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Characteristics of solution to singular fractional differential equation with two Riemann-Stieltjes integral boundary value conditions

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Abstract. The purpose of this paper is to study unique solution and iterative sequence of approximate solution for uniformly approaching unique solution to a new class of singular fractional differential equations with two kinds of Riemann-Stieltjes integral boundary value conditions by using some fixed point theorems. Because of different properties of the nonlinear terms and complexity of the boundary conditions in equations, we first probe several fixed point theorems of sum-type operators which expand many existing works in this research area. It is essential to point out that some conditions in our works greatly simplify the proof process of fixed point theorems. By applying the operator conclusions obtained in this paper, some sufficient conditions that guarantee the existence and uniqueness of solutions to singular differential equations are obtained, two iterative schemes that uniformly converge to the unique solution are given which provide computational methods of approximating solutions. As applications, some examples are provided to illustrate our main results.

Keywords: operator equation; mixed monotone operator; concave-convex operator; positive solution; fractional differential equation.

1. INTRODUCTION

In this article, we study a new form of fractional differential equations with four different nonlinear terms under two different complex integral boundary conditions and gain the sufficient conditions which guarantee the unique nontrivial solution and approximating iterative schemes of unique solution. Namely, we discuss the following problem:

$$D_{0^+}^{\eta_1+\eta_2}x(t) + k_1 f(t,x(t),x(t)) + k_2 g(t,x(t),x(t))$$
$$+ k_3 \phi(t,x(t),x(t)) + k_4 \rho(t,x(t)) = 0, \quad (1)$$

with two Riemann-Stieltjes integral boundary conditions

$$\begin{cases} x^{(i)}(0) = 0, \quad D_{0^+}^{\alpha_1}x(1) = \tau I_{0^+}^{\beta_1}x(\xi_1), \quad D_{0^+}^{\eta_1+j}x(0) = 0, \\ D_{0^+}^{\eta_1+\alpha_2}x(1) = \int_0^1 b(s)D_{0^+}^{\eta_1+\beta_2}x(s)\,dA(s), \end{cases} \quad (2)$$

$$\begin{cases} x^{(j)}(0) = 0, \quad D_{0^+}^{\alpha_2}x(1) = \int_0^1 b(s)D_{0^+}^{\beta_2}x(s)\,dA(s), \\ D_{0^+}^{\eta_2+i}x(0) = 0, \quad D_{0^+}^{\eta_2+\alpha_1}x(1) = \tau I_{0^+}^{\eta_2+\beta_1}x(\xi_1), \end{cases} \quad (3)$$

where $j = 0,1,\cdots,m-2$, $i = 0,1,\cdots,n-2$, $D_{0^+}^{\eta_1+\eta_2}$ stands for Riemann-Liouville (RL) fractional derivative, $n-1 < \eta_1 \le n$, $m-1 < \eta_2 \le m$, $\eta_1 - \alpha_1 - 1 > 0$, $\tau > 0$, $\beta_1 > 0$, $\alpha_2 - \beta_2 > 0$, $\eta_2 - \alpha_2 - 1 > 0$, $\eta_2 - \beta_2 - 1 > 0$, $f,g,\phi \in C\big((0,1)\times(0,+\infty)\times(0,+\infty),[0,+\infty)\big)$, $\rho \in C\big((0,1)\times(0,+\infty),[0,+\infty)\big)$, $k_i > 0$ ($i = 1,2,3,4$), $f(t,x,y)$, $g(t,x,y)$, $\phi(t,x,y)$, $\rho(t,x)$ are singular at $t = 0,1$ and $y = 0$.

As a matter of fact, FDEs with integral boundary value conditions are extensively applied in the description of a variety of practical situations and processes with memory and genetic characteristics [1–7] which explains why many authors have discussed existence, nonexistence and multiplicity questions for positive solutions to integral boundary value problems involving fractional derivative [8–14]. For instance, the existence conclusions of solution to the following RL equation in [8] are obtained by mean of the fixed point index theory

$$\begin{cases} D_{0^+}^{\eta}y(t) + h(t)f(t,y(t)), \quad 0 < t < 1, \quad 3 < \eta \le 4, \\ y^{(i)}(0) = 0, \quad i = 0,1,2, \quad y(1) = \lambda\int_0^{\xi} y(s)\,ds, \quad 0 \le \xi \le 1. \end{cases}$$

In [9], a Caputo FDE subject to integral boundary value conditions was investigated

$$\begin{cases} {}^c D_{0^+}^{\eta}y(t) + f(t,y(t),y'(t)) = 0, \quad 0 \le t \le 1, \quad 2 < \eta < 3, \\ y(0) = y''(0) = 0, \quad y(1) = \lambda\int_0^1 y(s)\,ds, \quad 0 < \lambda < 2. \end{cases}$$

By using a fixed point theorem due to Avery and Peterson, sufficient conditions for the existence and multiplicity of positive solution to this system were given.

From literature [15–18], some existence, nonexistence and multiplicity results of solution (or positive solution) when the nonlinear terms satisfy different requirements of superlinearity, sublinearity, and so forth. But the question of the unique solution and the computational methods of approximating solutions was not treated. Motivated by the above-mentioned work, our work in this paper is to unfold the existence and iteration of unique positive solution to the problem equation (1) with the conditions (2) or (3) by using some fixed point theorems of sum-type operator.

The new features of this paper are as follows: (a) We provide several kinds of fixed point methods of sum-type operators to discuss one of the most considerable qualitative aspects "existence and uniqueness" of solution for the system of equations (1) governed by the boundary conditions (2) or (3). The fixed point theorems of sum-type operators expand some existing results, such as [19–21]. (b) We study not only problem equation (1) with (2) or (3) admitting the existence and uniqueness of positive solutions but also two iterative schemes for dealing with approximate solutions that converge uniformly to the unique solution, which can provide computational methods for approximating solutions.

The rest of the content in this paper is organized as follows. In Section 2, some basic theory, works of Banach space and cone are reviewed. In Section 3, some conclusions of sum-type operators are presented. In Section 4, some theorems of existence-uniqueness solution for (1) with (2) or (3) are studied by the theoretical works of Section 3. In Section 5, some exact examples are given.

2. PRELIMINARIES

Let $(E, \|\cdot\|)$ be a real Banach space. The concept of cone, normal cone can be referred to [1–3]. Denoted a set $P_h = \{x \in E \mid \exists \mu, \nu > 0, \mu h \le x \le \nu h\}$.

Definition 1 [1]. Define an operator $A : P \times P \to P$, if $\forall u_i, v_i \in P$ $(i = 1, 2)$, when $u_1 \le u_2, v_1 \ge v_2$, there is $A(u_1, v_1) \le A(u_2, v_2)$, i.e. $A(u, v)$ is increasing in u, and decreasing in v, then A is called a mixed monotone operator. If $A(x, x) = x$, x is called a fixed point of A.

Definition 2 [2]. Let $D \subset E$ be a convex subset, $A : D \to E$ be an operator. If A satisfies $\forall x, y \in D$, $y \le x$,

$$A(tx + (1-t)y) \le tAx + (1-t)Ay, \ t \in (0,1),$$

then A is called a convex operator. If

$$A(tx + (1-t)y) \ge tAx + (1-t)Ay, \ t \in (0,1),$$

then, A is called a concave operator.

Definition 3 [3]. Define an operator $A : P \times P \to P$, if

$$A(tx) \ge tAx, \ \forall t \in (0,1), x \in P,$$

then, A is said to be sub-homogeneous operator.

Lemma 1 [4]. Let P be a normal cone and operator $T : P \times P \to P$. If T is a mixed monotone operator and

(L_1) $\exists h \in P$ $(h \ne \theta)$, such that $T(h, h) \in P_h$;

(L_2) $\exists \varphi(t) \in (t, 1]$, such that $T(tx, t^{-1}y) \ge \varphi(t)T(x, y), \forall t \in (0, 1), x, y \in P.$

Then we have

(C_1') $T : P_h \times P_h \to P_h$;

(C_2') $\exists u_0, v_0 \in P_h$, such that $r v_0 \le u_0 < v_0$, $u_0 \le T(u_0, v_0) \le T(v_0, u_0) \le v_0$, $r \in (0, 1)$;

(C_3') $T(x, x)$ has a unique solution $x^* \in P_h$;

(C_4') for any initial values $x_0, y_0 \in P_h$, constructing the iterative sequences:

$$x_n = T(x_{n-1}, y_{n-1}), \quad y_n = T(y_{n-1}, x_{n-1}), \quad n = 1, 2, \cdots,$$
$$x_n \to x^*, \ y_n \to x^*, \text{ as } n \to \infty.$$

3. FIXED POINT THEOREMS OF SUM OPERATOR

In this section, we use basic definition and lemma to investigate the properties and conclusions of a class of sum operators. Theorem 1 and Corollary 1 provide conclusions on the existence and uniqueness of solutions for a class of sum operators with parameters on P and P_h. What is more, the study on the properties of operator equation solutions is given in Remark 1 when the operator satisfies different convexities. In fact, the operator theorems studied have generality, as the parameters and operator properties change, the conclusions obtained can be simplified into some theorems in some literature, which are given in Remark 2, 3.

