jet INTL JOURNAL OF ELECTRONICS AND TELECOMMUNICATIONS, 2025, VOL. 71, NO. 2, PP. 537-543
Manuscript received March 3, 2025; revised May 2025. doi: 10.24425/ijet.2025.153602

Quaternion approach to the theory of 2-D causal systems

Kajetana M. Snopek

***Abstract*—The paper is devoted to the signal- and frequency characteristics of 2-D causal systems. Two different approaches to this problem are presented. First, the 2-D complex Fourier transformation of the 2-D causal impulse response $h(x_1, x_2)$ of a 2-D system defines the 2-D complex frequency response $H(f_1, f_2)$. The modulus $|H(f_1, f_2)|$ is the 2-D magnitude response and $\arg H(f_1, f_2)$ is the 2-D phase response of a system. On the other hand, applying the Pei's formula relating the 2-D complex Fourier transform with the 2-D right-sided quaternion Fourier transform we introduce a concept of the quaternion-valued frequency response $H_q(f_1, f_2)$ of a 2-D causal system. We define the 2-D magnitude system response $|H_q(f_1, f_2)|$ and three 2-D phase responses. These concepts constitute an original contribution of this paper. The theoretical aspects are illustrated with examples of magnitude and phase responses of a causal 2-D analog low-pass filter.**

***Keywords*—causal/anti-causal signal; causal system; Gabor's analytic signal; quaternion analytic signal; quaternion Fourier transform; dual symmetry; Pei's formula; impulse response; frequency response**

I. Introduction

THE theory of 1-D causal and anti-causal signals has been described in many books and papers (e.g. in [1-3]). It is known that in the 1-D analog and digital filter theory, causality of a system means its physical realizability and is equivalent to causality of its impulse response [4-9]. In 1-D analog and digital filter theory, the Fourier-, one-sided Laplace- and Z-transformations are applied [10]. On the other hand, we notice scarcity of publications concerning the 2-D filter theory. The most significant is the monograph [11] from 1985 of Kaczorek in which the theory of 2-D analog and digital linear systems is presented and perspectives of their practical usage are discussed. In [12], a hardware realization of a 2-D analog filter has been proposed and in [13] applied for real time video signal processing. In the conference paper [14], the authors presented some implications of the 2-D analog filtering on circuit theory. In [15], the problem of 2-D causal system identification is discussed. To our knowledge, there are no publications treating the topic of 2-D causal systems from the point of view of complex- and quaternion-valued functions. This paper tries to "fill this gap" and is the result of the author's long-term research on 2-D complex and hypercomplex analytic signals presented in detail in [2].

Kajetana M. Snopek is with the Warsaw University of Technology, Faculty of Electronics and Information Technology, Institute of Radioelectronics and Multimedia Technology, Poland (e-mail: kajetana.snopek@pw.edu.pl).

The organization of the paper is as follows. In Section II, we define causal and anti-causal signals in 1-D and 2-D. The Section III is devoted to the complex Fourier transform of 1-D and 2-D causal and anti-causal signals. Then in Section IV, we derive the spectrum of a 2-D causal signal in the quaternion form. To our knowledge, this result has not yet been published. Last Section VI is devoted to 2-D causal systems. We focus on their frequency representation and present formulas of magnitude- and phase responses. Our intention is to show an equivalence of both approaches (complex and quaternionic). The paper is illustrated with plots of impulse and frequency responses of a 2-D causal low-pass filter.

II. Causality in 1-D and 2-D

Let us start with formal definitions of causal and anti-causal signals in 1-D and 2-D domains. In Section IV these concepts will be used in definitions of causal impulse responses of 1-D and 2-D systems.

A. Causal and anti-causal signals in 1-D

Let us consider a 1-D *causal* signal $u(t), t \in \mathbb{R}$ defined as follows

$$u(t) = u_e(t)[1 + \operatorname{sgn} t] \tag{1}$$

where $u_e(t)$ is the *even component* of $u(t)$ defined as

$$u_e(t) = 0.5[u(t) + u(-t)] \tag{2}$$

and $\operatorname{sgn} t$ is the 1-D *signum distribution* given by

$$\operatorname{sgn} t = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t = 0 \\ -1 & \text{for } t < 0 \end{cases} \tag{3}$$

It is easy to notice that the support of the signal (1) is limited to the right half-axis $t \geq 0$, that is $u(t) = 0$ for $t < 0$. Analogously, we can define a 1-D *anti-causal* signal with the left-sided support, i.e., $u(t) = 0$ for $t > 0$.

B. Causal and anti-causal signals in 2-D

Let us generalize notions of causality and anti-causality for two dimensions. The 2-D signal plane (x_1, x_2) can be divided into four quadrants labelled with 1, 2, 3 and 4 respectively (see Figure 1). This numeration, originally proposed by Hahn in [16], differs from that commonly applied in analytic geometry and is compatible with the binary notation presented in Table I where 0 corresponds respectively to the sign " $+$ " and 1 to the sign " $-$ " of $x_i, i = 1,2$ using the reversed order. The advantage of this labelling becomes evident in higher dimensions, i.e., for $n \geq 3$. Similar notation of quadrants will be applied in this paper for all quadrants of the 2-D frequency plane.

The 2-D signal $u(x_1, x_2), (x_1, x_2) \in \mathbb{R}^2$ is said to be *causal* if and only if its support is limited to the quadrant 1 (see Figure 1). The 2-D signals with supports in quadrants 2, 3 and 4 are *anti-causal*. To distinguish every case, we will respectively call them *anti-causal(2)*, *anti-causal(3)* and *anti-causal(4)*.

TABLE I
LABELLING OF QUADRANTS IN THE 2-D SIGNAL DOMAIN

Quadrant label	Signs of x_1	Signs of x_2	Binary notation [a]	Reversed order
1	+	+	00	00
2	−	+	01	10
3	+	−	10	01
4	−	−	11	11

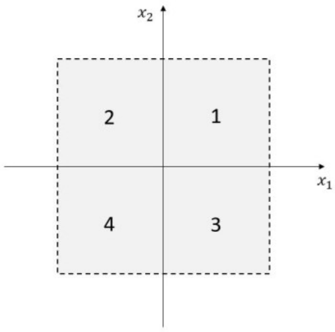

Fig. 1. The 2-D signal plane (x_1, x_2) divided into four quadrants labelled with 1,2,3 and 4.

