10.24425/acs.2025.155399

Archives of Control Sciences
Volume 35(LXXI), 2025
No. 2*, pages 351–373*

H_∞ positive filter-based control for positive linear systems

Montassar EZZINE, Mohamed DAROUACH,
Harouna SOULEY ALI and Hassani MESSAOUD

This paper deals with the design of a positive functional H_∞ filter-based controller for positive linear systems subject to bounded energy disturbances. We propose a new approach to numerically compute the controller, which is obtained via a function of the state to be estimated with the same order as the controller. The positive filter-based controller is obtained into two steps. First, we search for positive state feedback gain for the design of a control law such that the closed-loop is positive, stable and ensures an H_∞ performance requirement. This design of state-feedback gain is solved via constrained Linear Matrix Inequalities (LMIs). Then, we search in a second step for a positive functional filter-based controller permitting to reconstruct this control law, and so to estimate only a functional of the state useful for control purposes. The filter is positive, that is, it ensures the nonnegativity of the estimated states. The proposed procedure is based on the positivity of an augmented system composed of dynamics of both considered system and proposed filter-based controller and also, on the unbiasedness of the estimation error by solving a Sylvester equation. Then we derive conditions for the establishment of such filter-based controller in terms of an optimization problem, that can be solved via constrained LMIs. An algorithm that summarizes the different steps of the designed positive controller for positive linear systems is given. Finally, a numerical example is given to illustrate the effectiveness of the proposed method.

Key words: positive systems, positive functional H_∞ filters, linear systems, filter-based controller, LMI, optimization problem

Copyright © 2025. The Author(s). This is an open-access article distributed under the terms of the Creative Commons Attribution-NonCommercial-NoDerivatives License (CC BY-NC-ND 4.0 https://creativecommons.org/licenses/by-nc-nd/4.0/), which permits use, distribution, and reproduction in any medium, provided that the article is properly cited, the use is non-commercial, and no modifications or adaptations are made

M. Ezzine (corresponding author, e-mail: montassar.ezzine@ensi-uma.tn) is with Ecole Nationale des Sciences de l’Informatique, Campus Universitaire de la Manouba, 2010, Manouba, Tunisia.

M. Darouach (e-mail: mohamed.darouach@univ-lorraine.fr) and H. Souley Ali (e-mail: harouna.souley@univ-lorraine.fr) are with Université de Lorraine CRAN CNRS UMR 7039, IUT de Longwy, 186 Rue de Lorraine, 54400 Cosnes et Romain, France.

H. Messaoud (e-mail: hassani.messaoud@enim.rnu.tn) is with Laboratoire de Recheche LARATSI, Ecole Nationale d’Ingénieurs de Monastir, Tunisia.

Received 25.11.2024. Revised 13.05.2025.

1. Introduction

For many systems arising in practice, we have the constraint that their variables must be positive. For example, level of liquids in tanks, absolute temperature and concentration of substances in chemical processes. These examples belong to an important class of dynamical systems whose states are nonnegative for any nonnegative initial condition and any nonnegative input. In the literature, such systems are referred to be positive (see [3, 4, 17, 22, 27] or other items in the references). In view of these widespread applications, it is very important and necessary to investigate the study and design problems for positive linear systems (see [14, 19] and other items in the references). Note that positive systems differ from linear standard systems by the existence of positivity constraints. In fact, for example, if a system is controllable, the poles of the system can be placed arbitrarily, whereas for positive one this feature may not be true owing the positivity constraints on systems matrices.

On the other hand, in practice, many control processes require the availability on the components of the system state vector for the purpose of monitoring for example. This problem has motivated a great deal of work for observers'design for linear systems (see [5, 23, 26] and other items in the references) when the states are not measurable or not available. Mainly, the problem of functional observer design is equivalent to finding an observer, that estimates a linear combination of the states of a system using the input and output measurements. Such an estimator has the same order as the linear combination to be estimated. Note that, it has been the object of numerous studies for non-positive systems where the aim is only to minimize the estimation error (make the estimation error converge to zero) (see e.g. [5, 6, 9]). For positive systems, in addition to minimizing this error, positive observers must also guarantee the nonnegativity of the state estimates (see [7, 10–12]). This makes the positive observer design significantly more challenging (see [21, 25] and other items in the references).