Define an operator $T_5 = k_1 T_1 + k_2 T_2 + k_3 T_3 + k_4 T_4$ by

$$T_5(x, y) = k_1 T_1(x, y) + k_2 T_2(x, y) + k_3 T_3(x, y) + k_4 T_4 x, \ \forall x, y \in P.$$

Theorem 1. Let $T_1, T_2, T_3 : P \times P \to P$ be mixed monotone operators, and $T_4 : P \to P$ be an increasing sub-homogeneous operator, $k_i > 0$ $(i = 1, 2, 3, 4)$. Suppose that

(H_1) For any $\lambda \in (0, 1), x, y \in P, \exists \psi_1(\lambda), \psi_2(\lambda) \in (\lambda, 1]$ such that

$$T_1(\lambda x, \lambda^{-1} y) \ge \psi_1(\lambda) T_1(x, y),$$
$$T_2(\lambda x, \lambda^{-1} y) \ge \psi_2(\lambda) T_2(x, y),$$

and for any fixed $y \in P$, $T_3(\cdot, y)$ is concave; for any fixed $x \in P$, $T_3(x, \cdot)$ is convex;

(H_2) $\exists \frac{1}{2} \le a \le 1$ such that $T_3(\theta, \widetilde{l}h) \ge aT_3(\widetilde{l}h, \theta), \widetilde{l} \ge 1$;

(H_3) $\exists h_1 \in P_h$ $(h_1 \ne \theta)$ such that $T_1(h_1, h_1), T_2(h_1, h_1), T_3(h_1, h_1), T_4 h_1 \in P_h$;

(H_4) for any $x, y \in P$, $\exists p \in (0, 1)$ such that $k_1 T_1(x, y) + k_2 T_2(x, y) \ge \frac{p}{1-p} [k_3 T_3(x, y) + k_4 T_4 x]$.

The following conclusions hold:

(c1) $T_5(h, h) \in P_h$;

(c2) $T_5 : P_h \times P_h \to P_h$;

(c3) $T_5(x, x) = x$ has a unique solution $x^* \in P_h$;

(c4) $\exists u_0, v_0 \in P_h, r \in (0, 1)$, such that $r v_0 \le u_0 < v_0$,

$$u_0 \le T_5(u_0, v_0) \le T_5(v_0, u_0) \le v_0;$$

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(c5) for any initial values x_0, $y_0 \in P_h$, constructing the iterative sequences:

$$x_n = T_5(x_{n-1}, y_{n-1}), \quad y_n = T_5(y_{n-1}, x_{n-1}), \quad n = 1, 2, \cdots,$$

$x_n \to x^*$, $y_n \to x^*$, when $n \to \infty$.

Proof. The proof process may be divided into three steps.

Step 1: We prove that $T_5 \colon P \times P \to P$, and T_5 is a mixed monotone operator. Due to $T_1, T_2, T_3 \colon P \times P \to P$, $T_4 \colon P \to P$, and $k_i > 0$ ($i = 1, 2, 3, 4$), it is easy to verify $T_5 \colon P \times P \to P$. For every $x_1, y_1, x_2, y_2 \in P$, let $x_1 \leq x_2$, $y_1 \geq y_2$, since T_1, T_2, T_3 are mixed monotone operators, and T_4 is an increasing operator, we conclude T_5 is a mixed monotone operator.

Step 2: We state that $T_5(h, h) \in P_h$. By the condition (H_1), it is easy to obtain

$$T_1(\lambda^{-1}x, \lambda y) \leq \psi_1(\lambda)^{-1} T_1(x, y),$$
$$T_2(\lambda^{-1}x, \lambda y) \leq \psi_2(\lambda)^{-1} T_2(x, y), \quad T_4(\lambda^{-1}x) \leq \lambda^{-1} T_4 x. \tag{4}$$

For any $\lambda \in (0, 1)$, we let $y = \lambda \lambda^{-1} y + (1 - \lambda)\theta$, in view of the condition (H_1) and the definition of convex operator in Definition 2, we have

$$\lambda T_3(x, \lambda^{-1}y) \geq T_3(x, y) - (1 - \lambda)T_3(x, \theta). \tag{5}$$

Moreover, we can find a sufficiently large number $\widetilde{l}$ such that $x, y, \lambda^{-1}y \leq \widetilde{l}h$. By (5), (H_2), mixed monotonicity of T_3, and the definition of concave operator in Definition 2, there exists $\frac{1}{2} \leq a \leq 1$, such that

$$\begin{aligned}
T_3(\lambda x, \lambda^{-1}y) &= T_3(\lambda x + (1-\lambda)\theta, \lambda^{-1}y) \\
&\geq \lambda T_3(x, \lambda^{-1}y) + (1-\lambda)T_3(\theta, \lambda^{-1}y) \\
&\geq T_3(x, y) - (1-\lambda)T_3(x, \theta) + (1-\lambda)T_3(\theta, \lambda^{-1}y) \\
&\geq T_3(x, y) + (1-\lambda)[T_3(\theta, \widetilde{l}h) - T_3(\widetilde{l}h, \theta)] \\
&\geq T_3(x, y) + (1-\lambda)[T_3(\theta, \widetilde{l}h) - \frac{1}{a}T_3(\theta, \widetilde{l}h)] \\
&\geq [1 + (1-\lambda)(1 - \frac{1}{a})]T_3(x, y) \\
&= [(2 - \frac{1}{a}) + (\frac{1}{a} - 1)\lambda]T_3(x, y) \\
&\geq \lambda T_3(x, y).
\end{aligned}$$

Then, for any $\lambda \in (0, 1)$, it is easy to obtain

$$T_3(\lambda x, \lambda^{-1}y) \geq \lambda T_3(x, y), \quad T_3(\lambda^{-1}x, \lambda y) \leq \lambda^{-1} T_3(x, y), \tag{6}$$

From $h_1 \in P_h$ of the condition (H_3), there exists a constant $a_0 \in (0, 1)$ such that

$$a_0 h \leq h_1 \leq a_0^{-1} h. \tag{7}$$

What is more, by means of (H_3), there exist some constants $b_i \in (0, 1)$ ($i = 1, 2, \cdots, 8$) such that

$$b_1 h \leq T_1(h_1, h_1) \leq b_2 h, \quad b_3 h \leq T_2(h_1, h_1) \leq b_4 h,$$
$$b_5 h \leq T_3(h_1, h_1) \leq b_6 h, \quad b_7 h \leq T_4 h_1 \leq b_8 h. \tag{8}$$

From (4)–(5), (7)–(8), and the monotonicity of T_1, the assertion follows

$$\begin{aligned}
T_1(h, h) &\leq T_1(a_0^{-1}h_1, a_0 h_1) \leq \psi_1(a_0)^{-1}b_2 h, \\
T_1(h, h) &\geq T_1(a_0 h_1, a_0^{-1}h_1) \geq \psi_1(a_0)b_1 h.
\end{aligned} \tag{9}$$

In the similar way, we obtain

$$\begin{aligned}
\psi_2(a_0)b_3 h &\leq T_2(h, h) \leq \psi_2(a_0)^{-1}b_4 h, \\
a_0 b_5 h &\leq T_3(h, h) \leq a_0^{-1}b_6 h, \\
a_0 b_7 h &\leq T_4 h \leq a_0^{-1}b_8 h.
\end{aligned} \tag{10}$$

From (9)–(10), it holds that $T_1(h, h)$, $T_2(h, h)$, $T_3(h, h)$, $T_4 h \in P_h$. According to the definition of T_5 and (10)–(11), it is easy to obtain $T_5(h, h) \in P_h$.

Step 3: We verify that T_5 satisfies the condition (L_2) in Lemma 1. Through (H_4), it is easy to obtain that there exists $p \in (0, 1)$, such that

$$(1 - p)[k_1 T_1(x, y) + k_2 T_2(x, y)] \geq p[k_3 T_3(x, y) + k_4 T_4 x], \quad \forall x, y \in P.$$

Then by (H_1), $\forall t \in (0, 1)$, one observes

$$\begin{aligned}
T_5(tx, t^{-1}y) &\geq \psi(t)[k_1 T_1(x, y) + k_2 T_2(x, y)] + t[k_3 T_3(x, y) + k_4 T_4 x] \\
&= [p\psi(t) + (1-p)t][k_1 T_1(x, y) + k_2 T_2(x, y)] \\
&\quad + (1-p)(\psi(t) - t)[k_1 T_1(x, y) + k_2 T_2(x, y)] \\
&\quad + t[k_3 T_3(x, y) + k_4 T_4 x] \\
&\geq [p\psi(t) + (1-p)t][k_1 T_1(x, y) + k_2 T_2(x, y)] \\
&\quad + p(\psi(t) - t)[k_3 T_3(x, y) + k_4 T_4 x] \\
&\quad + t[k_3 T_3(x, y) + k_4 T_4 x] \\
&= [p\psi(t) + (1-p)t]T_5(x, y),
\end{aligned}$$

where $\psi(t) = \min\{\psi_1(t), \psi_2(t), t \in (0, 1)\}$. Due to $p \in (0, 1)$, $\psi(t) \in (t, 1]$, it is easy to obtain $p\psi(t) + (1-p)t \in (t, 1]$. Hence, there exists $\varphi(t) = p\psi(t) + (1-p)t \in (t, 1]$ such that

$$T_5(tx, t^{-1}y) \geq \varphi(t)T_5(x, y).$$

Thus, according to Lemma 1, we get the conclusions (c1)–(c5).

