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# Static analysis of a two-layer beam with a flexible interlayer connection

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**Abstract.** The aim of this research is to develop an algorithm that allows for the analysis of the influence of interlayer connections on the static response of double-layer beams. Multilayer beams, due to the fact that their construction uses the advantages of different materials, are widely used in the construction industry. The critical element of these types of structures is the connection between layers. The stiffness of this connection can significantly affect the static response of the system, and therefore its strength. The Euler-Bernoulli model was used to describe the double-layer beams. In the paper, the compliance of normal displacements in multilayer and double-layer beams was considered based on the research of other authors. It was also assumed that the tangential interactions at the connection of layers were tangential forces that are proportional to the relative tangential displacement (slip) of these layers. This general approach eliminates the need for a broader analysis of the connection (its description and structure) with regards to the applied "connector" between the layers. Using equilibrium equations and the adopted assumptions, a system of displacement equations was derived. This system is formed by three coupled second- and fourth-order differential equations, the exact solution of which is a non-trivial mathematical problem. This system was solved using the finite sine and cosine Fourier transform. Although the algorithm presented in the paper was used to solve a specific set of equations that describe a simply supported beam, the formulas derived in the paper allow for solving beams with other support schemes. The Fourier transformation method, after appropriate modifications (changing the boundary conditions and, in some cases, changing the sine to cosine transformation and vice versa), can be used to solve beams that have other support conditions. In order to verify the correctness and effectiveness of the described method, two numerical examples were solved. In the first one, the influence of the variable value of the tangential stiffness of the connection on the values and distribution of internal forces (including the influence on the values of normal stresses) was analyzed. In the second example, calculations were performed for an example taken from literature, and the obtained results were compared with the results obtained by other authors. The analyzed examples confirmed the significant influence of the tangential stiffness of the layer connection on the static response of the system. Furthermore, they confirmed the correctness and high accuracy of the method that was used to solve the problem.

**Key words:** two-layer beam, interlayer slip, static analysis.

## 1. INTRODUCTION

Composite elements are often used in building structures, for example in bridge and high-rise constructions. It is therefore worth researching multi-layered elements, which use the advantages of various materials. In recent years, a number of researchers have been studying multi-layered beams as ballastless track structures [1,2]. This is due to various reasons, such as the fact that ballastless track structures have become a main development direction for high-speed railway track structures worldwide. This is due to their high stability, high smoothness and low maintenance [1], and also because slab ballastless tracks, which have become the main form of subway track structures in China, have the advantages of a fast construction speed, good durability, less maintenance, and easy upgrading, [3]. An example of a double-beam model is a rail track, which can be represented as a two-layer system. In this system, the first layer represents rails, and the second one describes sleepers. They are both mathematically modelled by coupled and modified Euler- Bernoulli beam equations [2]. Other static models, such as columns, are also considered in the analysis [4, 5, 6, 7,8].

Connections between the layers in two-layer beams are sensitive elements that often have a significant impact on the beam's stresses and displacements, as well as on its reliability. These connections often undergo failure. In these places, slip may occur between the cooperating layers. This problem has been analyzed in many works, for example in papers [7, 9, 10,11,12] with regards to static and dynamic issues. Lu et al. [13] investigated the compressive response of multilayer columns with different interlayers with the use of experimental, analytical, and numerical approaches. Wu et al. [14] presented simply supported two-layer composite beams with an arbitrarily shaped interface that was not assumed to be a straight line. Girhammar and Gopu [4] presented closed-form solutions for the displacements and internal forces in partially composite beam-columns, which were developed for first- and second-order cases. Foraboschi [7] analyzed the buckling behaviour of laminated glass columns under axial compressive loads. The research highlighted the critical role of the thermoplastic interlayer in both transferring shear stresses and influencing the critical buckling load. Batista [5, 15] focused on the numerical and analytical analysis of a multi-layer beam, taking into account slip at the joint. He determined accurate finite element solutions and analytical approaches to describing a partial interaction, which is crucial for understanding the effects of slip on the structural behaviour of beams.

Le Grognec et al. [16] and Siciliano et al. [17] presented solutions that refer to the buckling and static analysis of two-layer Timoshenko beams with slip in the joint. They identified that proper slip modelling is crucial for predicting the critical loading conditions and behaviour of a beam under load. Ecsedi and Baksa [18, 19], as well as Monetto [20], indicated the difficulties associated with the analysis of beams with weak connections, and described the complexity of the mathematical models needed to describe the behaviour of such beams. The authors emphasized the importance of accurate theoretical and numerical analyses that can be used to predict the behaviour of beams under various conditions. Schnabl et al. [10] developed an analytical solution for two-layer beams, which incorporates both interlayer slip and shear deformation. The model was derived using Timoshenko beam theory, and provided an accurate representation of the mechanical behaviour under various loading conditions. The model that gives the exact analytical solution for the linear behaviour was also presented by Foraboschi [21].