In analogy to the 1-D case presented in subsection IIA., the 2-D causal signal can be represented as the product

$$u(x_1, x_2) = u_{ee}(x_1, x_2)[1 + \operatorname{sgn} x_1][1 + \operatorname{sgn} x_2] \quad (4)$$

where $u_{ee}(x_1, x_2)$ is the *even-even* component of $u(x_1, x_2)$ defined as

$$u_{ee}(x_1, x_2) = 0.25[u(x_1, x_2) + u(-x_1, x_2) + u(x_1, -x_2) + u(-x_1, -x_2)] \quad (5)$$

and $\operatorname{sgn} x_i, i = 1,2$, are defined similarly as in (3).

III. COMPLEX FOURIER TRANSFORM OF CAUSAL SIGNALS

This section concerns the Fourier transform (FT) of 1-D and 2-D causal signals given by (1) and (4) respectively.

A. *Fourier spectrum of 1-D causal signals*

Using linearity and multiplication property of the 1-D complex FT, we get the spectrum $U(f), f \in \mathbb{R}$ of the signal (1) in the following form

$$U(f) = U_e(f) + U_e(f) * \frac{1}{i\pi f} = U_e(f) - U_e(f) * \frac{i}{\pi f} \quad (6)$$

where $U_e(f)$ is the spectrum of the even component $u_e(t)$, "$*$" denotes a convolution in the frequency domain and $\frac{1}{i\pi f}$ is the FT of $\operatorname{sgn} t$. We can notice that $U(f)$ given by (6) is a conjugate *analytic function* (in the Gabor's sense [17]) since its imaginary part $U_e(f) * \frac{1}{\pi f}$ is the Hilbert transform $\mathcal{H}\{\cdot\}$ of $U_e(f)$, i.e.,

$$\mathcal{H}\{U_e(f)\} = U_e(f) * \frac{1}{\pi f} \quad (7)$$

Finally, we can express (6) as

$$U(f) = U_e(f) - i \cdot \mathcal{H}\{U_e(f)\} \quad (8)$$

In a very similar way as above, we can show that 1-D anti-causal signals have analytic spectra of the form $U_e(f) + i \cdot \mathcal{H}\{U_e(f)\}$, which is the conjugate of (8).

Let us remark that the formula (8) is in accordance with the dual symmetry property of the 1-D FT. It states that if the signal $u(t)$ has the spectrum $U(f)$ then the signal $U(t)$ has the FT $u(-f)$. In the considered case, we can write that the signal $U(t) = U_e(t) - i \cdot \mathcal{H}\{U_e(t)\}$ (conjugate analytic in the Gabor's sense) has the Fourier spectrum $u(f) = u_e(-f)[1 + \operatorname{sgn}(-f)] = u_e(f)[1 - \operatorname{sgn} f]$ (left-sided anti-causal). Then $U^*(t) = U_e(t) + i \cdot \mathcal{H}\{U_e(t)\}$ (analytic) has the spectrum $u^*(-f) = u_e(f)[1 + \operatorname{sgn} f]$ (right-sided causal).

Further, expressing the component $U_e(f)$ in (6) as a convolution with the Dirac delta distribution., i.e., $U_e(f) = U_e(f) * \delta(f)$, and using distributivity of convolution over addition, we can rewrite (6) in the form

$$U(f) = U_e(f) * \left[\delta(f) - \frac{i}{\pi f}\right] \quad (9)$$

or equivalently as

$$U(f) = U_e * \Psi_\delta^*(f) \quad (10)$$

where $\Psi_\delta^*(f)$ is the conjugate of the 1-D *complex delta distribution* $\Psi_\delta(f) = \delta(f) + \frac{i}{\pi f}$ defined by Hahn in [18] and presented in detail in [1]. $\Psi_\delta(f)$ is also an analytic function of frequency (in the Gabor's sense).

Let us point out some facts concerning causality in 1-D:

- a 1-D causal signal has a conjugate analytic spectrum;
- a 1-D anti-causal signal has an analytic spectrum.
- a 1-D analytic signal has a causal spectrum;
- a 1-D conjugate analytic signal has an anti-causal spectrum.

B. *Fourier spectrum of 2-D causal signals*

Let us now consider the 2-D causal signal given by (4). In order to derive its 2-D FT, we apply multiplication-to-convolution property of the 2-D Fourier transformation. The 2-D Fourier spectrum $U(f_1, f_2)$ of the signal (4) has the form of a double convolution of the spectrum $U_{ee}(f_1, f_2)$ of the even-even component (5) with the 2-D Fourier transform of $(1 + \operatorname{sgn} x_1 + \operatorname{sgn} x_2 + \operatorname{sgn} x_1 \cdot \operatorname{sgn} x_2)$, that is

$$U(f_1, f_2) = U_{ee}(f_1, f_2) ** \mathcal{F}\{1 + \operatorname{sgn} x_1 + \operatorname{sgn} x_2 + \operatorname{sgn} x_1 \cdot \operatorname{sgn} x_2\} \quad (11)$$

where $\mathcal{F}\{1 + \operatorname{sgn} x_1 + \operatorname{sgn} x_2 + \operatorname{sgn} x_1 \cdot \operatorname{sgn} x_2\} = \delta(f_1, f_2) + \frac{\delta(f_2)}{i\pi f_1} + \frac{\delta(f_1)}{i\pi f_2} - \frac{1}{\pi^2 f_1 f_2}$. Then,

$$U(f_1,f_2) = U_{ee}(f_1,f_2) ** \left[\delta(f_1,f_2) - i\frac{\delta(f_2)}{\pi f_1} - i\frac{\delta(f_1)}{\pi f_2} - \frac{1}{\pi^2 f_1 f_2}\right] \quad (12)$$