Corollary 1. Let $T_1, T_2, T_3 \colon P_h \times P_h \to P_h$ be mixed monotone operators, and $T_4 \colon P_h \to P_h$ be an increasing sub-homogeneous operator, $k_i > 0$ ($i = 1, 2, 3, 4$). Suppose that (H_2) holds and

(H_1') for any $\lambda \in (0, 1), x, y \in P_h$, $\exists \psi_1(\lambda)$, $\psi_2(\lambda) \in (\lambda, 1]$ such that

$$T_1(\lambda x, \lambda^{-1}y) \geq \psi_1(\lambda)T_1(x, y), \ T_2(\lambda x, \lambda^{-1}y) \geq \psi_2(\lambda)T_2(x, y),$$

and for any fixed $y \in P_h$, $T_3(\cdot, y)$ is concave; for any fixed $x \in P_h$, $T_3(x, \cdot)$ is convex;

(H_4') for any $x, y \in P_h$, $\exists p \in (0, 1)$ such that $k_1 T_1(x, y) + k_2 T_2(x, y) \geq \frac{p}{1-p}[k_3 T_3(x, y) + k_4 T_4 x]$.

Thus, the conclusions (c1)–(c5).

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Proof. According to the definition of the set P_h, it can be inferred that $P_h \subset P$. It can be observed that the conditions in Corollary 1 are given on the subset P_h of P. Therefore, according to the proof in Theorem 1, the same conclusion can be drawn. □

Remark 1. In Theorem 1, the properties of T_1, T_2 in the condition (H_1) turn into the condition (l_1) or (l_2) or (l_3):

(l_1) for any $\lambda \in (0,1)$, $x, y \in P$, $\exists \alpha_1, \alpha_2 \in (0,1)$ such that

$$T_1(\lambda x, \lambda^{-1} y) \geq \lambda^{\alpha_1} T_1(x,y),$$
$$T_2(\lambda x, \lambda^{-1} y) \geq \lambda^{\alpha_2} T_2(x,y),$$

(l_2) for any $\lambda \in (0,1)$, $x, y \in P$, $\exists \eta_1(\lambda), \eta_2(\lambda) \in (0,1]$ such that

$$T_1(\lambda x, \lambda^{-1} y) \geq [1 + \lambda - \lambda^{\eta_1(\lambda)}] T_1(x,y),$$
$$T_2(\lambda x, \lambda^{-1} y) \geq [1 + \lambda - \lambda^{\eta_2(\lambda)}] T_2(x,y),$$

(l_3) for any $\lambda \in (0,1)$, $x, y \in P$, $\exists \eta_3(\lambda), \eta_4(\lambda) > 0$ such that $\lambda[1 + \eta_3(\lambda)] < 1$, $\lambda[1 + \eta_4(\lambda)] < 1$ and

$$T_1(\lambda x, \lambda^{-1} y) \geq \lambda[1 + \eta_3(\lambda)] T_1(x,y),$$
$$T_2(\lambda x, \lambda^{-1} y) \geq \lambda[1 + \eta_4(\lambda)] T_2(x,y),$$

then the conclusions $(c1)$–$(c5)$ still hold.

Remark 2. In Theorem 1, if we take $k_1 = k_4 = 1$, $k_2 = k_3 = 0$, or $k_1 = k_4 = 1$, $T_2 = T_3 = \theta$, and the property of T_1 in the condition (H_1) turns into

$$\exists \alpha_1 \in (0,1) \text{ such that } T_1(\lambda x, \lambda^{-1} y) \geq \lambda^{\alpha_1} T_1(x,y),$$

the corresponding result can be reduced to the Theorem 2.1 of [19].

Remark 3. In Theorem 1, in the situation of $k_1 = k_2 = 1$, $k_3 = k_4 = 0$, or $k_1 = k_2$, $T_3 = T_4 = \theta$, the corresponding results are consistent with the result of [20].

4. UNIQUE SOLUTION TO THE FDES

In this part, using our main theoretical results, we study the existence, uniqueness solution and approximating iterative schemes of unique solution to equation (1) with the conditions (2) or (3). The brief process of the research is as follows. Firstly, we discuss the equivalent integral equations of two singular differential equations (Lemma 2, Lemma 4), then the properties of the Green's function in the equivalent integral equations are obtained (Lemma 3). Thirdly, based on the obtained operator equation theorem (Theorem 1), the sufficient conditions for unique solution of two fractional differential equations are given (Theorem 2, Theorem 3).

In the following, we will work in $E = C[0,1]$ with the norm $\|x\| = \sup_{0 \leq t \leq 1} |x(t)|$, and $P = \{x \in E : x(t) \geq 0, t \in [0,1]\}$. Evidently, $(E, \|\cdot\|)$ is a Banach space, P is a normal cone and the normality constant is 1. Define

$$P_h = \left\{ x \in C[0,1] : \exists J \in (0,1), Jh(t) \leq x \leq \frac{1}{J} h(t), t \in [0,1] \right\}.$$

To simplify, let

$$M_1 = \Gamma(\eta_1)\Gamma(\eta_1 + \beta_1) - \tau \Gamma(\eta_1)\Gamma(\eta_1 - \alpha_1)\xi_1^{\eta_1 + \beta_1 - 1},$$

$$M_2 = \frac{1}{\Gamma(\eta_2 - \alpha_2)} - \frac{\int_0^1 b(s)s^{\eta_2 - \beta_2 - 1}\,\mathrm{d}A(s)}{\Gamma(\eta_2 - \beta_2)},$$

$$g_1(t,s) = \frac{t^{\eta_1 - 1}}{M_1}(1-s)^{\eta_1 - \alpha_1 - 1}\Gamma(\eta_1 + \beta_1),$$

$$g_2(t,s) = \frac{(t-s)^{\eta_1 - 1}}{\Gamma(\eta_1)},$$

$$g_3(t,s) = \tau \frac{\Gamma(\eta_1 - \alpha_1)t^{\eta_1 - 1}(\xi_1 - s)^{\eta_1 + \beta_1 - 1}}{M_1}.$$

Lemma 2. If $\sigma(t) = k_1 f(t, x(t), x(t)) + k_2 g(t, x(t), x(t)) + k_3 \phi(t, x(t), x(t)) + k_4 \rho(t, x(t)) \in C[0,1]$, $M_1 > 0$, $M_2 > 0$, equation (1) with condition (2) has the following equivalent integral equation

$$x(t) = \int_0^1 K(t,s) \int_0^1 G(s,\tau)\sigma(\tau)\,\mathrm{d}\tau\,\mathrm{d}s,$$

where

$$G(t,s) = G_1(t,s) + t^{\eta_2 - 1}\int_0^1 G_2(\tau,s)b(\tau)\,\mathrm{d}A(\tau),$$

$$G_1(t,s) = \begin{cases} \dfrac{t^{\eta_2 - 1}(1-s)^{\eta_2 - \alpha_2 - 1} - (t-s)^{\eta_2 - 1}}{\Gamma(\eta_2)}, & 0 \leq s \leq t \leq 1, \\[3mm] \dfrac{t^{\eta_2 - 1}(1-s)^{\eta_2 - \alpha_2 - 1}}{\Gamma(\eta_2)}, & 0 \leq t \leq s \leq 1, \end{cases}$$

$$G_2(t,s) = \begin{cases} \dfrac{(1-s)^{\eta_2 - \alpha_2 - 1}t^{\eta_2 - \beta_2 - 1} - (t-s)^{\eta_2 - \beta_2 - 1}}{\Gamma(\eta_2)\Gamma(\eta_2 - \beta_2)M_2}, \\ \qquad\qquad\qquad\qquad 0 \leq s \leq t \leq 1, \\[3mm] \dfrac{(1-s)^{\eta_2 - \alpha_2 - 1}t^{\eta_2 - \beta_2 - 1}}{\Gamma(\eta_2)\Gamma(\eta_2 - \beta_2)M_2}, & 0 \leq t \leq s \leq 1, \end{cases}$$

$$K(t,s) = \begin{cases} g_1(t,s) - g_2(t,s) - g_3(t,s), & 0 \leq s \leq t \leq 1, \ s \leq \xi_1, \\ g_1(t,s) - g_2(t,s), & 0 \leq \xi_1 \leq s \leq t \leq 1, \\ g_1(t,s) - g_3(t,s), & 0 \leq t \leq s \leq \xi_1 \leq 1, \\ g_1(t,s), & 0 \leq t \leq s \leq 1, \ \xi_1 \leq s. \end{cases}$$

