]	✓	✓	✓	✓	×	×
[48]	✓	✓	✓	✓	×	×
Proposed work	✓	✓	✓	✓	✓	✓

7. Conclusion

In this work, we have presented a novel study on transportation problems in a complex Pythagorean fuzzy setting. First, the work has highlighted some major drawbacks of the existing score function value of CPyFNs in practical scenarios. We have defined an advanced score function value by considering the membership, nonmembership, and hesitancy degrees of a CPyF to establish a more robust ranking order among CPyFNs. Next, the work has introduced the notion of CPyFTPs and developed mathematical optimization models corresponding to different situations. The solution algorithm has also been developed based on VAM and MODI methods with the help of the suggested advanced score value function. We have also considered a real-life problem to demonstrate the practicality and efficiency of the developed algorithm in practical situations. In future work, we intend to explore the potential of the developed algorithm in multi-objective TPs. The sustainability parameters are also very important to consider during transportation management. We will also extend the developed optimization transportation models with different sustainable parameters such as energy efficiency, emissions reduction, use of renewable energy, and waste reduction.

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