In paper [6], Girhammar presented a new two-dimensional model of composite beams with interlayer slips. The solution includes the effect of shear deformation and is two-dimensionally exact, as it does not introduce the Euler–Bernoulli hypothesis of deformation that is usually assumed in one-dimensional theory. The results obtained from the present two-dimensional method are compared with those available in the literature that are based on one-dimensional theory. Faella, Martinelli, and Nigro [22] presented an "exact" closed-form solution for the stiffness matrix and equivalent nodal forces in steel-concrete composite beams with partial interaction. The study builds on Newmark's theory and proposes a 1D finite element that allows for efficient, linear elastic analysis using only one element per beam member. Girhammar [23] presented a simplified static analysis method for composite beams with interlayer slip, which is similar to Eurocode 5 but more general. The presented method is suitable for various boundary and loading conditions, and typically yields errors below 5%, except for shear stresses, where errors range from 10% to 20%. Foraboschi [7] proposed a closed-form exact analytical solution of a two-layer beam with nonlinear interlayer slip. The model incorporated fully developed non-linear equations to accurately describe the behaviour of the beam under various loading conditions, with a focus on interlayer slip. Udovč et al. [24] introduced a new model for analyzing two-layer spatial beams with inter-layer slip in longitudinal and transverse directions. The model incorporates shear deformations and uses deformation-based finite elements to avoid locking issues. This approach improves accuracy and stability in composite beam analysis, particularly in cases where inter-layer slip significantly impacts structural behaviour.

Bochicchio et al. [25] presented the different behaviour of the nonlinear system with regards to double-beam linear systems.

This paper analyzes the influence of the connections between layers on the static response of a simply supported two-layer beam, while also taking into account the slip between layers. The results clearly illustrate the magnification of the displacements and the mutual slip between layers due to the reduction in the shear stiffness of the connection between both layers. To solve the described problem, an approximation method was applied, with a trigonometric Fourier series being used to expand the displacement functions. Although the paper only presents the solution for a simply supported beam, it is important to note that this method is universal and allows for the solving of beams with other types of supports. As demonstrated in the analyzed examples, the method is characterized by high accuracy when compared to other approximate solution methods. A further advantage of this method is the semi-analytical form of the derived solutions. According to the authors, the method presented in the paper, as well as the obtained results, can be a valuable guide when designing sandwich beams that have an increased strength and reliability. This in turn will support the development of more efficient and durable composite structures.

## 2. DESCRIPTION OF THE MODEL

Let us consider a beam composed of two connected layers with a cross-section and a longitudinal section, as shown in Figure 1. In the presented considerations, it was assumed that the normal displacements of both beams were consistent and, based on the research of other authors, it was also assumed that the tangential interactions at the connection of both layers were tangential forces that are proportional to the relative tangential displacement (slip) of both layers.

The state of the longitudinal displacements, together with the mutual shift of the layers and the cross-sectional forces acting on the infinitesimal fragment of the beam, are presented in Figures 2a and 2b, respectively.

The slip of the layers, relative to each other, is described by Eq. (1):

$$u_s(x) = u_2(x) - u_1(x) + h\frac{dw(x)}{dx}, \tag{1}$$

where: $h = h_1 + h_2$ (see Fig.1.)

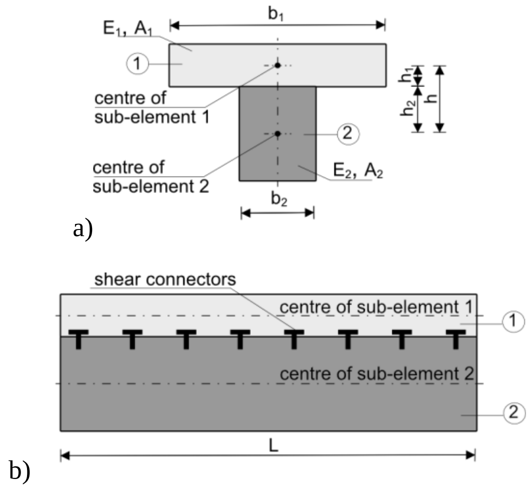



**Fig.1.** Geometry of a) the cross-section and b) the longitudinal section of a two-layer element.

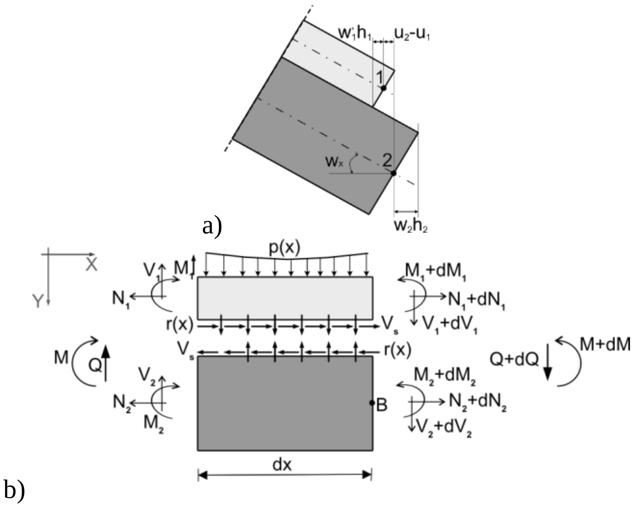



**Fig.2.** a) Mutual displacements of the components of a two-layer beam; b) Internal forces in the cross-section of the two-layer beam and at the interface of the component layers along section dx.