We can notice that

$$U_{ee}(f_1,f_2) ** \delta(f_1,f_2) = U_{ee}(f_1,f_2) \quad (13)$$

$$U_{ee}(f_1,f_2) ** i\frac{\delta(f_2)}{\pi f_1} = \mathcal{H}_1\{U_{ee}(f_1,f_2)\} = \mathcal{H}_1\{U_{ee}\} \quad (14)$$

$$U_{ee}(f_1,f_2) ** i\frac{\delta(f_1)}{\pi f_2} = \mathcal{H}_2\{U_{ee}(f_1,f_2)\} = \mathcal{H}_2\{U_{ee}\} \quad (15)$$

$$U_{ee}(f_1,f_2) ** \frac{1}{\pi^2 f_1 f_2} = \mathcal{H}\{U_{ee}(f_1,f_2)\} = \mathcal{H}\{U_{ee}\} \quad (16)$$

where $\mathcal{H}_i\{U_{ee}\}$ are 2-D partial Hilbert transforms w.r.t. to f_i of $U_{ee}(f_1,f_2)$ and $\mathcal{H}\{U_{ee}\}$ is the 2-D total Hilbert transform of $U_{ee}(f_1,f_2)$. The complete theory concerning partial and total Hilbert transforms is presented in [1], [2], [18]. Finally, (12) gets the form

$$U(f_1,f_2) = U_{ee} - \mathcal{H}\{U_{ee}\} - i[\mathcal{H}_1\{U_{ee}\} + \mathcal{H}_2\{U_{ee}\}] \quad (17)$$

where we applied the simplified notation $\{U_{ee}\}$ for $\{U_{ee}(f_1,f_2)\}$.

In Table II we collected causal (the first row) and anti-causal 2-D signals and their spectra (i denotes the corresponding quadrant of the signal support). We can notice that the spectrum of the 2-D causal signal defined in (4) is the conjugate of the spectrum of an anti-causal(4) signal (the 4th row of Table II) and the spectra of other anti-causal 2-D signals (with supports in quadrants 2. and 3.) also form a conjugate pair.

TABLE II
CAUSAL/ANTI-CAUSAL SIGNALS AND THEIR SPECTRA IN 2-D

i	Signal $u(x_1,x_2)$	Spectrum $U(f_1,f_2)$
1	$u_{ee}(1+\mathrm{sgn}x_1)(1+\mathrm{sgn}x_2)$ *causal*	$U_{ee} - \mathcal{H}\{\cdot\} - i[\mathcal{H}_1\{\cdot\} + \mathcal{H}_2\{\cdot\}]$
2	$u_{ee}(1-\mathrm{sgn}x_1)(1+\mathrm{sgn}x_2)$ *anti-causal(2)*	$U_{\mathrm{ee}} + \mathcal{H}\{\cdot\} + i[\mathcal{H}_1\{\cdot\} - \mathcal{H}_2\{\cdot\}]$
3	$u_{ee}(1+\mathrm{sgn}x_1)(1-\mathrm{sgn}x_2)$ *anti-causal(3)*	$U_{\mathrm{ee}} + \mathcal{H}\{\cdot\} - i[\mathcal{H}_1\{\cdot\} - \mathcal{H}_2\{\cdot\}]$
4	$u_{ee}(1-\mathrm{sgn}x_1)(1-\mathrm{sgn}x_2)$ *anti-causal(4)*	$U_{\mathrm{ee}} - \mathcal{H}\{\cdot\} + i[\mathcal{H}_1\{\cdot\} + \mathcal{H}_2\{\cdot\}]$

Here once again, we can refer to the dual symmetry property of the 2-D Fourier transformation. According to it, if a signal $u(x_1,x_2)$ has the 2-D FT $U(f_1,f_2)$, then the signal $U(x_1,x_2)$ has the spectrum $u(-f_1,-f_2)$. Let us consider the Fourier pair from the first row of Table II. We can state that the signal $U_{ee}(x_1,x_2) - \mathcal{H}\{U_{ee}(x_1,x_2)\} - i[\mathcal{H}_1\{U_{ee}(x_1,x_2)\} + \mathcal{H}_2\{U_{ee}(x_1,x_2)\}]$ has an anti-causal(4) spectrum $u_{ee}(-f_1,-f_2)(1-\mathrm{sgn}f_1)(1-\mathrm{sgn}f_2) = u_{ee}(f_1,f_2)(1-\mathrm{sgn}f_1)(1-\mathrm{sgn}f_2)$. In an analogous way, we can derive next Fourier pairs.

The spectrum given by (17) is *analytic* in the sense of the Hahn's theory of single-quadrant analytic functions presented in [16]. According to this theory, a 2-D function is *analytic* if its 2-D FT has a single-quadrant support. Hahn defined four different analytic signals denoted with $\psi_i(x_1,x_2)$ (i denotes the corresponding quadrant of the spectrum support). All these four signals collected in Table III form two conjugate pairs, namely $\psi_1 = \psi_4^*$ and $\psi_2 = \psi_3^*$.

Let us summarize all results concerning spectrum properties of causal/anti-causal signals in 2-D:

- the 2-D causal signal (4) has an analytic spectrum of the form corresponding to ψ_4 (that is a conjugate of the spectrum of the anti-causal(4) signal)
- the 2-D anti-causal(3) signal has an analytic spectrum of the form corresponding to ψ_3 (that is a conjugate of the spectrum of the anti-causal(2) signal).

TABLE III
HAHN'S ANALYTIC SIGNALS IN 2-D

i	$\psi_i(x_1,x_2)$
1	$u - \mathcal{H}\{u\} + i[\mathcal{H}_1\{u\} + \mathcal{H}_2\{u\}]$
2	$u + \mathcal{H}\{u\} - i[\mathcal{H}_1\{u\} - \mathcal{H}_2\{u\}]$
3	$u + \mathcal{H}\{u\} + i[\mathcal{H}_1\{u\} - \mathcal{H}_2\{u\}]$
4	$u - \mathcal{H}\{u\} - i[\mathcal{H}_1\{u\} + \mathcal{H}_2\{u\}]$

IV. QUATERNION APPROACH TO THE FREQUENCY REPRESENTATION OF 2-D CAUSAL SIGNALS

In this section, we develop the formula of the quaternion Fourier spectrum of the 2-D causal signal (4). Let us recall the definition of the 2-D Fourier transform $U(f_1,f_2)$ of a 2-D real-valued signal $u(x_1,x_2)$:

$$U(f_1,f_2) = \iint u(x_1,x_2)e^{-2\pi i(f_1x_1+f_2x_2)}dx_1dx_2 \quad (18)$$

Then, the right-sided quaternion Fourier transform $U_q(f_1,f_2)$ of $u(x_1,x_2)$, defined by Ell in [19], has the form

$$U_q(f_1,f_2) = \iint u(x_1,x_2)e^{-2\pi i f_1 x_1}e^{-2\pi j f_2 x_2}dx_1dx_2 \quad (19)$$

where j is the second (beside i and k) quaternion imaginary unit. The details concerning quaternions, multidimensional quaternion-valued signals and their spectra are to be found in [2], [20].