Notice that, to ensure stability and optimal performance, it is imperative to consider the positivity constraint when controlling positive systems. Failure to do so may lead the mathematical model of these systems to venture into infeasible regions that are beyond the reach of the real system. Consequently, this can result in a loss of stability or performance when implementing the controller in the actual plant. Observer-based control is usually used when all the states of a system are not accessible (see [13, 28–30]). It is interesting to recall that feedback principle is an important concept in control theory especially for positive systems: based on algebraic approaches in [2, 8], via quadratic programming problem in [20], based on Linear Matrix Inequality (LMI) in [16], using analytical methods in [18] and via linear programming (LP) in e.g. [1]. Indeed, many different control strategies are based on the assumption that all internal states of the control object are available for feedback. In most cases, however, only a few parts of the states or some functions of the states can be measured. Then the need for techniques which make it possible not only to estimate states, but also to derive control laws that guarantee stability and positivity when using the estimated states instead of the true ones, is crucial. Despite the recent advances to the best of our knowledge, little attention has been paid on positive functional H_∞ filter-based controller design for positive linear systems subject to bounded energy disturbances, which motivates the present work. This is an important problem finding its way into multiple engineering applications such as in fault detection and it is an important research topic since these kinds of controllers are very important in practice as they possess real physical meaning. This is another motivation for the present work.

Notice that in this paper the only information we have on the disturbances is to be of bounded energy, which is of practical interest. Our purpose is to minimize the influence of such disturbances on the estimation error using H_∞ technique (see [15]). As far as the authors' knowledge, the proposed approach is original.

Motivated by these facts, we consider in this paper a new problem of designing positive functional H_∞ filter-based controller for positive linear standard systems subject to bounded energy disturbances. We propose a new approach to address numerically the computation of the solution, where the order of the filter-based controller is equal to the state feedback that is found to fulfill the control purpose. The positive filter-based controller is obtained into two steps. First, we search for positive state feedback gain for the design of a control law such that the closed-loop is positive, stable and ensures an H_∞ performance, using constrained LMI technique. Then, we search in a second step for a positive observer-based controller permitting to reconstruct this control law, and so to estimate only a function of the state useful for control purposes. The proposed procedure is based on the positivity of an augmented system composed of dynamics of both considered system and proposed filter-based controller and, on the unbiasedness of the estimation error by the resolution of a Sylvester equation. Then existence conditions of such filter-based controller are formulated in terms of an optimization problem. An algorithm that summarizes the different steps of the proposed controller based on positive functional filters for positive linear systems design is given. Finally, a numerical example and simulation results are given to illustrate the effectiveness of the proposed design method.

Recall that the most interest of our approach, is the use of a functional filter (not a full order one) which estimates the desired control law, without estimating all the state of the system contrary to full order filter-based control approach. The interesting is the reduction of the order of the controller since the designed functional filter is of the same order as the functional to be estimated, i.e. the dimension of the control input $u(t)$.

Notations: We shall use throughout the paper the following notations. $\Re$ denotes the set of real numbers, $\Re_+^n$ denotes the nonnegative orthant of the n-dimensional real space $\Re^n$ and $\Re^{m\times n}$ is the set of $m \times n$ matrices for which all entries belong to $\Re$. For a matrix $A \in \Re^{n\times m}$, a_{ij} denotes the element located at the ith row $(i \leqslant n)$ and jth column $(j \leqslant m)$. A matrix A is said to be nonnegative, denoted by $A \succ 0$, if $\forall(i, j),\ a_{ij} \geqslant 0$. It is said to be positive, if $\forall(i, j),\ a_{ij} \geqslant 0, \exists(i, j),\ a_{ij} > 0$. (Note that definitions of nonnegative and positive matrices are equivalent, except when a nonnegative matrix is identically zero which is the degenerate case and is of no interest. So, we consider that these two definitions are equivalent in general cases). A matrix A is said to be negative, denoted by $A \prec 0$, if $\forall(i, j),\ a_{ij} \leqslant 0$. $A > 0$ (respectively, $A < 0$) means that the matrix A is positive definite (respectively, negative definite), $A \geqslant 0$ (respectively, $A \leqslant 0$) means that the matrix A is positive semidefinite (respectively, negative semidefinite). For a real matrix A, A^T denotes the transpose. A^- denotes any generalized inverse of matrix A, i.e. verifies $AA^-A = A$. $\lambda_i(A)$ designs the ith eigenvalue of matrix A. $Re(\lambda_i(A))$ designs the real part of $\lambda_i(A)$. A is Hurwitz matrix if all real parts of its eigenvalues are in the left half plane. diag (v) denotes the diagonal matrix formed from the vector v. Ones (n) denotes an $n \times 1$ vector of ones. I and 0 are the identity matrix and the zero matrix of appropriate dimensions. $v \succ 0$ denotes a vector v such that for all its coordinates it holds, $\forall i,\ v_i \geqslant 0$.