The tangential forces acting on the beam in the plane of the connection (shear forces) are therefore described by the following formula (see e.g. [5], [23], [26]):

$$V_s(x) = k_s u_s(x) = k_s\left[u_2(x) - u_1(x) + h\frac{dw(x)}{dx}\right], \tag{2}$$

where constant $k_s$ is the shear stiffness of the connection of both layers.
From the equilibrium equations for the selected beam's element, the following relationships can be obtained:
$\Sigma X = 0$
- for the upper beam - for the lower beam

$$\frac{dN_1(x)}{dx} + k_s u_s(x) = 0 \qquad \frac{dN_2(x)}{dx} - k_s u_s(x) = 0 \tag{3}$$

$\Sigma Y = 0$
- for the upper beam - for the lower beam

$$\frac{dV_1(x)}{dx} = -p(x) + r(x) \qquad \frac{dV_2(x)}{dx} = -r(x) \tag{4}$$

which, when added, gives the following equations:

$$\frac{dV(x)}{dx} = -p(x), \qquad V(x) = V_1(x) + V_2(x) \tag{5}$$

$\Sigma M = 0$

- for the upper beam - for the lower beam

$$V_1(x) = \frac{dM_1(x)}{dx} + V_s(x)h_1, V_2(x) = \frac{dM_2(x)}{dx} + V_s(x)h_2 \tag{6}$$

which, when added, gives the following equations:

$$V(x) = \frac{dM_B(x)}{dx} + V_s(x)h, M_B(x) = M_1(x) + M_2(x). \tag{7}$$

The following constitutive relations are important for the Bernoulli–Euler beam:

$$M_1(x) = -EI_1\frac{d^2w_1(x)}{dx^2}, \quad M_2(x) = -EI_2\frac{d^2w_2(x)}{dx^2} \tag{8}$$

and

$$N_1(x) = EA_1\frac{du_1(x)}{dx}, \quad N_2(x) = EA_2\frac{du_2(x)}{dx}. \tag{9}$$

When taking into consideration that $w_1(x) = w_2(x) = w(x)$ (the condition of compliance of displacements that are perpendicular to the beam's axis), the following is obtained:

$$M_B(x) = M_1(x) + M_2(x) = -EI\frac{d^2w(x)}{dx^2}, \tag{10}$$

where $EI = EI_1 + EI_2$.
After using relationship (5) and formulas (7) and (10), the following is obtained:

$$\frac{dV(x)}{dx} = \frac{d^2M(x)}{dx^2} + h\frac{dV_s(x)}{dx} = -p(x). \tag{11}$$

By substituting constitutive relations (8) and (9) in equilibrium Eq. (3), (5), and (7), the following displacement equations describing the analyzed model of the two-layer beam are obtained:

$$\begin{cases} EI\frac{d^4w(x)}{dx^4} + hk_s\frac{du_1(x)}{dx} - hk_s\frac{du_2(x)}{dx} - h^2k_s\frac{d^2w(x)}{dx^2} = \\ \qquad = p(x) \\ EA_1\frac{d^2u_1(x)}{dx^2} + k_s u_2(x) - k_s u_1(x) + hk_s\frac{dw(x)}{dx} = 0 \\ EA_2\frac{d^2u_2(x)}{dx^2} - k_s u_2(x) + k_s u_1(x) - hk_s\frac{dw(x)}{dx} = 0. \end{cases} \tag{12}$$

In the case of the static scheme analyzed in the paper, i.e. a simply supported beam ( ), and due to the fact that the beam is not subjected to any external physical load, the following relation $N_1(x) + N_2(x) = 0$ occurs. The internal forces are defined as follows:

- bending moments:

$$M(x) = M_1(x) + M_2(x) - N_1(x)h, \tag{13}$$

- shear forces:

$$Q(x) = V_1(x) + V_2(x) = \frac{dM_B(x)}{dx} + V_s(x)h. \tag{14}$$

## 3. SOLUTION OF THE PROBLEM

The subject of further analysis will be a simply supported beam, with support points located at the ends of beam No. 2 - on its lower edge.

The boundary conditions for this case are presented in Eq. (15) to (17):

$$w(0) = w(L) = 0, \ \left.\frac{d^2w(x)}{dx^2}\right|_{x=0} = \left.\frac{d^2w(x)}{dx^2}\right|_{x=L} = 0, \tag{15}$$

$$\left.\frac{du_1(x)}{dx}\right|_{x=0} = \left.\frac{du_1(x)}{dx}\right|_{x=L} = 0, \tag{16}$$

$$u_2(0) = h_2 \left.\frac{dw(x)}{dx}\right|_{x=0}, \qquad \left.\frac{du_2(x)}{dx}\right|_{x=L} = 0. \tag{17}$$

The analyzed issue is solved by developing the sought functions into a Fourier series. With such boundary conditions, we are looking for solutions in the form of the following series:

$$\begin{cases} u_1(x) = {\sum_{n=0}^{\infty}}' u_{1n} \cos \alpha_n x \\ u_2(x) = {\sum_{n=0}^{\infty}}' u_{2n} \cos \alpha_n x \ , \\ w(x) = \sum_{n=1}^{\infty} w_n \sin \alpha_n x \end{cases} \tag{18}$$

where:

$$\alpha_n = \frac{n\pi}{L}, \qquad {\sum_{n=0}^{\infty}}' a_n = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n.$$