The formula relating (18) and (19), proposed by Pei in [21], is

$$U_q(f_1,f_2) = U(f_1,f_2)\frac{1-k}{2} + U(f_1,-f_2)\frac{1+k}{2} \quad (20)$$

Let us now come back to the formula (17) of the 2-D FT of a 2-D causal signal. We can notice that

$$U_{ee}(f_1,-f_2) = U_{ee}(f_1,f_2) \quad (21)$$

$$\mathcal{H}\{U_{ee}(f_1,-f_2)\} = U_{\mathrm{ee}}(f_1,-f_2) ** \frac{1}{\pi^2 f_1(-f_2)} = -\mathcal{H}\{U_{\mathrm{ee}}(f_1,f_2)\} \quad (22)$$

$$\mathcal{H}_1\{U_{ee}(f_1,-f_2)\} = U_{ee}(f_1,-f_2) ** \frac{\delta(-f_2)}{\pi f_1} = \mathcal{H}_1\{U_{ee}(f_1,f_2)\} \quad (23)$$

$$\mathcal{H}_2\{U_{ee}(f_1,-f_2)\} = U_{ee}(f_1,-f_2) ** \frac{\delta(f_1)}{\pi(-f_2)} = -\mathcal{H}_2\{U_{ee}(f_1,f_2)\} \quad (24)$$

Introducing (21)-(24) into (17), the spectrum $U(f_1,-f_2)$ gets the following form

$$U(f_1,-f_2) = U_{ee} + \mathcal{H}\{U_{ee}\} - i[\mathcal{H}_1\{U_{ee}\} - \mathcal{H}_2\{U_{ee}\}] \quad (25)$$

where, similarly as in (17), we used the simplified notation $\{U_{ee}\}$ instead of $\{U_{ee}(f_1,f_2)\}$.

Let us now introduce the expressions (14) and (25) into the Pei's formula (20). Then, we get the 2-D quaternion Fourier spectrum of a 2-D causal signal $u(x_1,x_2)$ as follows

$$\begin{aligned} &U_q(f_1,f_2) \\ &= [U_{ee} - \mathcal{H}\{U_{ee}\} - i(\mathcal{H}_1\{U_{ee}\} + \mathcal{H}_2\{U_{ee}\})]\frac{1-k}{2} \\ &+[U_{ee} + \mathcal{H}\{U_{ee}\} - i(\mathcal{H}_1\{U_{ee}\} - \mathcal{H}_2\{U_{ee}\})]\frac{1+k}{2} \end{aligned} \quad (26)$$

After simple calculations involving multiplication rules working in the algebra of quaternions: $i \cdot k = -j$ (see[2]) we obtain the 2-D quaternion Fourier spectrum of the 2-D causal signal (4) in the following form

$$U_q(f_1,f_2) = U_{ee} - i \cdot \mathcal{H}_1\{U_{ee}\} - j \cdot \mathcal{H}_2\{U_{ee}\} + k \cdot \mathcal{H}\{U_{ee}\} \quad (27)$$

We can notice that the quaternion Fourier spectrum (27) is a 2-D quaternion-valued function in which we recognize the same components as in (17). In Table IV, we collected quaternion spectra of causal and anti-causal 2-D signals from Table II with supports in successive quadrants of the signal plane (for simplicity $\{\cdot\}$ denotes $\{U_{\mathrm{ee}}\}$). We observe that they do not form conjugate pairs and it is not surprising having in mind the properties of the algebra of quaternions (see e.g. [2]). However, all quaternion Fourier spectra from Table IV are expressed by formulas corresponding to *quaternion analytic functions* introduced by Bülow and Sommer in [22], [23] and, for comparison, displayed in Table V. The theory of these functions is also presented in detail in [2] and [20].

TABLE IV
CAUSAL/ANTI-CAUSAL SIGNALS AND THEIR QUATERNION SPECTRA IN 2-D

i	Signal $u(x_1,x_2)$	Spectrum $U_q(f_1,f_2)$
1	$u_{ee}(1+\mathrm{sgn}x_1)(1+\mathrm{sgn}x_2)$ *causal*	$U_{ee} - i\cdot\mathcal{H}_1\{\cdot\} - j\cdot\mathcal{H}_2\{\cdot\} + k\cdot\mathcal{H}\{\cdot\}$
2	$u_{ee}(1-\mathrm{sgn}x_1)(1+\mathrm{sgn}x_2)$ *anti-causal(2)*	$U_{ee} + i\cdot\mathcal{H}_1\{\cdot\} - j\cdot\mathcal{H}_2\{\cdot\} - k\cdot\mathcal{H}\{\cdot\}$
3	$u_{ee}(1+\mathrm{sgn}x_1)(1-\mathrm{sgn}x_2)$ *anti-causal(3)*	$U_{ee} - i\cdot\mathcal{H}_1\{\cdot\} + j\cdot\mathcal{H}_2\{\cdot\} - k\cdot\mathcal{H}\{\cdot\}$
4	$u_{ee}(1-\mathrm{sgn}x_1)(1-\mathrm{sgn}x_2)$ *anti-causal(4)*	$U_{ee} + i\cdot\mathcal{H}_1\{\cdot\} + j\cdot\mathcal{H}_2\{\cdot\} + k\cdot\mathcal{H}\{\cdot\}$