2. Preliminaries

In the first part of this section, we give basic results on positive linear standard systems. Then, we recall by Lemma 2, another result that will be essential in the proposed design, namely in the state-feedback gain synthesis and filter-based controller design. For that, let us consider the following continuous linear systems of the form:

$$\dot{x}(t) = \widetilde{A}x(t) + \widetilde{B}u(t), \tag{1a}$$

$$y(t) = \widetilde{C}x(t) + \widetilde{D}u(t), \tag{1b}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^p$ and $y(t) \in \Re^m$ are the system state, (exogenous) input and the external output respectively. $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$ and $\widetilde{D}$ are system matrices with compatible dimensions.

The following Definition 1 and Lemma 1 are a slight generalization of the well known definition of positive systems (see [14, 19]).

Definition 1. *A linear system is said to be positive (internally positive: Metzlerian system) if its state and output are both nonnegative ($x(t) \in \Re^n_+$, $y(t) \in \Re^m_+$ $\forall t \geqslant 0$) for any nonnegative input and nonnegative initial state.*

Definition 2. *[24] A square real matrix M is called a Metzler matrix if its off-diagonal elements are nonnegative, i.e. $m_{ij} \geqslant 0$, $i \neq j$.*

Lemma 1. *System (1) is positive if and only if $\widetilde{A}$ is a Metzler matrix and $\widetilde{B} \in \Re^{n\times p}_+$, $\widetilde{C} \in \Re^{m\times n}_+$ and $\widetilde{D} \in \Re^{m\times p}_+$ are nonnegative matrices: ($\widetilde{B} \succ 0$, $\widetilde{C} \succ 0$ and $\widetilde{D}$).*

Lemma 2. *[15] For the continuous linear systems (1), consider a continuous-time transfer function $T(s)$ of (not necessarily minimal) realization $T(s) = \widetilde{D} + \widetilde{C}(sI - \widetilde{A})^{-1}\widetilde{B}$. The following statements are equivalent, for a given $\gamma > 0$:*

- *The relation below is satisfied:*
$$\|\widetilde{D} + \widetilde{C}(sI - \widetilde{A})^{-1}\widetilde{B}\|_\infty < \gamma \tag{2}$$
and $\widetilde{A}$ is stable in the continuous-time sense.
- *There exists a symmetric positive definite solution X to the LMI:*
$$\begin{pmatrix} \widetilde{A}^T X + X\widetilde{A} & X\widetilde{B} & \widetilde{C}^T \\ \widetilde{B}^T X & -\gamma I & \widetilde{D}^T \\ \widetilde{C} & \widetilde{D} & -\gamma I \end{pmatrix} < 0. \tag{3}$$

3. Problem statement

Let us consider the following linear multivariable continuous-time system described by

$$\dot{x}(t) = Ax(t) + Bu(t) + D_1 w(t), \tag{4a}$$
$$z(t) = F_1 x(t), \tag{4b}$$
$$y(t) = Cx(t), \tag{4c}$$

where $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^p$, $p \leqslant n$ is the input, $w(t) \in \Re^{q_1}$ represents the disturbance vector which is assumed to be of bounded energy, $y(t) \in \Re^m$, $m \leqslant n$ is the output vector and $z(t) \in \Re^{m_z}$, $m_z \leqslant n$ is the controlled outputs. A, B, D_1, F_1 and C are known constant matrices of appropriate dimensions.

Further, it is assumed through the paper that:

Assumption 1.

1) *rank* $F_1 = m_z$, $m_z \leqslant n$,
2) *rank* $C = m$, $m \leqslant n$.

www.czasopisma.pan.pl PAN www.journals.pan.pl

Remark 1. *Assumption (1) is not restrictive and means that the measurements are independent; we can always by some transformation obtain this case. The same remark can be applied to the functional to be estimated.*

Let us now assume that the considered linear system defined in (4) is positive, for any nonnegative input ($u(t)$ and $w(t)$) and nonnegative initial state. From Lemma 1, we have A is a Metzler matrix and $B \succ 0$, $D_1 \succ 0$, $F_1 \succ 0$ and $C \succ 0$.