Proof. Let $D_{0^+}^{\eta_1} x(t) = u(t)$. Equation (1) and some conditions of (2)

$$\begin{cases} D_{0^+}^{\eta_1 + j} x(0) = 0, & j = 0, 1, \cdots, m - 2, \\[2mm] D_{0^+}^{\eta_1 + \alpha_2} x(1) = \int_0^1 b(s)D_{0^+}^{\eta_1 + \beta_2} x(s)\,\mathrm{d}A(s), \end{cases}$$

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can turn into

$$
\begin{cases}
D_{0^+}^{\eta_2} u(t) + \sigma(t) = 0, \\
u^{(j)}(0) = 0, \quad j = 0, 1, \cdots, m-2, \\
D_{0^+}^{\alpha_2} u(1) = \int_0^1 b(s) D_{0^+}^{\beta_2} u(s) \, dA(s).
\end{cases}
\tag{11}
$$

Integrating η_2 times to the first formula of equation (11), we can obtain

$$
u(t) = -I_{0^+}^{\eta_2} \sigma(t) + \widetilde{c}_1 t^{\eta_2-1} + \widetilde{c}_2 t^{\eta_2-2} + \cdots + \widetilde{c}_m t^{\eta_2-m}.
\tag{12}
$$

From $u^{(j)}(0) = 0$ ($j = 0, 1, \cdots, m-2$), we see that $\widetilde{c}_m = \widetilde{c}_{m-1} = \cdots = \widetilde{c}_2 = 0$. Then by

$$
D_{0^+}^{\alpha} t^{\eta-1} = \frac{\Gamma(\eta)}{\Gamma(\eta-\alpha)} t^{\eta-\alpha-1}
$$

and

$$
D_{0^+}^{\alpha_2} u(1) = \int_0^1 b(s) D_{0^+}^{\beta_2} u(s) \, dA(s),
$$

one gets

$$
\widetilde{c}_1 = \int_0^1 \frac{(1-s)^{\eta_2-\alpha_2-1}}{\Gamma(\eta_2)\Gamma(\eta_2-\alpha_2)M_2} \sigma(s) \, ds
$$
$$
- \int_0^1 \int_0^s \frac{(s-\tau)^{\eta_2-\beta_2-1}}{\Gamma(\eta_2)\Gamma(\eta_2-\beta_2)M_2} b(s)\sigma(\tau) \, d\tau \, dA(s).
\tag{13}
$$

Substituting (13) in (12), and from the definition of M_2, we can get

$$
u(t) = -\int_0^t \frac{(t-s)^{\eta_2-1}}{\Gamma(\eta_2)} \sigma(s) \, ds + \int_0^1 \frac{t^{\eta_2-1}(1-s)^{\eta_2-\alpha_2-1}}{\Gamma(\eta_2)} \sigma(s) \, ds
$$
$$
+ \int_0^1 \int_0^1 \frac{t^{\eta_2-1}(1-\tau)^{\eta_2-\alpha_2-1} s^{\eta_2-\beta_2-1}}{\Gamma(\eta_2)\Gamma(\eta_2-\beta_2)M_2} b(s)\sigma(\tau) \, d\tau \, dA(s)
$$
$$
- \int_0^1 \int_0^s \frac{t^{\eta_2-1}(s-\tau)^{\eta_2-\beta_2-1}}{\Gamma(\eta_2)\Gamma(\eta_2-\beta_2)M_2} b(s)\sigma(\tau) \, d\tau \, dA(s)
$$
$$
= \int_0^1 G(t,s)\sigma(s) \, ds.
$$

i.e., $D_{0^+}^{\eta_1} x(t) = u(t) = \int_0^1 G(t,s)\sigma(s) \, ds.$

Integrating η_1 times with the above formula, we have

$$
x(t) = -I_{0^+}^{\eta_1} u(t) + \widetilde{a}_1 t^{\eta_1-1} + \widetilde{a}_2 t^{\eta_1-2} + \cdots + \widetilde{a}_n t^{\eta_1-n}.
\tag{14}
$$

According to $x^{(i)}(0) = 0$ ($i = 0, 1, \cdots, n-2$), there is $\widetilde{a}_n = \widetilde{a}_{n-1} = \cdots = \widetilde{a}_2 = 0$. It follows from $D_{0^+}^{\alpha_1} x(1) = \tau I_{0^+}^{\beta_1} x(\xi_1)$ that

$$
\widetilde{a}_1 = \int_0^1 \frac{\Gamma(\eta_1+\beta_1)(1-s)^{\eta_1-\alpha_1-1}}{M_1} u(s) \, ds
$$
$$
- \int_0^{\xi_1} \frac{\tau\Gamma(\eta_1-\alpha_1)(\xi_1-s)^{\eta_1+\beta_1-1}}{M_1} u(s) \, ds.
\tag{15}
$$

Replace $\widetilde{a}_1$ of (15) with (14) in the equation, there is

$$
x(t) = \int_0^1 K(t,s) u(s) \, ds,
$$

Then from $u(t)$ and $\sigma(t)$, we can get the conclusion.

Lemma 3. If $M_1 > 0, M_2 > 0$, the following properties are established

(i) for any $t, s \in [0,1]$, $G(t,s), K(t,s)$ are continuous.

(ii) for any $t, s \in [0,1]$, we have

$$
0 \leq l_1(s) t^{\eta_2-1} \leq G(t,s) \leq L_1 t^{\eta_2-1},
$$

where

$$
L_1 = \frac{1}{\Gamma(\eta_2)} + \int_0^1 \frac{\tau^{\eta_2-\beta_2-1}}{\Gamma(\eta_2)\Gamma(\eta_2-\beta_2)M_2} b(\tau) \, dA(\tau),
$$
$$
l_1(s) = \left[1 - (1-s)^{\alpha_2-\beta_2}\right](1-s)^{\eta_2-\alpha_2-1}
$$
$$
\cdot \frac{(1-s)^{\eta_2-\alpha_2-1}[1-(1-s)^{\alpha_2}]}{\Gamma(\eta_2)}
$$
$$
+ \int_0^1 \frac{\tau^{\eta_2-\beta_2-1}}{\Gamma(\eta_2)\Gamma(\eta_2-\beta_2)M_2} b(\tau) \, dA(\tau).
$$

(iii) for any $t, s \in [0,1]$, we have

$$
0 \leq l_2(s) t^{\eta_1-1} \leq K(t,s) \leq L_2 t^{\eta_1-1},
$$

where

$$
L_2 = \frac{\Gamma(\eta_1)\Gamma(\eta_1+\beta_1)}{M_1},
$$
$$
l_2(s) = \frac{\tau\Gamma(\eta_1-\alpha_1)\xi_1^{\eta_1+\beta_1-1}}{M_1}(1-s)^{\eta_1-\alpha_1-1}[1-(1-s)^{\alpha_1}].
$$

Proof. From the definition of $G(t,s), K(t,s)$, we can know that (i) holds.

From the definition of $G_1(t,s)$, $G_2(t,s)$, for any $t, s \in [0,1]$, we compute

$$
0 \leq \frac{t^{\eta_2-1}(1-s)^{\eta_2-\alpha_2-1}[1-(1-s)^{\alpha_2}]}{\Gamma(\eta_2)} \leq G_1(t,s) \leq \frac{t^{\eta_2-1}}{\Gamma(\eta_2)},
$$

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$$0 \leq \frac{(1-s)^{\eta_2-\alpha_2-1}t^{\eta_2-\beta_2-1}[1-(1-s)^{\alpha_2-\beta_2}]}{\Gamma(\eta_2)\Gamma(\eta_2-\beta_2)M_2} \leq G_2(t,s)$$

$$\leq t^{\eta_2-\beta_2-1}\frac{1}{\Gamma(\eta_2)\Gamma(\eta_2-\beta_2)}.$$

From the definition of $G(t,s)$, for any $t,\ s \in [0,1]$, it holds that

$$G(t,s) = G_1(t,s) + t^{\eta_2-1}\int_0^1 G_2(\tau,s)b(\tau)\,\mathrm{d}A(\tau) \geq l_1(s)t^{\eta_2-1},$$

and $G(t,s) \leq L_1 t^{\eta_2-1}$, i.e., (ii) holds.