After substituting the above expansions of functions $u_1, u_2,$ and $w$, and the expansions of their derivatives $u_1', u_2', u_1'', u_2'', w', w''$, and $w^{IV}$ (see formulas (A.2)-(A.6)) into the system of differential equations (12), and after comparing the coefficients on the left and right sides of this system, an infinite system of algebraic equations is obtained:

$$\begin{cases} EI\alpha_n^4 w_n - hk_s\alpha_n u_{1n} + hk_s\alpha_n u_{2n} + h^2k_s\alpha_n^2 w_n = \\ \quad = p_n - EI\ \frac{2}{L}\alpha_n^3\ [(-1)^n w(L) - w(0)] \\ \quad +EI\ \frac{2}{L}\alpha_n\ [(-1)w''(L) - w''(0)] \\ \quad +h^2k_s\ \frac{2}{L}\alpha_n\ [\ (-1)^n w(L) - w(0)\ ]\ , \\ -EA_1\alpha_n^2 u_{1n} + k_s u_{2n} - k_s u_{1n} + hk_s\alpha_n w_n = \\ \quad = -EA_1\ \frac{2}{L}[(-1)^n\ u_1'(L) - u_1'(0)] \\ \quad -hk_s\frac{2}{L}[(-1)^n w(L) - w(0)]\ , \\ -EA_2\alpha_n^2 u_{2n} - k_s u_{2n} + k_s u_{1n} - hk_s\alpha_n w_n = \\ \quad = -EA_2\frac{2}{L}[(-1)^n\ u_2'(L) - u_2'(0)] \\ \quad +hk_s\frac{2}{L}[(-1)^n w(L) - w(0)]; \\ \quad \text{for } n \geq 0\ . \end{cases} \tag{19}$$

Taking into account the boundary conditions (15)-(17) in equations (19), the following is obtained:

for $n \geq 1$:

$$\begin{cases} EI\alpha_n^4 w_n - hk_s\alpha_n u_{1n} + hk_s\alpha_n u_{2n} + h^2k_s\alpha_n^2 w_n = \\ \quad = p_n \\ -EA_1\alpha_n^2 u_{1n} + k_s u_{2n} - k_s u_{1n} + hk_s\alpha_n w_n = 0 \\ -EA_2\alpha_n^2 u_{2n} - k_s u_{2n} + k_s u_{1n} - hk_s\alpha_n w_n = \\ \quad = EA_2\frac{2}{L}u_2'(0) \end{cases} \tag{20}$$

for $n = 0$:

$$\begin{cases} k_s u_{20} - k_s u_{10} = 0 \\ -k_s u_{20} + k_s u_{10} = EA_2\frac{2}{L}u_2'(0), \end{cases} \tag{21}$$

where:

$$p_n = \frac{2}{L}\int_0^L p(x)\sin\alpha_n x\, dx, \ \ u_2'(0) = \left.\frac{du_2(x)}{dx}\right|_{x=0}.$$

As a result of solving the system of Eq. (21), the following is obtained:

$$u_{10} = u_{20}, \quad u_2'(0) = 0. \tag{22}$$

After substituting $u_2'(0) = 0$ into the system of Eq. (20), and after solving it, the following is obtained:

$$\begin{cases} u_{1n} = \frac{EA_2hk_sL^5p_n}{(n\pi)^3\left[\left((EA_1+EA_2)EI+EA_1EA_2h^2\right)k_sL^2+EA_1EA_2EI(n\pi)^2\right]} \\ u_{2n} = -\frac{EA_1hk_sL^5p_n}{(n\pi)^3\left[\left((EA_1+EA_2)EI+EA_1EA_2h^2\right)k_sL^2+EA_1EA_2EI(n\pi)^2\right]} \\ w_n = -\frac{L^4\left((EA_1+EA_2)k_sL^2+EA_1EA_2EI(n\pi)^2\right)p_n}{(n\pi)^4\left[\left((EA_1+EA_2)EI+EA_1EA_2h^2\right)k_sL^2+EA_1EA_2EI(n\pi)^2\right]} \end{cases} \tag{23}$$

Expansion factors $u_{20} = u_{10}$ are determined using the boundary condition $u_2(0) = h_2 \left.\frac{dw(x)}{dx}\right|_{x=0}$

$$u_2(0) = \frac{1}{2}u_{20} + \sum_{n=1}^{\infty} u_{2n} = h_2 \sum_{n=1}^{\infty} \alpha_n w_n \quad (24)$$

and from this, the following is obtained:

$$u_{10} = u_{20} = 2\sum_{n=1}^{\infty}(\alpha_n h_2 w_n - u_{2n}). \quad (25)$$

Calculating coefficients $w_n$, $u_{1n}$ and $u_{2n}$ allows the sought displacement functions to be determined and, after using the relationships that determine the coefficients of the derivatives of these functions, also the internal forces. Normal stresses $\sigma_x$ were determined using known formula:

$$\sigma_x^i(y_i) = \frac{N_i}{EA_i} + \frac{M_i}{EI_i} y_i. \quad (26)$$

## 4. NUMERICAL EXAMPLE

The influence of the stiffness (shear) of the connection on the displacement and effort state of the two-layer system was examined by analyzing the following two numerical examples. The analysis involved the simply supported beam described in Chapter 3, which has its cross-section shown in Figure 1. The solutions for this beam are achieved by numerical integration using Wolfram Mathematica [27].