Our aim is to design a positive functional H_∞ filter-based controller with the following structure

$$\dot{\varphi}(t) = N\varphi(t) + Jy(t) + Hu(t), \tag{5a}$$

$$\hat{u}(t) = \varphi(t) + Ey(t), \tag{5b}$$

where $\varphi(t) \in \Re^p$ is the filter state, $\hat{u}(t)$ is the output of the filter and matrices N, J, H and E are to be designed.

We consider, in this paper, the usual H_∞ performance index given by:

$$J_{zw} = \int_0^\infty (z^T z - \gamma^2 w^T w)\,\mathrm{d}t \tag{6}$$

for a given, minimum performance index $\gamma > 0$.

Then, the problem to be investigated can be stated as follows:

Objective 1. *Our main purpose is to build for the positive linear system (4) subject to positive bounded energy disturbances, a reduced order positive linear functional H_∞ filter-based controller into two steps:*

i) First, we search for a positive state feedback gain $K \succ 0$ for the design of a control law u satisfying $lim_{t\to\infty} u - Kx = 0$, such that the closed-loop of subsystem (4a)-(4b) is positive, stable and satisfies the H_∞ performance requirement; $J_{zw} < 0$ for a given $\gamma > 0$.

ii) Then, we search in a second step for a positive filter-based controller permitting to reconstruct this control law u, so to estimate only a functional of the state useful for control purposes, such that the estimation error $e(t) = \hat{u}(t) - Kx(t)$ is stable and converges asymptotically to zero as $t \to \infty$ by minimizing the influence of disturbances $w(t)$ on $e(t)$, i.e. $\dfrac{\|e\|_2}{\|w\|_2} < \gamma_1$ for a given $\gamma_1 > 0$.

Notice the order of this controller is equal to the dimension of the function to be estimated.

Definition 3. *The functional filter defined in (5) is called an H_∞ positive linear functional filter-based controller of system (4) if the output of the filter $\hat{u}(t) \in \Re_+^p$*

www.czasopisma.pan.pl PAN www.journals.pan.pl
POLSKA AKADEMIA NAUK

for all $t \geqslant 0$ and converges asymptotically to the functional $Kx(t)$ as $t \to \infty$ by minimizing the effect of disturbances $w(t)$ on the estimation error $e(t)$.

4. Positive filter-based controller

Here, we propose to solve the positive filter-based control problem into the two steps given in Objective 1.

4.1. Positive state-feedback synthesis

Let us assume that the state variable $x(t)$ can be directly measured. We propose in this part to design a state feedback control law of the following form:

$$u(t) = Kx(t) \tag{7}$$

such that the closed-loop system given by:

$$\dot{x}(t) = (A + BK)x(t) + D_1 w(t), \tag{8a}$$

$$z(t) = F_1 x(t) \tag{8b}$$

is positive, stable and satisfies the H_∞ performance requirement, where $K \succ 0$ is the state feedback gain. Using Lemma 1 and Definition 1, one can state the following lemma:

Lemma 3. *System (8) is positive if and only if $(A + BK)$ is a Metzler matrix and $D_1 \in \Re^{n\times q_1}$, $F_1 \in \Re^{m_z\times n}$ are nonnegative matrices: ($D_1 \succ 0$ and $F_1 \succ 0$).*

Then, the problem reduces to find conditions satisfying:

1) $A + BK$ is a Metzler matrix, with $D_1 \succ 0$ and $F_1 \succ 0$.
2) $A + BK$ is a Hurwitz matrix.
3) The closed-loop system (8) satisfies the H_∞ performance requirement.

So, we are in position to state the main result of this section.

Theorem 1. *By considering the positive linear system (4), where $A = [a_{ij}] \in \Re^{n\times n}$ and $B = [b_{ij}] \in \Re^{n\times p}$. A controller of the form in (7) satisfying that the closed-loop system given by (8) is positive, stable and satisfies the H_∞ performance $J_{zw} < 0$ for a given $\gamma > 0$, if there exists a nonnegative vector $v = [v_1, \ldots, v_n]^T \in \Re^n_+$ and $Y = [y_{ij}] \in \Re^{p\times n}_+$ such that the following LMI*

$$\begin{pmatrix} \Upsilon & D_1 & VF_1^T \\ (D_1)^T & -\gamma I & 0 \\ F_1 V & 0 & -\gamma I \end{pmatrix} < 0 \tag{9}$$

under the inequality

$$a_{ij}v_j + \sum_{z=1}^{p} b_{iz}y_{zj} \succ 0, \quad 1 \leqslant i \neq j \leqslant n \tag{10}$$

is satisfied,

where $V = diag(v)$ *and* $\Upsilon = VA^T + Y^T B^T + AV + BY$.