From the definition of $K(t,s)$, and $M_1 > 0$, when $0 \leq s \leq t \leq 1$, $s \leq \xi_1$, we observe that

$$M_1\Gamma(\eta_1)K(t,s) = \Gamma(\eta_1)\Gamma(\eta_1+\beta_1)t^{\eta_1-1}(1-s)^{\eta_1-\alpha_1-1}$$

$$- \Gamma(\eta_1)\Gamma(\eta_1+\beta_1)(t-s)^{\eta_1-1} + \tau\Gamma(\eta_1)\Gamma(\eta_1-\alpha_1)\xi_1^{\eta_1+\beta_1-1}$$

$$\cdot (t-s)^{\eta_1-1} - \tau\Gamma(\eta_1)\Gamma(\eta_1-\alpha_1)t^{\eta_1-1}(\xi_1-s)^{\eta_1+\beta_1-1}$$

$$\geq \tau\Gamma(\eta_1)\Gamma(\eta_1-\alpha_1)\xi_1^{\eta_1+\beta_1-1}t^{\eta_1-1}(1-s)^{\eta_1-\alpha_1-1}$$

$$\cdot [1-(1-s)^{\alpha_1}] \geq 0,$$

and

$$M_1\Gamma(\eta_1)K(t,s) \leq \Gamma(\eta_1)\Gamma(\eta_1+\beta_1)t^{\eta_1-1}(1-s)^{\eta_1-\alpha_1-1}$$

$$\leq \Gamma(\eta_1)\Gamma(\eta_1+\beta_1)t^{\eta_1-1}.$$

Similarly, when $0 \leq \xi_1 \leq s \leq t \leq 1$, $0 \leq t \leq s \leq \xi_1 \leq 1$, $0 \leq t \leq s \leq 1, \xi_1 \leq s$, we can draw the same conclusion. Hence, for any $t,s \in [0,1]$, we infer that (iii) holds.

Theorem 2. Let $f,g,\phi\colon C((0,1)\times(0,+\infty)\times(0,+\infty),[0,+\infty))$, $\rho\colon C((0,1)\times(0,+\infty),[0,+\infty))$, and $f(t,u,v)$, $g(t,u,v)$, $\phi(t,u,v),\rho(t,u)$ are singular at $t=0,1$ and $v=0$. Assume that

(r_1) for $t\in(0,1),x,y\in(0,+\infty)$, for any fixed t,y, $f(t,x,y)$, $g(t,x,y),\phi(t,x,y),\rho(t,x)$ are nondecreasing in x; for any fixed $t,\ x,\ f(t,x,y),g(t,x,y),\phi(t,x,y)$ are nonincreasing in y;

(r_2) for any $\lambda,t\in(0,1),x,y\in(0,+\infty)$, there exist $\psi_1(\lambda)\in(\lambda,1]$, $\psi_2(\lambda)\in(\lambda,1]$ such that

$$f(t,\lambda x,\lambda^{-1}y) \geq \psi_1(\lambda)f(t,x,y),$$

$$g(t,\lambda x,\lambda^{-1}y) \geq \psi_2(\lambda)g(t,x,y), \quad \rho(t,\lambda x) \geq \lambda\rho(t,x),$$

and for fixed $t,\ y$, $\phi(t,\cdot,y)$ is concave; for fixed $t,\ x$, $\phi(t,x,\cdot)$ is convex;

(r_3) $\exists\ \frac{1}{2} \leq a \leq 1$ such that $\phi(t,\theta,\widetilde{l}h) \geq a\phi(t,\widetilde{l}h,\theta)$, where $\widetilde{l} \geq 1, h(t) = t^{\eta_1-1}$;

(r_4) $\displaystyle\int_0^1 s^{\eta_2-1}\int_0^1 \psi_1(\tau^{\eta_1-1})^{-1}f(\tau,1,1)\,\mathrm{d}\tau\,\mathrm{d}s < +\infty$,

$\displaystyle\int_0^1 s^{\eta_2-1}\int_0^1 \psi_2(\tau^{\eta_1-1})^{-1}g(\tau,1,1)\,\mathrm{d}\tau\,\mathrm{d}s < +\infty$,

$\displaystyle\int_0^1 s^{\eta_2-1}\int_0^1 \tau^{1-\eta_1}\phi(\tau,1,1)\,\mathrm{d}\tau\,\mathrm{d}s < +\infty$,

$\displaystyle\int_0^1 s^{\eta_2-1}\int_0^1 \tau^{1-\eta_1}\rho(\tau,1)\,\mathrm{d}\tau\,\mathrm{d}s < +\infty$;

(r_5) $\exists p\in(0,1)$ such that $k_1 f(t,x,y) + k_2 g(t,x,y) \geq \dfrac{p}{1-p}[k_3\phi(t,x,y)+k_4\rho(t,x)]$.

Then

(T1) there exist $u_0,v_0\in P_h$, $r\in(0,1)$ such that $rv_0 \leq u_0 < v_0$, and

$$u_0 \leq \int_0^1 K(t,s)\int_0^1 G(s,\tau)[k_1 f(\tau,u_0(\tau),v_0(\tau)) + k_2 g(\tau,u_0(\tau),$$

$$v_0(\tau)) + k_3\phi(\tau,u_0(\tau),v_0(\tau)) + k_4\rho(\tau,u_0(\tau))]\,\mathrm{d}\tau\,\mathrm{d}s$$

$$\leq \int_0^1 K(t,s)\int_0^1 G(s,\tau)[k_1 f(\tau,v_0(\tau),u_0(\tau)) + k_2 g(\tau,v_0(\tau),$$

$$u_0(\tau)) + k_3\phi(\tau,v_0(\tau),u_0(\tau)) + k_4\rho(\tau,v_0(\tau))]\,\mathrm{d}\tau\,\mathrm{d}s \leq v_0.$$

(T2) equation (1) with the boundary value condition (1) has a unique positive solution x^* in P_h, where $h(t) = t^{\eta_1-1}$.

(T3) for any initial values $x_0,\ y_0\in P_h$, constructing successively the iterative sequences

$$x_{n+1}(t) = \left[\int_0^1 K(t,s)\int_0^1 G(s,\tau)k_1 f(\tau,x_n(\tau),y_n(\tau))\right.$$

$$+ k_2 g(\tau,x_n(\tau),y_n(\tau)) + k_3\phi(\tau,x_n(\tau),y_n(\tau))$$

$$+ \left.k_4\rho(\tau,x_n(\tau))\right]\mathrm{d}\tau\,\mathrm{d}s,$$

$$y_{n+1}(t) = \int_0^1 K(t,s)\int_0^1 G(s,\tau)\Big[k_1 f(\tau,y_n(\tau),x_n(\tau))$$

$$+ k_2 g(\tau,y_n(\tau),x_n(\tau)) + k_3\phi(\tau,y_n(\tau),x_n(\tau))$$

$$+ k_4\rho(\tau,y_n(\tau))\Big]\mathrm{d}\tau\,\mathrm{d}s,$$

$n = 0,1,2,\cdots,x_{n+1}\to x^*,\ y_{n+1}\to x^*$, when $n\to\infty$.

Proof. Define some operators $T_1,T_2,T_3\colon P_h\times P_h\to E$, $T_4\colon P_h\to E$ by

$$T_1(x,y)(t) = \int_0^1 K(t,s)\int_0^1 G(s,\tau)f(\tau,x(\tau),y(\tau))\,\mathrm{d}\tau\,\mathrm{d}s,$$

$$T_2(x,y)(t) = \int_0^1 K(t,s)\int_0^1 G(s,\tau)g(\tau,x(\tau),y(\tau))\,\mathrm{d}\tau\,\mathrm{d}s,$$

$$T_3(x,y)(t) = \int_0^1 K(t,s)\int_0^1 G(s,\tau)\phi(\tau,x(\tau),y(\tau))\,\mathrm{d}\tau\,\mathrm{d}s,$$

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$$T_4 x(t) = \int_0^1 K(t,s) \int_0^1 G(s,\tau)\rho(\tau,x(\tau))\,\mathrm{d}\tau\,\mathrm{d}s.$$

Hence, from Lemma 2, x is the solution of equations (1)–(2) if and only if x is the solution of $x = k_1 T_1(x,x) + k_2 T_2(x,x) + k_3 T_3(x,x) + k_4 T_4 x$. We divide the proof into five steps.

Step 1: Due to $f(t,x,y)$, $g(t,x,y)$, $\phi(t,x,y)$, $\rho(t,x) \geq 0$, and $G(t,s), K(t,s) \geq 0$ of Lemma 3, it is easy to verify $T_1, T_2, T_3 : P_h \times P_h \to P$, $T_4 : P_h \to P$. Moreover, it can be verified that T_1, T_2, T_3 are monotone operators, and T_4 is nondecreasing operator by monotonicity of f, g, ϕ and ρ in the condition (r_1).