### 4.1. Example 1.

The dimensions of this beam are: $L = 2.0\text{m}, b_1 = 0.30\text{m}, b_2 = 0.05\text{m}, H_1 = 2h_1 = 0.05\text{m}, H_2 = 2h_2 = 0.15\text{m}$. The material parameters are $E_1 = E_2 = 10^{10}\text{Pa}$. The beam was loaded at mid-span ($x = L/2$) with a concentrated force $P = 1\text{kN}$, and the coefficients, developed into a sine Fourier series, are given by Eq. (26):

$$p_n = \frac{2}{L}\int_0^L P\delta\left(x - \frac{L}{2}\right) \sin \alpha_n x dx = \frac{2P}{L}\sin\frac{n\pi}{2}, \quad (26)$$

for $n = 1,2,3, \ldots$

The calculations were performed for different values of the stiffness of the connection between layers $k_s$. The dependence between the maximum displacement $w(L/2)$ and the value of parameter $k_s \in \langle 0, 10^9 \rangle$ Pa is shown in Figure 3.

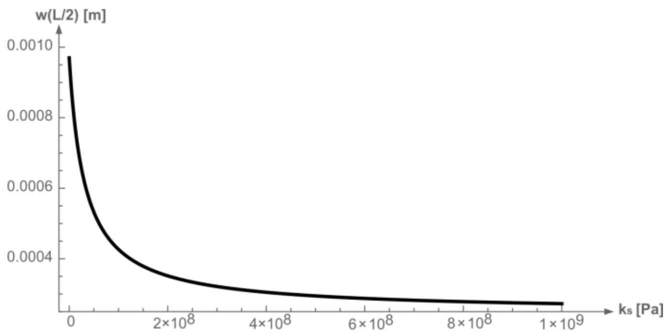



**Fig.3.** Dependence between displacement w(L/2) and the value of the $k_s$ parameter.

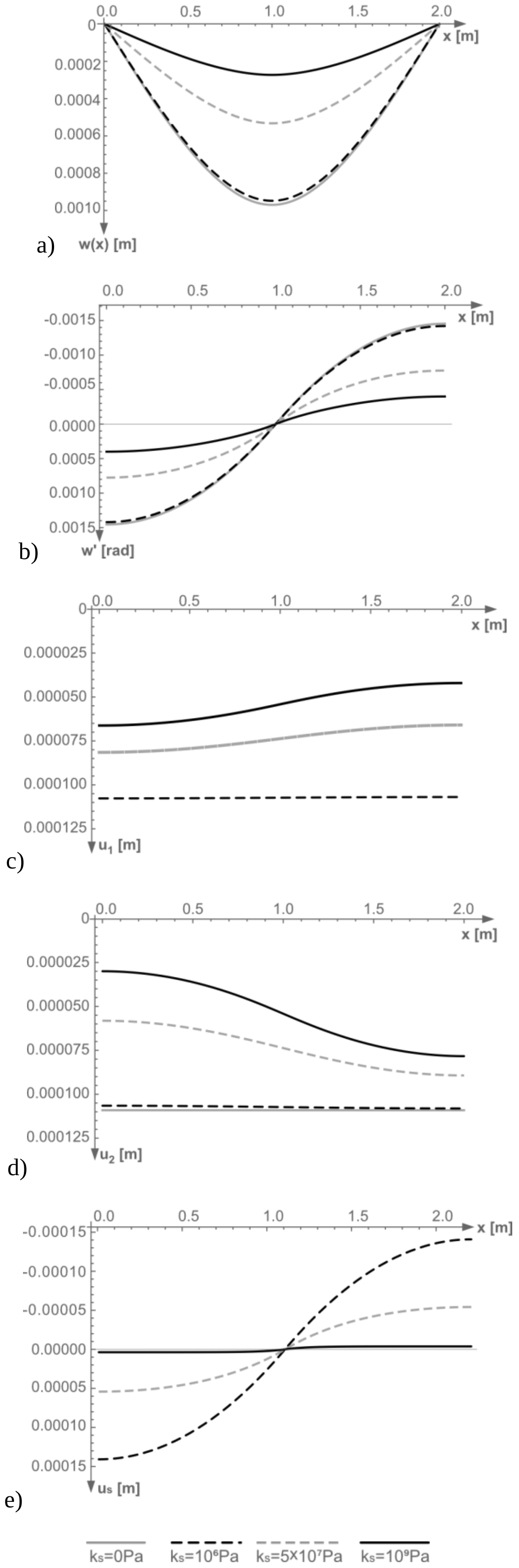



**Fig.4.** Dependence between: a) displacement w(x), b) rotation angle w'(x), c) displacement $u_1(x)$, d) displacement $u_2(x)$, e) the mutual slip between layers $u_s$ and the value of the $k_s$ parameter.