TABLE V
QUATERNION ANALYTIC SIGNALS IN 2-D

i	$\psi_q(x_1,x_2)$
1	$u + i\cdot\mathcal{H}_1\{u\} + j\cdot\mathcal{H}_2\{u\} + k\cdot\mathcal{H}\{u\}$
2	$u - i\cdot\mathcal{H}_1\{u\} + j\cdot\mathcal{H}_2\{u\} - k\cdot\mathcal{H}\{u\}$
3	$u + i\cdot\mathcal{H}_1\{u\} - j\cdot\mathcal{H}_2\{u\} - k\cdot\mathcal{H}\{u\}$
4	$u - i\cdot\mathcal{H}_1\{u\} - j\cdot\mathcal{H}_2\{u\} + k\cdot\mathcal{H}\{u\}$

Basing on all formulas presented in Tables IV and V, we can conclude that:

- the 2-D causal signal (4) has the quaternion spectrum described by the formula corresponding to the quaternion analytic function with the spectrum support in the quadrant 4. of the frequency plane
- the spectrum of the 2-D anti-causal(2) signal has the form corresponding to the quaternion analytic signal with the spectrum support in the quadrant 3
- the spectrum of the 2-D anti-causal(3) signal has the form corresponding to the quaternion analytic signal with the spectrum support in the quadrant 2
- the spectrum of the 2-D anti-causal(4) signal has the from corresponding to the quaternion analytic signal with the spectrum support in the quadrant 1.

V. CAUSALITY OF 2-D LINEAR TIME-INVARIANT SYSTEMS

The 1-D causal systems are subject of many publications, e.g. [4]-[9], and we do not intend to recall this theory here. We focus on 2-D causal (linear time-invariant) analog systems. Basing on theory presented in previous sections, it will be easy to "translate" all notions into the "language" of the system theory.

We say that a 2-D analog system is *causal* if and only if its 2-D impulse response $h(x_1,x_2)$ is a causal function of the form given by (4), i.e.,

$$h(x_1,x_2) = h_{ee}(x_1,x_2)[1+\mathrm{sgn}\,x_1][1+\mathrm{sgn}\,x_2] \quad (28)$$

where $h_{ee}(x_1,x_2)$ is the *even-even* component of $h(x_1,x_2)$ defined by (5).

A. 2-D complex frequency response of a 2-D causal system

The 2-D complex Fourier transform of the 2-D causal impulse response defines the *2-D complex frequency response* $H(f_1,f_2)$ of a 2-D system (compare with (17)), i.e.,

$$H(f_1,f_2) = H_{ee} - \mathcal{H}\{H_{ee}\} - i[\mathcal{H}_1\{H_{ee}\} + \mathcal{H}_2\{H_{ee}\}] \quad (29)$$

The 2-D *magnitude response* $A(f_1,f_2)$ of a 2-D causal system is the absolute value of (29) and has the following form

$$\begin{aligned} &A(f_1,f_2) = |H(f_1,f_2)| \\ &= \sqrt{(H_{ee} - \mathcal{H}\{H_{ee}\})^2 + (\mathcal{H}_1\{H_{ee}\} + \mathcal{H}_2\{H_{ee}\})^2} \\ &= \sqrt{H_{ee}^2 + \mathcal{H}^2\{\cdot\} + \mathcal{H}_1^2\{\cdot\} + \mathcal{H}_2^2\{\cdot\} + 2[\mathcal{H}_1\{\cdot\}\mathcal{H}_2\{\cdot\} - H_{ee}\mathcal{H}\{\cdot\}]} \end{aligned} \quad (30)$$

where, for simplicity, $\{\cdot\}$ denotes $\{H_{ee}\}$. Notice that in case of a *separable* 2-D impulse response, i.e., $h(x_1,x_2) = h_1(x_1)h_2(x_2)$, we have $2[\mathcal{H}_1\{\cdot\}\mathcal{H}_2\{\cdot\} - H_{ee}\mathcal{H}\{\cdot\}] = 0$ (see e.g. [1], [16]).

Then, the 2-D *phase response* $\varphi(f_1,f_2)$ of a 2-D causal system is given by

$$\varphi(f_1,f_2) = -\mathrm{atan2}\left(\frac{\mathcal{H}_1\{H_{ee}\}+\mathcal{H}_2\{H_{ee}\}}{H_{ee}-\mathcal{H}\{H_{ee}\}}\right) \quad (31)$$

B. 2-D quaternion frequency response of a 2-D causal system

Applying the quaternion approach presented in Section IV, we can define the 2-D *quaternion frequency response* $H_q(f_1,f_2)$ of a 2-D analog causal system in the form

$$H_q(f_1, f_2) = H_{ee} - i \cdot \mathcal{H}_1\{H_{ee}\} - j \cdot \mathcal{H}_2\{H_{ee}\} + k \cdot \mathcal{H}\{H_{ee}\} \quad (32)$$

Let us now define the *magnitude response* $A_q(f_1, f_2)$ as the absolute value of the 2-D quaternion frequency response (32), i.e.,

$$A_q(f_1, f_2) = |H_q(f_1, f_2)| = \sqrt{H_{ee}^2 + \mathcal{H}_1^2\{H_{ee}\} + \mathcal{H}_2^2\{H_{ee}\} + \mathcal{H}^2\{H_{ee}\}} \quad (33)$$

In this case, it is possible to define three *phase responses* $\varphi_i(f_1, f_2), i = 1,2,3$ as generalizations of three phase (Euler) angles of a quaternion [22] as follows

$$\varphi_1(f_1, f_2) = \frac{1}{2}\operatorname{atan2}\left(\frac{2(H_{ee}\mathcal{H}_1\{\cdot\} + \mathcal{H}\{\cdot\}\mathcal{H}_2\{\cdot\})}{H_{ee}^2 - \mathcal{H}_1^2\{\cdot\} + \mathcal{H}_2^2\{\cdot\} - \mathcal{H}^2\{\cdot\}}\right) \quad (34)$$

$$\varphi_2(f_1, f_2) = \frac{1}{2}\operatorname{atan2}\left(\frac{2(H_{ee}\mathcal{H}_2\{\cdot\} + \mathcal{H}\{\cdot\}\mathcal{H}_1\{\cdot\})}{H_{ee}^2 + \mathcal{H}_1^2\{\cdot\} - \mathcal{H}_2^2\{\cdot\} - \mathcal{H}^2\{\cdot\}}\right) \quad (35)$$

$$\varphi_3(f_1, f_2) = \frac{1}{2}\operatorname{asin}\left(\frac{2(H_{ee}\{\cdot\}\mathcal{H}\{\cdot\} - \mathcal{H}_1\{\cdot\}\mathcal{H}_2\{\cdot\})}{H_{ee}^2 + \mathcal{H}_1^2\{\cdot\} + \mathcal{H}_2^2\{\cdot\} + \mathcal{H}^2\{\cdot\}}\right) \quad (36)$$