Step 2: We illustrate that $T_1, T_2, T_3 : P_h \times P_h \to P_h$, and $T_4 : P_h \to P_h$. Since $\phi(t,x,\cdot)$ is convex, we conclude

$$\phi(t,x,y) \leq \lambda\phi(t,x,\lambda^{-1}y) + (1-\lambda)\phi(t,x,\theta), \quad \lambda \in (0,1),$$

For $x, y \in P_h$, there exists sufficiently large $\widetilde{l}$ such that $x, y, \lambda^{-1}y \leq \widetilde{l}h$. Since $\phi(t,\cdot,y)$ is concave, by the condition (r_3), and the monotonicity of ϕ, we infer that

$$\begin{aligned}
\phi(t,\lambda x,\lambda^{-1}y) &\geq \lambda\phi(t,x,\lambda^{-1}y) + (1-\lambda)\phi(t,\theta,\lambda^{-1}y) \\
&\geq \phi(t,x,y) + (1-\lambda)(\phi(t,\theta,\widetilde{l}h) - \phi(t,\widetilde{l}h,\theta)) \\
&\geq [1 + (1-\lambda)(1-\frac{1}{a})]\phi(t,x,y) \\
&= [(2-\frac{1}{a}) + (\frac{1}{a}-1)\lambda]\phi(t,x,y) \geq \lambda\phi(t,x,y).
\end{aligned}$$

Then, we can deduce

$$\begin{aligned}
\phi(t,\lambda x,\lambda^{-1}y) &\geq \lambda\phi(t,x,y), \\
\phi(t,\lambda^{-1}x,\lambda y) &\leq \lambda^{-1}\phi(t,x,y).
\end{aligned} \tag{16}$$

In view of (r_2), we deduce

$$\begin{aligned}
f(t,\lambda^{-1}x,\lambda y) &\leq \psi_1(\lambda)^{-1}f(t,x,y), \\
g(t,\lambda^{-1}x,\lambda y) &\leq \psi_2(\lambda)^{-1}g(t,x,y), \\
\rho(t,\lambda^{-1}x) &\leq \lambda^{-1}\rho(t,x).
\end{aligned} \tag{17}$$

Taking $x = y = 1$ in (16)–(17) and the condition (r_2), we have

$$\begin{aligned}
&f(t,\lambda,\lambda^{-1}) \geq \psi_1(\lambda)f(t,1,1), f(t,\lambda^{-1},\lambda) \leq \psi_1(\lambda)^{-1}f(t,1,1), \\
&g(t,\lambda,\lambda^{-1}) \geq \psi_2(\lambda)g(t,1,1), g(t,\lambda^{-1},\lambda) \leq \psi_2(\lambda)^{-1}g(t,1,1), \\
&\phi(t,\lambda,\lambda^{-1}) \geq \lambda\phi(t,1,1), \phi(t,\lambda^{-1},\lambda) \leq \lambda^{-1}\phi(t,1,1), \\
&\rho(t,\lambda) \geq \lambda\rho(t,1), \quad \rho(t,\lambda^{-1}) \leq \lambda^{-1}\rho(t,1).
\end{aligned}$$

For $x, y \in P_h$, there exists $J \in (0,1)$ such that $Jh \leq x, y \leq J^{-1}h$. Then, we obtain

$$\begin{aligned}
f(t,x(t),y(t)) &\leq f(t,J^{-1}t^{\eta_1-1},Jt^{\eta_1-1}) \leq f(t,J^{-1}t^{1-\eta_1},Jt^{\eta_1-1}) \\
&\leq \psi_1(t^{\eta_1-1})^{-1}\psi_1(J)^{-1}f(t,1,1), \\
f(t,x(t),y(t)) &\geq f(t,Jt^{\eta_1-1},J^{-1}t^{\eta_1-1}) \geq f(t,Jt^{\eta_1-1},J^{-1}t^{1-\eta_1}) \\
&\geq \psi_1(t^{\eta_1-1})\psi_1(J)f(t,1,1).
\end{aligned}$$

In the similar way, the following inequalities hold

$$\psi_2(t^{\eta_1-1})\psi_2(J)g(t,1,1) \leq g(t,x(t),y(t)) \leq \psi_2(t^{\eta_1-1})^{-1},$$
$$\psi_2(J)^{-1}g(t,1,1), t^{\eta_1-1}J\phi(t,1,1) \leq \phi(t,x(t),y(t)) \leq t^{1-\eta_1},$$
$$J^{-1}\phi(t,1,1), t^{\eta_1-1}J\rho(t,1) \leq \rho(t,x(t)) \leq t^{1-\eta_1}J^{-1}\rho(t,1).$$

Using Lemma 3, and the condition (r_4), one observes

$$\begin{aligned}
T_1(x,y)(t) &\leq \int_0^1 L_2 t^{\eta_1-1} \int_0^1 L_1 s^{\eta_2-1}\psi_1(\tau^{\eta_1-1})^{-1}\psi_1(J)^{-1} \\
&\quad \cdot f(\tau,1,1)\,\mathrm{d}\tau\,\mathrm{d}s = L_1 L_2 \psi_1(J)^{-1}t^{\eta_1-1}\int_0^1 s^{\eta_2-1} \\
&\quad \cdot \int_0^1 \psi_1(\tau^{\eta_1-1})^{-1}f(\tau,1,1)\,\mathrm{d}\tau\,\mathrm{d}s < +\infty.
\end{aligned} \tag{18}$$

Similarly, it gives

$$T_2(x,y)(t) < +\infty, \quad T_3(x,y)(t) < +\infty, \quad T_4 x(t) < +\infty.$$

Assume that $J_1 \in (0,1)$ and

$$\begin{aligned}
J_1 < \min\bigg\{ & \left(L_1 L_2 \psi_1(J)^{-1}\int_0^1 s^{\eta_2-1}\int_0^1 \psi_1(\tau^{\eta_1-1})^{-1}f(\tau,1,1)\,\mathrm{d}\tau\,\mathrm{d}s\right)^{-1}, \\
& \psi_1(J)\int_0^1 l_2(s)s^{\eta_2-1}\int_0^1 l_1(\tau)\psi_1(\tau^{\eta_1-1})f(\tau,1,1)\,\mathrm{d}\tau\,\mathrm{d}s \\
& \left(L_1 L_2 \psi_2(J)^{-1}\int_0^1 s^{\eta_2-1}\int_0^1 \psi_2(\tau^{\eta_1-1})^{-1}g(\tau,1,1)\,\mathrm{d}\tau\,\mathrm{d}s\right)^{-1}, \\
& \psi_2(J)\int_0^1 l_2(s)s^{\eta_2-1}\int_0^1 l_1(\tau)\psi_2(\tau^{\eta_1-1})g(\tau,1,1)\,\mathrm{d}\tau\,\mathrm{d}s \\
& \left(L_1 L_2 J^{-1}\int_0^1 s^{\eta_2-1}\int_0^1 \tau^{1-\eta_1}\phi(\tau,1,1)\,\mathrm{d}\tau\,\mathrm{d}s\right)^{-1}, \\
& J\int_0^1 l_2(s)s^{\eta_2-1}\int_0^1 l_1(\tau)\tau^{\eta_1-1}\phi(\tau,1,1)\,\mathrm{d}\tau\,\mathrm{d}s \\
& \left(L_1 L_2 J^{-1}\int_0^1 s^{\eta_2-1}\int_0^1 \tau^{1-\eta_1}\rho(\tau,1)\,\mathrm{d}\tau\,\mathrm{d}s\right)^{-1}, \\
& J\int_0^1 l_2(s)s^{\eta_2-1}\int_0^1 l_1(\tau)\tau^{\eta_1-1}\rho(\tau,1)\,\mathrm{d}\tau\,\mathrm{d}s \bigg\}.
\end{aligned}$$

By means of J_1, it holds that

$$J_1 t^{\eta_1-1} \leq T_1(x,y)(t) \leq J_1^{-1}h(t),$$

i.e., $T_1 : P_h \times P_h \to P_h$. Similarly, $T_2, T_3 : P_h \times P_h \to P_h$, $T_4 : P_h \to P_h$.

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Step 3: From the assumption (r_2), there exist $\psi_1(\lambda) \in (\lambda, 1]$, $\psi_2(\lambda) \in (\lambda, 1]$ such that

$$
T_1(\lambda x, \lambda^{-1} y)(t) \geq \int_0^1 K(t,s) \int_0^1 G(s,\tau) \psi_1(\lambda)
$$
$$
\cdot f(\tau, x(\tau), y(\tau)) \, \mathrm{d}\tau \, \mathrm{d}s = \psi_1(\lambda) T_1(x, y)(t),
$$
$$
T_2(\lambda x, \lambda^{-1} y)(t) \geq \int_0^1 K(t,s) \int_0^1 G(s,\tau) \psi_2(\lambda)
$$
$$
\cdot g(\tau, x(\tau), y(\tau))) \, \mathrm{d}\tau \, \mathrm{d}s = \psi_2(\lambda) T_2(x, y)(t).
$$

What is more, for fixed $t \in (0,1)$, $y \in P$, for any $\iota \in (0,1)$, $x_1, x_2 \in P$, we can find

$$
T_3(\iota x_1 + (1-\iota) x_2, y)(t) \geq \int_0^1 K(t,s) \int_0^1 G(s,\tau) [\iota \phi(\tau, x_1(\tau),
$$
$$
y(\tau)) + (1-\iota) \phi(\tau, x_2(\tau), y(\tau))] \, \mathrm{d}\tau \, \mathrm{d}s
$$
$$
= \iota \int_0^1 K(t,s) \int_0^1 G(s,\tau) \phi(\tau, x_1(\tau), y(\tau)) \, \mathrm{d}\tau \, \mathrm{d}s
$$
$$
+ (1-\iota) \int_0^1 K(t,s) \int_0^1 G(s,\tau) \phi(\tau, x_2(\tau), y(\tau)) \, \mathrm{d}\tau \, \mathrm{d}s
$$
$$
= \iota T_3(x_1, y) + (1-\iota) T_3(x_2, y)(t).
$$