The displacement graphs $w(x)$ for different values of $k_s = 0; 10^6; 5 \cdot 10^7;\ 10^9$Pa are presented in Figure 4a. In the conducted numerical analyses, the values of the relative displacement of the layers (slip) in the plane of the contact were also determined, i.e. the $u_s(x)$ function (see Eq. (1)). The calculations were made by assuming $k_s = 10^6$ Pa; $5 \cdot 10^7$Pa; $10^9$Pa. The graphs of the functions, which are components of the formula (which defines the relative displacement $u_s(x)$, i.e. function $w'(x), u_1(x),\ u_2(x)$), and the graph of the function $u_s(x)$, are presented in Figures 4b, 4c, 4d and 4e, respectively.

The diagrams of internal forces: bending moments and shear forces, were determined using formulas (13) and (14). The calculations were performed for the values $k_s = 10^6$ Pa; $5 \cdot 10^7$ Pa; $10^9$Pa. The obtained results are presented in Figures 5a – 5c. Each of the figures also presents diagrams of the elements included in the formulas that were used for the calculations. In Figure 5a, these are the corresponding functions $M_1(x) + M_2(x)$ , in Figure 5b $-N_1(x) \cdot h$, and in Figure 5c $M(x)$. In Figures 6a - 6c, these are $\frac{dM_B(x)}{dx}$, $V_s(x) \cdot h$, and $Q(x)$, respectively.

The determined graphs confirm the correctness of the derived formulas, as they are consistent with the generally known (for the analyzed static scheme) internal force diagrams.

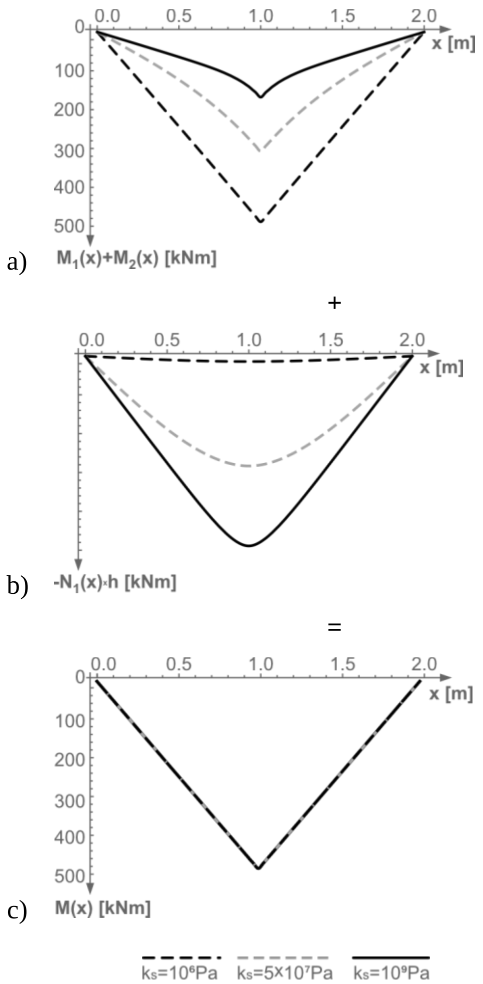



**Fig.5.** Bending moments: a) component $M_1(x)+M_2(x)$, b) component $-N_1(x) \cdot h$, c) diagram of bending moments $M(x)$.

The analysis also involved the assessment of the stress level of the structure. Due to the limited volume of the paper, the obtained results, i.e. normal stress distribution charts (see Eq.(25)), are presented for three values of parameter $k_s = 0$ Pa, $5 \cdot 10^7$Pa; $10^{14}$ Pa (see Figs. 7 and 8). These stresses were determined in cross-sections $x = L/4$ and $x = L/2$. To confirm the correctness of the calculations, analyses were carried out for the parameter $k_s = 10^{14}$Pa. There is practically no slippage between layers, and the beam behaves like a monolithic beam. The obtained results are presented in Figure 8.

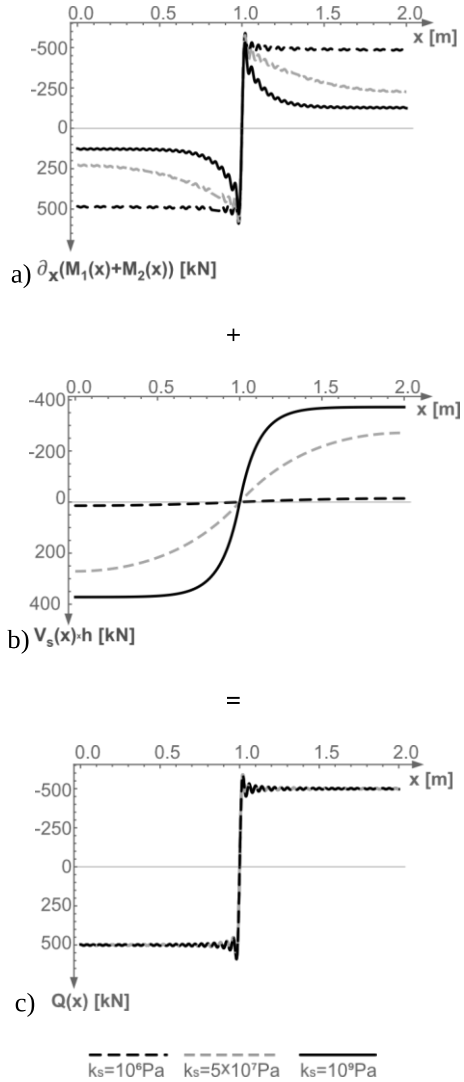
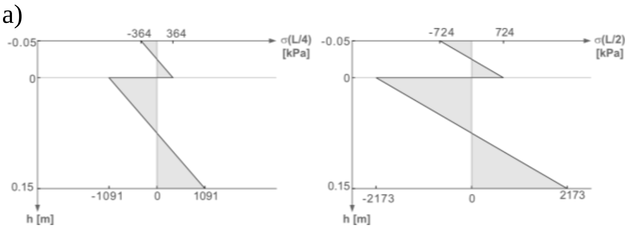