Let us illustrate the above notions with the example of a 2-D low-pass causal filter with the 2-D separable causal impulse response given by

$$h(x_1, x_2) = e^{-(\alpha x_1 + \beta x_2)} \cdot \mathbf{1}(x_1, x_2),\ \alpha, \beta > 0 \quad (37)$$

where $\mathbf{1}(x_1, x_2) = \mathbf{1}(x_1) \cdot \mathbf{1}(x_2)$ is the 2-D step function and $\mathbf{1}(x_i) = \frac{1}{2}(1 + \operatorname{sgn} x_i)$ with $\operatorname{sgn} x_i$ defined in (3). The Figure 2 displays $h(x_1, x_2)$ for $\alpha = \beta = 10^4$ and we can observe that it is evidently causal with the support in the first quadrant of the signal plane (x_1, x_2). Next Figure 3 shows the even-even component of (37). Then, Figures 4-7 respectively show the following components of the frequency response of the considered system: H_{ee} – 2-D Fourier transform of the even-even component of $h(x_1, x_2)$ and its full and partial Hilbert transforms $\mathcal{H}\{H_{ee}\}, \mathcal{H}_1\{H_{ee}\}, \mathcal{H}_2\{H_{ee}\}$. All components are given by the following formulas

$$H_{ee}(f_1, f_2) = \frac{\alpha\beta}{(4\pi^2 f_1^2 + \alpha^2)(4\pi^2 f_2^2 + \beta^2)} \quad (38)$$

$$\mathcal{H}\{H_{ee}(f_1, f_2)\} = \frac{4\pi^2 f_1 f_2}{(4\pi^2 f_1^2 + \alpha^2)(4\pi^2 f_2^2 + \beta^2)} \quad (39)$$

$$\mathcal{H}_1\{H_{ee}(f_1, f_2)\} = \frac{2\pi\beta f_1}{(4\pi^2 f_1^2 + \alpha^2)(4\pi^2 f_2^2 + \beta^2)} \quad (40)$$

$$\mathcal{H}_2\{H_{ee}(f_1, f_2)\} = \frac{2\pi\alpha f_2}{(4\pi^2 f_1^2 + \alpha^2)(4\pi^2 f_2^2 + \beta^2)} \quad (41)$$

In Fig. 4, we can notice the even symmetry of $H_{ee}(f_1, f_2)$ w.r.t. f_1 and f_2. Then, the full Hilbert transform from Fig.5 is an odd function w.r.t. f_1 and f_2. The partial Hilbert transform $\mathcal{H}_1\{H_{ee}(f_1, f_2)\}$ from Fig. 6 is an odd function w.r.t. f_1 and an even function w.r.t. f_2 opposite to $\mathcal{H}_2\{H_{ee}(f_1, f_2)\}$ from Fig. 7. In Figures 8 and 9, we respectively show the real part $H_{ee} - \mathcal{H}\{H_{ee}\}$ and the imaginary part $-[\mathcal{H}_1\{H_{ee}\} + \mathcal{H}_2\{H_{ee}\}]$ of the complex frequency response given by (29).

Since the 2-D impulse response (37) of the considered filter is a separable function, i.e., $h(x_1, x_2) = e^{-\alpha x_1}\mathbf{1}(x_1) \cdot e^{-\beta x_2}\mathbf{1}(x_2)$, the 2-D magnitude response (30) is exactly the same as (33). It is shown in Fig. 10.

Using the formula (31), we have calculated the phase response of the system and visualized the obtained result in Figure 11.

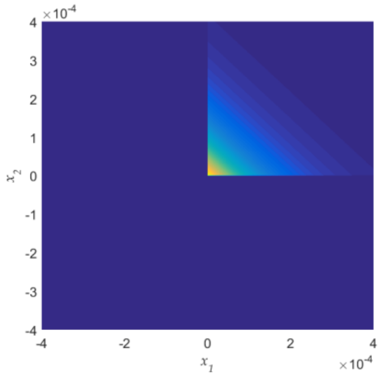

Fig. 2. The 2-D causal impulse response (37); $\alpha = \beta = 10^4$.

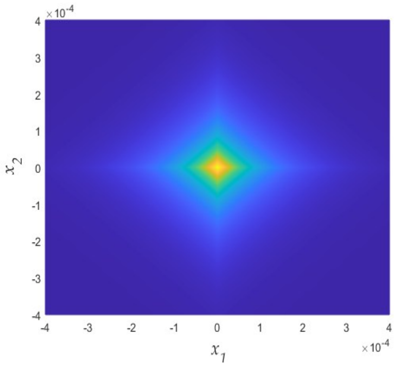

Fig. 3. The even-even component $h_{ee}(x_1, x_2)$ of $h(x_1, x_2)$ from Fig. 2.

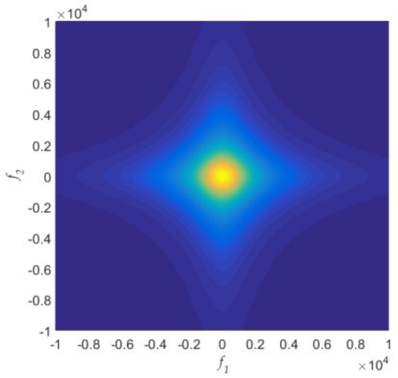

Fig. 4. The 2-D Fourier transform $H_{ee}(f_1, f_2)$ of $h_{ee}(x_1, x_2)$ from Fig. 3.

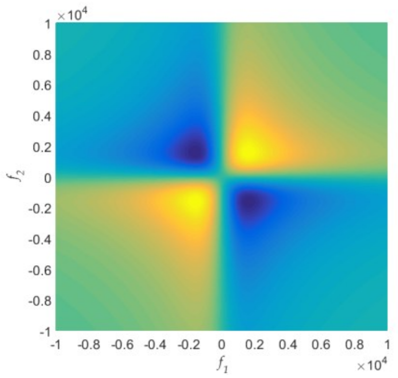

Fig. 5. The full Hilbert transform $\mathcal{H}\{H_{ee}(f_1, f_2)\}$ of H_{ee} from Fig. 4.