For fixed $t \in (0,1)$, $x \in P$, for any $\iota \in (0,1)$, $y_1, y_2 \in P$, it gives

$$
T_3(x, \iota y_1 + (1-\iota) y_2)(t) \leq \int_0^1 K(t,s) \int_0^1 G(s,\tau) [\iota \phi(\tau, x(\tau),
$$
$$
y_1(\tau)) + (1-\iota) \phi(s, x(s), y_2(s))] \, \mathrm{d}\tau \, \mathrm{d}s
$$
$$
= \iota \int_0^1 K(t,s) \int_0^1 G(s,\tau) \phi(\tau, x(\tau), y_1(\tau)) \, \mathrm{d}\tau \, \mathrm{d}s
$$
$$
+ (1-\iota) \int_0^1 K(t,s) \int_0^1 G(s,\tau) \phi(\tau, x(\tau), y_2(\tau)) \, \mathrm{d}\tau \, \mathrm{d}s
$$
$$
= \iota T_3(x, y_1) + (1-\iota) T_3(x, y_2)(t).
$$

Hence, for fixed $y \in P$, $T_3(\cdot, y)$ is concave; for fixed $x \in P$, $T_3(x, \cdot)$ is convex. Besides, for any $\lambda \in (0,1)$, we derive that

$$
T_4(\lambda x)(t) \geq \int_0^1 K(t,s) \int_0^1 G(s,\tau) \lambda \rho(\tau, x(\tau)) \, \mathrm{d}\tau \, \mathrm{d}s
$$
$$
= \lambda T_4 x(t),
$$

i.e., T_4 is a sub-homogeneous operator.

Step 4: By the condition (r_3), there exists $\frac{1}{2} \leq a \leq 1$ such that

$$
T_3(\theta, \widetilde{l}h)(t) = \int_0^1 K(t,s) \int_0^1 G(s,\tau) \phi(\tau, \theta, \widetilde{l}h(\tau)) \, \mathrm{d}\tau \, \mathrm{d}s
$$
$$
\geq \int_0^1 K(t,s) \int_0^1 G(s,\tau) a \phi(\tau, \widetilde{l}h(\tau), \theta) \, \mathrm{d}\tau \, \mathrm{d}s
$$
$$
= a T_3(\widetilde{l}h, \theta)(t),
$$

which prove the condition (H_2) of Corollary 1.

Step 5: It can be concluded from (r_5) that

$$
k_1 T_1(x, y)(t) + k_2 T_2(x, y)(t)
$$
$$
= \int_0^1 K(t,s) \int_0^1 G(s,\tau) [k_1 f(\tau, x(\tau), y(\tau))
$$
$$
+ k_2 g(\tau, x(\tau), y(\tau))] \, \mathrm{d}\tau \, \mathrm{d}s
$$
$$
\geq \frac{p}{1-p} \int_0^1 K(t,s) \int_0^1 G(s,\tau) [k_3 \phi(\tau, x(\tau), y(\tau))
$$
$$
+ k_4 \rho(\tau, x(\tau))] \, \mathrm{d}\tau \, \mathrm{d}s
$$
$$
= \frac{p}{1-p} [k_3 T_3(x, y)(t) + k_4 T_4 x(t)], \quad p \in (0,1), \forall t \in (0,1),
$$

which implies (H_4') of Corollary 1. $\qquad\square$

Based on the above five steps, from Corollary 1, the conclusions (T1)–(T3) hold.

Lemma 4. If $\sigma(t) = k_1 f(t, x(t), x(t)) + k_2 g(t, x(t), x(t)) + k_3 \phi(t, x(t), x(t)) + k_4 \rho(t, x(t)) \in C[0,1]$, $M_1 > 0$, $M_2 > 0$, the equation (1) with condition (3) has the following equivalent integral equation

$$
x(t) = \int_0^1 G(t,s) \int_0^1 K(s,\tau) \big[k_1 f(\tau, x(\tau), x(\tau)) + k_2 g(\tau, x(\tau), x(\tau))
$$
$$
+ k_3 \phi(\tau, x(\tau), x(\tau)) + k_4 \rho(\tau, x(\tau)) \big] \, \mathrm{d}\tau \, \mathrm{d}s,
$$

where $G(t,s)$, $K(t,s)$ defined as Lemma 2.

Theorem 3. Let $f, g, \phi \colon C((0,1) \times (0,+\infty) \times (0,+\infty), [0,+\infty))$, $\rho \colon C((0,1) \times (0,+\infty), [0,+\infty))$, and $f(t,u,v)$, $g(t,u,v)$, $\phi(t,u,v)$, $\rho(t,u)$ are singular at $t = 0, 1$ and $v = 0$. Assume that (r_1)–(r_2), (r_5) hold and

(r_9) $\exists \frac{1}{2} \leq a \leq 1$ such that $\phi(t, \theta, \widetilde{l}h) \geq a \phi(t, \widetilde{l}h, \theta)$, where $\widetilde{l} \geq 1$, $h(t) = t^{\eta_2 - 1}$;

(r_{10})
$$
\int_0^1 s^{\eta_1 - 1} \int_0^1 \psi_1(\tau^{\eta_2 - 1})^{-1} f(\tau, 1, 1) \, \mathrm{d}\tau \, \mathrm{d}s < +\infty,
$$
$$
\int_0^1 s^{\eta_1 - 1} \int_0^1 \psi_2(\tau^{\eta_2 - 1})^{-1} g(\tau, 1, 1) \, \mathrm{d}\tau \, \mathrm{d}s < +\infty,
$$
$$
\int_0^1 s^{\eta_1 - 1} \int_0^1 \tau^{1 - \eta_2} \phi(\tau, 1, 1) \, \mathrm{d}\tau \, \mathrm{d}s < +\infty,
$$

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$$\int_0^1 s^{\eta_1-1} \int_0^1 \tau^{1-\eta_2} \rho(\tau,1)\,\mathrm{d}\tau\,\mathrm{d}s < +\infty.$$

Then

(T6) there exist $u_0, v_0 \in P_h$, $r \in (0,1)$ such that $rv_0 \le u_0 < v_0$, and

$$u_0 \le \int_0^1 G(t,s) \int_0^1 K(s,\tau)\big[k_1 f(\tau,u_0(\tau),v_0(\tau))$$
$$+ k_2 g(\tau,u_0(\tau),v_0(\tau)) + k_3\phi(\tau,u_0(\tau),v_0(\tau))$$
$$+ k_4\rho(\tau,u_0(\tau))\big]\,\mathrm{d}\tau\,\mathrm{d}s$$
$$\le \int_0^1 G(t,s) \int_0^1 K(s,\tau)\big[k_1 f(\tau,v_0(\tau),u_0(\tau))$$
$$+ k_2 g(\tau,v_0(\tau),u_0(\tau)) + k_3\phi(\tau,v_0(\tau),u_0(\tau))$$
$$+ k_4\rho(\tau,v_0(\tau))\big]\,\mathrm{d}\tau\,\mathrm{d}s \le v_0;$$

(T7) the equation (1) with the boundary value condition (3) has a unique positive solution x^* in P_h, where $h(t) = t^{\eta_2-1}$.

(T8) for any initial values $x_0,\ y_0 \in P_h$, constructing successively the iterative sequences

$$x_{n+1}(t) = \int_0^1 G(t,s) \int_0^1 K(s,\tau)\big[k_1 f(\tau,x_n(\tau),y_n(\tau))$$
$$+ k_2 g(\tau,x_n(\tau),y_n(\tau)) + k_3\phi(\tau,x_n(\tau),y_n(\tau))$$
$$+ k_4\rho(\tau,x_n(\tau))\big]\,\mathrm{d}\tau\,\mathrm{d}s,$$
$$y_{n+1}(t) = \int_0^1 G(t,s) \int_0^1 K(s,\tau)\big[k_1 f(\tau,y_n(\tau),x_n(\tau))$$
$$+ k_2 g(\tau,y_n(\tau),x_n(\tau)) + k_3\phi(\tau,y_n(\tau),x_n(\tau))$$
$$+ k_4\rho(\tau,y_n(\tau))\big]\,\mathrm{d}\tau\,\mathrm{d}s,$$

$n = 0,1,2,\cdots, x_{n+1} \to x^*,\ y_{n+1} \to x^*$, when $n \to \infty$.