**Fig.6.** Shear forces: a) component $dM_B(x)/dx$, b) component $V_s(x) \cdot h$, c) shear forces $Q(x)$.

a)



b)

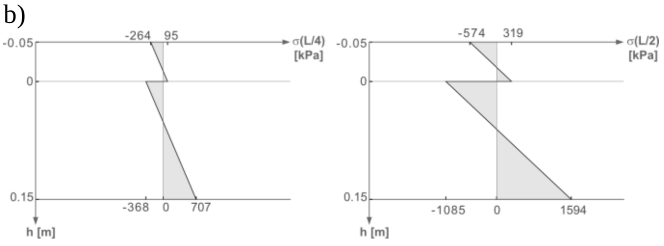



**Fig.7.** Normal stress diagram in cross-section x=L/4 (on the left) and x=L/2 (in the right) when a) $k_s$=0 Pa, b) $k_s$=5·10⁷ Pa,

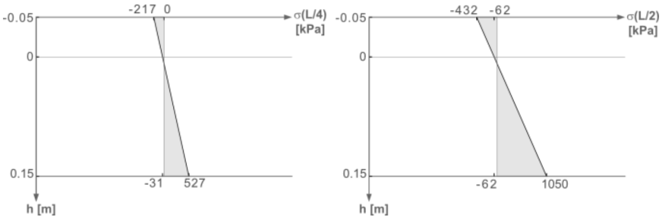



**Fig.8.** Normal stress diagram in cross-section x=L/4 (on the left) and x=L/2 (in the right) when $k_s$=$10^{14}$ Pa.

### 4.2. Example 2.

To verify the accuracy of the obtained solutions, the presented method was applied to solve an example taken from the literature (see [4, 28, 29]). In this example, the subject of analysis is a simply supported beam with a cross-section - defined in Example 1. The material parameters of the beam are: $E_1 = 12 \cdot 10^9$Pa, $E_2 = 8 \cdot 10^9$Pa, and the stiffness of the connection between the two layers $k_s = 5 \cdot 10^7$Pa. Beams of various spans were analyzed, assuming $L/H = 4, 5, 10, 20$ (where $H = 2h = 0.2$m). The beam was subjected to a uniformly distributed load over its entire length $q(x) = 1\,\mathrm{kN/m}$. In this case, the coefficients of the load expansion in the sinusoidal series are determined by the following formula for $n = 1,2,3, \ldots$ :

$$p_n = \frac{2}{L}\int_0^L q\ \sin \alpha_n x dx = 2q\,\frac{1-\cos n\pi}{n\pi} \tag{27}$$

The obtained results for the maximum displacements of the beam at point $x = L/2$ are presented in Table 1 and compared with the results presented in the cited papers.

**TABLE 1.** Maximum vertical displacement [mm]

| L/H | 20 | 10 | 5 | 4 |
|---|---|---|---|---|
| **Paper [4]** | 7.5599 | 0.7172 | 0.0665 | 0.0296 |
| **Paper [28]** | 7.6204 | 0.7315 | 0.0700 | 0.0318 |
| **Paper [29]** | 7.5590 | 0.7169 | 0.0665 | 0.0296 |
| **This paper** | 7.5599 | 0.7172 | 0.0665 | 0.0296 |

## 5. DISCUSSIONS

The obtained results indicate the changes in stiffness within the range of $\langle 0,\ 5 \cdot 10^8 \rangle$ Pa have a significant impact on the displacements $w(x)$ of the two-layer system. If the difference in the displacements calculated for $k_s = 0$ Pa and $k_s = 10^{14}$Pa is taken as a comparative value, then the changes in these displacements when $k_s \in \langle 0,\ 5 \cdot 10^8 \rangle$Pa are within the range of 93.55%. For the value $k_s \in \langle 5 \cdot 10^8, 10^{14} \rangle$ Pa, these changes are within the range of 6.45%. This result allows for a rational estimation of the limit of the strengthening of the connection between layers, above which such a reinforcement causes only minor strength “effects”.

It is also worth paying attention to the graphs of the normal stresses calculated for $k_s = 10^{14}$Pa (Fig. 8). In the case of such a high stiffness of the connection, the system should "behave" like a monolithic beam. For such a beam, when $E_1 = E_2$, the normal stress diagrams in a given cross-section change linearly. The linear course obtained in this paper for $k_s = 10^{14}$Pa verifies the correctness of the formulated two-layer beam model.