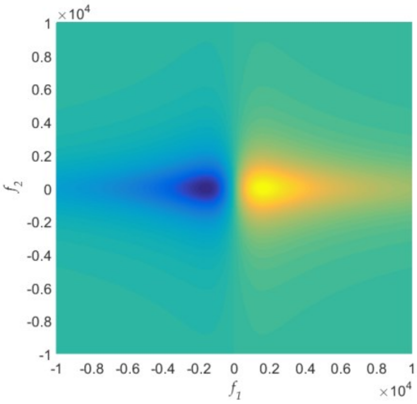

Fig. 6. The partial Hilbert transform $\mathcal{H}_1\{H_{ee}(f_1,f_2)\}$ of H_{ee} from Fig.4.

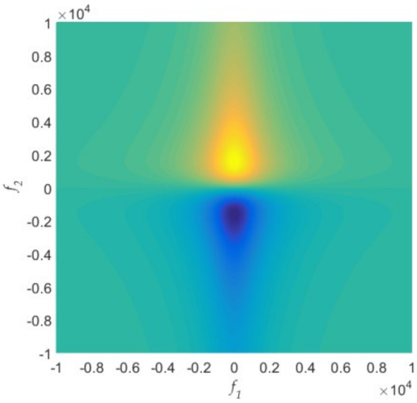

Fig. 7. The partial Hilbert transform $\mathcal{H}_2\{H_{ee}(f_1,f_2)\}$ of H_{ee} from Fig.4.

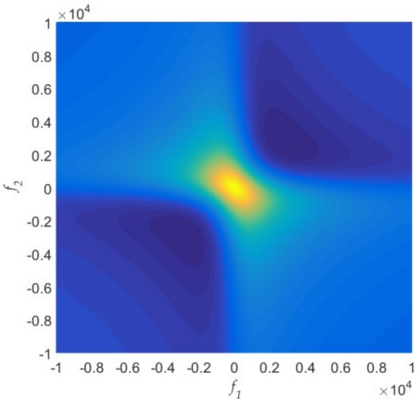

Fig. 8. The real part $H_{ee} - \mathcal{H}\{H_{ee}\}$ of (29) (complex approach)

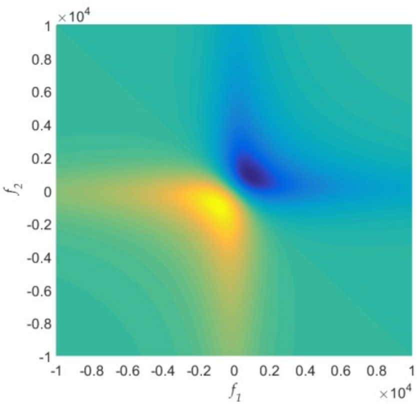

Fig. 9. The imaginary part $-[\mathcal{H}_1\{H_{ee}\} + \mathcal{H}_2\{H_{ee}\}]$ of (29) (complex approach)

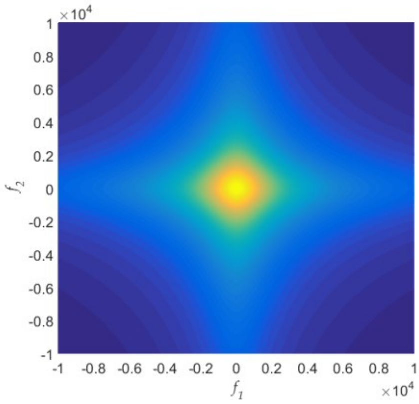

Fig. 10. The 2-D magnitude response $A(f_1,f_2) \equiv A_q(f_1,f_2)$ of the 2-D low-pass filter

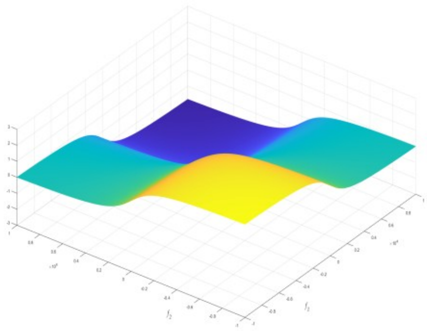

Fig. 11. The 2-D phase response $\varphi(f_1,f_2)$ given by (31) (complex approach)

In next two Figures 12 and 13 we show two non-zero phase responses of the system given by (34)–(35). The phase response $\varphi_3(f_1,f_2)$ given by (36) is zero because the impulse response $h(x_1,x_2)$ is a separable function.

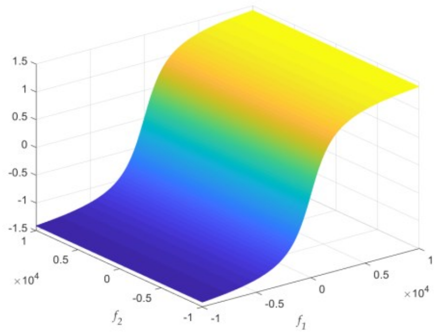

Fig. 12. The 2-D phase response $\varphi_1(f_1,f_2)$ given by (34) (quaternion approach)

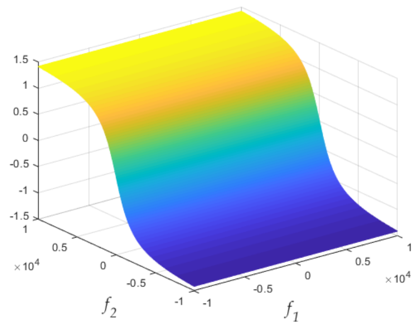

Fig. 13. The 2-D phase response $\varphi_2(f_1,f_2)$ given by (35) (quaternion approach)

In this computer experiment, we have shown that it is possible to study the properties of 2-D causal systems simultaneously in signal and frequency domains using a classical approach involving the 2-D Fourier transformation or a quaternion approach. Our intention was to show the equivalence of both approaches. Evidently, all presented formulas can be easily implemented for other causal 2-D systems.