5. APPLICATIONS

Consider the following equation:

$$\begin{cases} D_{0^+}^{\frac{41}{12}}x(t) + 2\big[(x+4)^{\frac{1}{3}} + 4x^{-\frac{1}{4}} + t^{-\frac{1}{8}}(1-t)^{-\frac{1}{9}} + 3\big] \\ \qquad + (x+3)^{\frac{1}{4}} + 3x^{-\frac{1}{4}} + t^{-\frac{1}{9}}(1-t)^{-\frac{1}{10}} \\ \qquad + \frac{1}{2}\big[-\frac{3}{5}e^{-x} + \frac{9}{4}e^{-y} + t^{-\frac{1}{10}}(1-t)^{-\frac{1}{11}} + 1\big] \\ \qquad + \frac{1}{3}\big[(x+1)^{\frac{1}{6}} + t^{-\frac{1}{12}}(1-t)^{-\frac{1}{13}}\big] = 0, \qquad (19) \\ x(0) = 0, \quad D_{0^+}^{\frac{5}{3}}x(0) = 0, \quad D_{0^+}^{\frac{7}{4}}x(1) = 2I_{0^+}^{\frac{1}{8}}x\left(\frac{1}{2}\right), \\ D_{0^+}^{\frac{27}{15}}x(1) = \int_0^1 s^{\frac{1}{2}} D_{0^+}^{\frac{11}{6}}x(s)\,\mathrm{d}s^2, \end{cases}$$

where $\eta_1 = \frac{5}{3}$, $\eta_2 = \frac{7}{4}$, $\alpha_1 = \frac{1}{7}$, $\alpha_2 = \frac{1}{5}$, $\beta_1 = \frac{1}{8}$, $\beta_2 = \frac{1}{6}$, $\tau = 2$, $b(s) = s^{\frac{1}{2}}$, $A(s) = s^2$, $\xi_1 = \frac{1}{2}$, $k_1 = 2$, $k_2 = 1$, $k_3 = \frac{1}{2}$, $k_4 = \frac{1}{3}$, and

$$f(t,x,y) = (x+4)^{\frac{1}{3}} + 4y^{-\frac{1}{4}} + t^{-\frac{1}{8}}(1-t)^{-\frac{1}{9}} + 3,$$
$$g(t,x,y) = (x+3)^{\frac{1}{4}} + 3y^{-\frac{1}{4}} + t^{-\frac{1}{9}}(1-t)^{-\frac{1}{10}},$$
$$\phi(t,x,y) = -\frac{3}{5}e^{-x} + \frac{9}{4}e^{-y} + t^{-\frac{1}{10}}(1-t)^{-\frac{1}{11}} + 1,$$
$$\rho(t,x) = (x+1)^{\frac{1}{6}} + t^{-\frac{1}{12}}(1-t)^{-\frac{1}{13}}.$$

Then equation (19) has unique positive solution $x^* \in P_h$, where $h(t) = t^{\frac{2}{3}}$. In addition, the other conclusions in Theorem 2 hold.

Proof. The proof process can be divided into the following steps.

$(\widetilde{a1})$ $\eta_2 - \alpha_2 - 1 = \frac{13}{4} - \frac{11}{5} - 1 > 0$, $\eta_2 - \beta_2 - 1 = \frac{13}{4} - \frac{7}{6} - 1 > 0$, $\alpha_2 - \beta_2 = \frac{1}{5} - \frac{1}{6} > 0$, and

$$M_1 = \Gamma\left(\tfrac{5}{3}\right)\Gamma\left(\tfrac{5}{3} + \tfrac{1}{8}\right) - 2\Gamma\left(\tfrac{5}{3}\right)\Gamma\left(\tfrac{5}{3} - \tfrac{1}{7}\right)\tfrac{1}{2}^{\frac{5}{3} + \frac{1}{8} - 1} > 0,$$

$$M_2 = \frac{1}{\Gamma\left(\tfrac{7}{4} - \tfrac{1}{5}\right)} - \frac{\int_0^1 s^{\frac{1}{2}} s^{\frac{7}{4} - \frac{1}{6} - 1}\,\mathrm{d}s^2}{\Gamma\left(\tfrac{7}{4} - \tfrac{1}{6}\right)} > 0.$$

$(\widetilde{b1})$ from the definition of f, g, ϕ, ρ, we can get that $f(t,x,y)$, $g(t,x,y)$, $\phi(t,x,y)$, $\rho(t,x)$ are nondecreasing in x, $f(t,x,y)$, $g(t,x,y)$, $\phi(t,x,y)$ are nonincreasing in y, and $f(t,u,v)$, $g(t,u,v)$, $\rho(t,u)$ are singular at $t = 0,1$ and $v = 0$, $\phi(t,u,v)$ is singular at $t = 0,1$.

$(\widetilde{c1})$ there exist $\psi_1(\lambda) = \lambda^{\frac{1}{3}}$, $\psi_2(\lambda) = \lambda^{\frac{1}{4}}$ such that

$$f(t,\lambda x,\lambda^{-1}y) \ge \lambda^{\frac{1}{3}}\big[(x+4)^{\frac{1}{3}} + 4y^{-\frac{1}{4}} + t^{-\frac{1}{8}}(1-t)^{-\frac{1}{9}} + 3\big]$$
$$= \psi_1(\lambda)f(t,x,y),$$
$$g(t,\lambda x,\lambda^{-1}y) \ge \lambda^{\frac{1}{4}}\big[(x+3)^{\frac{1}{4}} + 3y^{-\frac{1}{4}} + t^{-\frac{1}{9}}(1-t)^{-\frac{1}{10}}\big]$$
$$= \psi_2(\lambda)g(t,x,y).$$

From $\phi''_{xx}(t,x,y) = -\frac{3}{5}e^{-x} \le 0$, $\phi''_{yy}(t,x,y) = \frac{9}{4}e^{-y} \ge 0$, we can know that $\phi(t,\cdot,y)$ is concave; $\phi(t,x,\cdot)$ is convex. For any $\lambda, t \in (0,1)$, $x,y \in (0,+\infty)$, one gets

$$\rho(t,\lambda x) \ge \lambda\big[(x+1)^{\frac{1}{6}} + t^{-\frac{1}{12}}(1-t)^{-\frac{1}{13}}\big] \ge \lambda\rho(t,x).$$

$(\widetilde{d1})$ Let $x,\ y \le M_3$, M_3 is a sufficiently large number. Take $a = \frac{1}{9}$, we deduce

$$\phi(t,\theta,M_3) \ge \frac{1}{9}\left[-\frac{3}{5}e^{-M_3} + \frac{9}{4} + t^{-\frac{1}{10}}(1-t)^{-\frac{1}{11}} + 1\right] = a\phi(t,M_3,\theta).$$

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$(\widetilde{e1})$ We can get $\int\limits_0^1 s^{\eta_2-1}\int\limits_0^1 \psi_1(\tau^{\eta_1-1})^{-1}f(\tau,1,1)\,\mathrm{d}\tau\,\mathrm{d}s =$

$\int\limits_0^1 s^{\frac{3}{4}}\int\limits_0^1 \tau^{-\frac{2}{9}}\left[5^{\frac{1}{3}}+\tau^{-\frac{1}{8}}(1-\tau)^{-\frac{1}{9}}+7\right]\mathrm{d}\tau\,\mathrm{d}s < +\infty$. Similarly,

$\int\limits_0^1 s^{\eta_2-1}\int\limits_0^1 \psi_2(\tau^{\eta_1-1})^{-1}g(\tau,1,1)\,\mathrm{d}\tau\,\mathrm{d}s < +\infty,$

$\int\limits_0^1 s^{\eta_2-1}\int\limits_0^1 \tau^{1-\eta_1}\phi(\tau,1,1)\,\mathrm{d}\tau\,\mathrm{d}s < +\infty,$

$\int\limits_0^1 s^{\eta_2-1}\int\limits_0^1 \tau^{1-\eta_1}\rho(\tau,1)\,\mathrm{d}\tau\,\mathrm{d}s < +\infty;$

$(\widetilde{f1})$ Let $p = \dfrac{3}{4}$. We obtain

$$k_1 f(t,x,y)+k_2 g(t,x,y) \geq 3\cdot\frac{1}{2}\left[-\frac{3}{5}e^{-x}+\frac{9}{4}e^{-y}\right.$$
$$\left.+t^{-\frac{1}{10}}(1-t)^{-\frac{1}{11}}+1\right]+3\cdot\frac{1}{3}\left[(x+1)^{\frac{1}{6}}+t^{-\frac{1}{12}}(1-t)^{-\frac{1}{13}}\right]$$
$$=\frac{p}{1-p}\left[k_3\phi(t,x,y)+k_4\rho(t,x)\right].$$

According to Theorem 2, equation (19) has unique positive solution $x^* \in P_h$, where $h(t) = t^{\frac{2}{3}}$, and $(T1),(T3)$ hold. □

6. CONCLUSIONS

This paper mainly discusses the existence and uniqueness of solution as well as the iterative sequences for uniformly approximating the unique solution for two types of singular FDEs with RS integral boundary conditions. The main conclusions are as follows. (i) Considering that nonlinear terms in abstract differential equations in some real field have different convexity and monotonicity, in order to better study these equations, this paper discuss the fixed point theorems of sum operators that contain differential properties. We obtain the corresponding properties of solution to operator equation with parameters $T_5(x,y) = k_1 T_1(x,y)+k_2 T_2(x,y)+k_3 T_3(x,y)+k_4 T_4 x$ on cone P and set P_h. Based on different parameter values and operator properties, the operator conclusions obtained generalize some existing literature results. (ii) Using the conclusions of nonlinear operators in the study, two types of fractional differential equations are discussed. Several sufficient conditions of unique solutions, the existence of maximum and minimum solutions, and the iterative approximation sequences of unique solutions are given.

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