The correctness of the obtained results is also confirmed by the obtained (and widely known) diarams of the internal forces (see Fig.7).

The primary confirmation of the correctness of the derived solution is the consistency of the results obtained in Example 2 with the results presented by other authors (see Tab. 1). Particular attention should be paid to the full consistency of the results (given the adopted accuracy of the presentation of the results) with the exact results obtained by analytical methods for the Euler model (see [19]).

## 6. CONCLUSIONS

In this paper, the authors focused on the analysis of the main beams of a layered system. They omitted the "precise" analysis of an "element" or layer connecting these elements. It was assumed that the interaction is of a tangential nature, and is proportional to the value of the "slip" between the layers. As already mentioned, such a general approach allows the detailed considerations concerning the description of the connection between layers to be omitted, in turn making them more general.

The consequence of this approach is an approximate description of this interaction. The basic goal of the research, i.e. developing an effective mathematical algorithm for solving and analyzing such problems, has been achieved according to the authors. The applied method, which uses finite Fourier transforms, allows for the exact solution of equations that describe such problems.

The results clearly illustrate the enlargement of displacements and the mutual slippage between layers due to the reduction of the shear stiffness $k_s$- of the connection between the two layers. The obtained dependence is nonlinear (see Fig. 3). The results also show that the share of bending strains in relation to axial strains in generating the total bending moment depends on the value of the stiffness parameter $k_s$, and it decreases with an increase of its value (see Fig. 5). The analogous conclusion concerns the influence of bending deformability in relation to the share of the tangential force $V_s$ on the values of the resultant shear force (see Fig. 6).

The goal of this research was to better understand the influence of interlayer connections on the strength and reliability of sandwich beams, in turn providing valuable design guidelines for engineers working with composite structures.

The methodology involves the application of mathematical analysis methods to evaluate the effects of interlayer stiffness on a beam's performance. The correctness of the derived solutions was confirmed by: (1) the diagrams of the internal forces; (2) the behaviour of a system that has high stiffness of the connection and which acts like a monolithic beam; (3) the full consistency of the results with the exact results obtained by analytical methods.

## 7. RESEARCH PERSPECTIVE

In this study, only the case of a simply supported beam was analyzed, as this is the most common model of any civil engineering structure or any structural element of a building. Future research by the authors will focus on beams with other types of supports that are commonly encountered in practice, such as clamped–pinned, clamped–free, and others. For some static schemes, to reduce the number of additional (i.e., non-identically satisfied) equations that define the boundary conditions, it will be necessary to apply a cosine series for the approximation. Particularly interesting results are expected in static schemes where both ends of the beam/beams are constrained from moving along the beam's axis. Furthermore, a subsequent stage will involve solving dynamic problems for this type of beam using the described method. These investigations will further enhance the understanding of interlayer connections, and also their impact on the structural behaviour of layered beams.

## APPENDIX A

If we develop the function $f(x)$ in the interval $< 0, L >$ into a sine series

$$\begin{cases} f(x) = \sum_{n=1}^{\infty} f_n \sin \alpha_n x, \\ f_n = \frac{2}{L}\int_0^L f(x) \sin \alpha_n x dx, \end{cases} \quad \alpha_n = \frac{n\pi}{L}. \tag{A1}$$

This function's derivatives are defined by the following formulas:

$$\begin{cases} f'(x) = {\sum_{n=0}^{\infty}}' f_n' \cos \alpha_n x\,, \\ f''(x) = \sum_{n=1}^{\infty} f_n'' \sin \alpha_n x\,, \\ f^{(4)}(x) = \sum_{n=1}^{\infty} f_n^{(4)} \sin \alpha_n x\,, \end{cases} \tag{A2}$$

where $n = 0, 1, 2, 3, \ldots$ and

$$\sum_{n=0}^{\infty}{}' a_n = \frac{1}{2}a + \sum_{n=1}^{\infty} a_n,$$

$$\begin{cases} f_n' = \frac{2}{L}[(-1)^n f(L) - f(0)] + \alpha_n f_n, \\ f_n'' = -\frac{2}{L}\alpha_n [(-1)^n f(L) - f(0)] - \alpha_n^2 f_n \\ f_n^{(4)} = \frac{2}{L}\alpha_n^3 [(-1)^n f(L) - f(0)] - \\ -\frac{2}{L}\alpha_n [(-1) f''(L) - f''(0)] + \alpha_n^4 f_n \end{cases} \tag{A3}$$

If we expand the function $f(x)$ in the interval $< 0, L >$ into a cosine series

$$\begin{cases} f(x) = {\sum_{n=0}^{\infty}}' f_n \cos \alpha_n x\,, \\ f_n = \frac{2}{L}\int_0^L f(x) \cos \alpha_n x dx\,. \end{cases} \tag{A4}$$

This function's derivatives are defined by the following formulas:

$$f''(x) = {\sum_{n=0}^{\infty}}' f_n'' \cos \alpha_n x, \tag{A5}$$

where for $n = 0, 1, 2, 3, \ldots$:

$$\begin{cases} f_n' = -\alpha_n f_n, \\ f_n'' = \frac{2}{L}[(-1)^n f'(L) - f'(0)] - \alpha_n^2 f_n. \end{cases} \tag{A6}$$

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