Conclusion

In this paper, we described two different approaches to the theory of 2-D causal systems. The whole theory was presented as a generalization of the theory of 1-D causal/anti-causal signals. We defined 2-D causal/anti-causal signals and presented formulas describing their complex and quaternion Fourier spectra. In the complex case, we referred to the Hahn's theory of 2-D analytic signals and to the symmetry property of the 2-D Fourier transformation. In the quaternion approach, the symmetry relations between signal- and frequency domains are more complicated but here also we noticed some interesting relations. The most significant is the definition of the 2-D quaternion frequency system response accompanied with the magnitude- and three phase responses of a 2-D causal system.

The paper concerns the 2-D analog signals and systems but implementation of the described theory for a digital case is possible and can be a subject of future research.

References

[1] S. L. Hahn, "Hilbert Transforms in Signal Processing", Artech House Boston|London, ISBN: 9780890068861, 1996.
https://uk.artechhouse.com/Hilbert-Transforms-in-Signal-Processing-P425.aspx

[2] S. L. Hahn, K. M. Snopek, "Complex and Hypercomplex Analytic Signals. Theory and Applications", Artech House Boston|London, ISBN: 9781630811327, 2017.
https://ieeexplore.ieee.org/document/9100928

[3] A. V. Oppenheim, A. S. Willsky, S. H. Nawab, "Signals & Systems", 2nd ed., Pearson Education Limited, ISBN: 9780138147570, 1998.

[4] L. B. Jackson, "Digital Filters and Signal Processing", 2nd ed., Kluwer Academic Publishers, U.S.A., ISBN: 9781475724585, 1986.
https://doi.org/10.1007/978-1-4757-2458-5

[5] P. Denbigh, "System Analysis & Signal Processing", Springer New York, NY, ISBN: 9780201178609, 1995.

[6] R. Vich, "Z Transform Theory and Applications", Springer Dordrecht, ISBN: 9789027719171, 1987.

[7] M. S. Roden, "Analog and Digital Communication Systems", 5th ed., Discovery Press, ISBN: 9780964696976, 2003.

[8] R. L. Allen., D. W. Mills., "Signal Analysis – Time, Frequency, Scale and Structure", IEEE Press, ISBN: 9780471660378, 2004.
https://doi.org/10.1002/047166037X

[9] I. A. Glover, P. M. Grant, "Digital Communications", Prentice Hall Europe, ISBN: 9780273718307, 2009.

[10] A. D. Poularikas (Ed.), "The Transforms and Applications Handbook", 3rd ed., CRC Press, ISBN: 9781315218915, 2018.
https://doi.org/10.1201/9781315218915

[11] T. Kaczorek, "Two-Dimensional Linear Systems", Springer-Verlag, Berlin, Germany, ISBN: 9783540150862, 1985 .
https://doi.org/10.1007/BFb0005617

[12] M. A. Sid-Ahmed, "Two-dimensional analog filters: a new form of realization," IEEE Transactions on Circuits and Systems, vol. 36, no. 1, pp. 153-154, January 1989.
https://doi.org/10.1109/31.16586

[13] H. J. Kaufman, M. A. Sid-Ahmed, "2-D analog filters for real time video signal processing," IEEE Transactions on Consumer Electronics, vol. 36, no. 2, pp. 137-141, May 1990.
https://doi.org/10.1109/30.54280

[14] S. Erfani, M. Ahmadi, N. Bayan, "Two-dimensional analog filtering and its implications on circuit theory," presented at 2014 IEEE 5th Latin American Symposium on Circuits and Systems, Santiago, Chile, pp. 1-5, 2014.
https://doi.org/10.1109/LASCAS.2014.6820262

[15] J. Chen, B. Huang, F. Ding, "Identification of two-dimensional causal systems with missing output data via expectation–maximization algorithm," IEEE Trans. Industrial Inform., vol.17, no. 8, pp. 5185–5196, August 2021.
https://doi.org/10.1109/TII.2020.3025581

[16] S. L. Hahn, "Multidimensional complex signals with single-orthant spectra," Proc. IEEE, vol. 80, no. 8; pp. 1287–1300, August 1992.
https://doi.org/10.1109/5.158601

[17] D. Gabor, "Theory of communications," J. Inst. EE, Part III, vol. 93, no. 26, pp. 429–457, November 1946.
https://doi.org/10.1049/ji-3-2.1946.0074

[18] S. L. Hahn, "The N-dimensional complex delta distribution," IEEE Trans. Signal Process., vol. 44, no. 7; pp. 1833–1837, July 1996.
https://doi.org/10.1109/78.510632

[19] T. A. Ell, "Hypercomplex Spectral Transformations", Ph. D. dissertation, Univ. of Minnesota, Minneapolis, 1992.
https://www.proquest.com/dissertations-theses/hypercomplex-spectral-transformations/docview/304027651/se-2?accountid=27375

[20] K. M. Snopek, "Complex and hypercomplex multidimensional analytic signals – the theory and chosen properties," in Advances in Signal Processing: Reviews, Sergey Y. Yurish (Ed.), International Frequency Sensor Association (IFSA) Publishing, vol. 2., pp. 21–51, ISBN: 978-8409288304, 2021.
https://repo.pw.edu.pl/docstore/download/@WUT314d4e5defb84e41a7d45106c75df466/Advances_in_Signal_Processing_Vol_2.pdf

[21] S.-C. Pei, J.-J. Ding, J.-H. Chang, "Efficient implementation of quaternion Fourier transform, convolution, and correlation by 2-D complex FFT," IEEE Trans. Sig. Proc., vol. 49, no. 11; pp. 2783–2797, November 2001.
https://doi.org/10.1109/78.960426.

[22] T. Bülow, G. Sommer, "Hypercomplex Signals – a novel approach of the analytic signal to the multidimensional case," IEEE Trans. Sig. Proc., vol. 49, no. 11; pp. 2844–2852, November 2001.
https://doi.org/10.1109/78.960432

[23] T. Bülow, "Hypercomplex Spectral Signal Representations for the Processing and Analysis of Images," Selbstverlag des Instituts für Informatik, Kiel, 1999.
https://nbn-resolving.org/urn:nbn:de:gbv:8:1-zs-00000190-a9