Under the above conditions, the desired state-feedback gain $K = [k_{ij}] \in \Re^{p\times n}$ *can be computed by:*

$$K = Y\,V^{-1}. \tag{11}$$

Proof. The closed-loop system given by (8) is:

1) **Stable** and **satisfies** the H_∞ performance $J_{zw} < 0$ for a given $\gamma > 0$ if, from Lemma 2, relation (3) holds. Then, for a symmetric positive definite matrix X, premultiplying and postmultiplying equation (3) by $\begin{pmatrix} X^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$ we obtain the following inequality, for $\widetilde{D} = 0$:

$$\begin{pmatrix} X^{-1}\widetilde{A}^T + \widetilde{A}X^{-1} & \widetilde{B} & X^{-1}\widetilde{C}^T \\ \widetilde{B}^T & -\gamma I & 0 \\ \widetilde{C}X^{-1} & 0 & -\gamma I \end{pmatrix} < 0 \tag{12}$$

which is equivalent to (9), with $\widetilde{A} = A + BK$, $\widetilde{B} = D_1$, $\widetilde{C} = F_1$ (compare system (8) to (1)) and by defining $V = X^{-1}$ and $Y = KV$.

2) **Positive** if $A+BK$ is a Metzler matrix, with $D_1 \succ 0$ and $F_1 \succ 0$ (see Lemma 3). Matrices D_1 and F_1 are nonnegative since the considered initial system (4) is positive. It is easy to see that $A + BK$ is a Metzler matrix implies that for $i \neq j$

$$a_{ij} + \sum_{z=1}^{p} b_{iz}k_{zj} \succ 0, \; 1 \leqslant i \neq j \leqslant n \tag{13}$$

which coincides with relation (10), by defining $k_{zj} = y_{zj}v_j^{-1}$. This completes the proof.

Remark 2. *Note that, Theorem 1 presents a simple approach to solve numerically the computation of the state feedback gain problem. In fact, conditions (9)–(10), that permit to find the desired state feedback gain and so the desired state feedback control law (7) guaranteeing the closed-loop system to be positive, stable and satisfying the* H_∞ *performance, are all LMIs. So, they can be solved by using standard numerical software.*

Once the state feedback gain obtained, recall that all the states are not accessible. Then, we propose a functional filter-based controller to reconstruct the obtained control law, as mentioned in second step of given Objective 1.

4.2. Positive filter-based controller existence conditions

Before providing the second result of the paper, let us compute the estimation error:

$$e(t) = \hat{u}(t) - Kx(t) \tag{14a}$$

$$= \varphi(t) + (EC - K)x(t). \tag{14b}$$

So, its dynamics can be written as

$$\dot{e}(t) = \dot{\varphi}(t) + (EC - K)\dot{x}(t) \tag{15a}$$

$$= Ne(t) + (H + ECB - KB)u(t) + (NK - NEC + JC + ECA - KA)x(t) + (ECD_1 - KD_1)w(t). \tag{15b}$$

Furthermore, to guarantee the positivity of the designed filter-based controller, we propose to compute the following augmented system consisting of the dynamics of (4) and (5). In fact, it can be given by:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{\hat{u}}(t) \end{pmatrix} = \kappa \begin{pmatrix} x(t) \\ \hat{u}(t) \end{pmatrix} + \begin{pmatrix} B \\ H + ECB \end{pmatrix} u(t) + \begin{pmatrix} D_1 \\ ECD_1 \end{pmatrix} w(t), \tag{16}$$

where $\kappa = \begin{pmatrix} A & 0 \\ JC + ECA - NEC & N \end{pmatrix}$.

By applying Lemma 1 on augmented system (16), we can give the following result:

Lemma 4. *(5) is a positive filter of system (4) if:*

1) N *is Metzler,*
2) $JC + ECA - NEC \succ 0$,
3) $H + ECB \succ 0$,
4) $ECD_1 \succ 0$.

The problem of the positive functional filter design is to determine N, J, H and E such that

i) (5) is a positive filter-based controller of system (4), stable and unbiased if $w(t) = 0$, i.e. the filter does not depend explicitly on the system state vector $x(t)$ and the input $u(t)$.

ii) the H_∞ performance requirement is satisfied, i.e. $\dfrac{\|e\|_2}{\|w\|_2} < \gamma_1$, for a given $\gamma_1 > 0$.

We are now ready to state the second result of the paper, namely the existence conditions of the proposed functional filter-based controller. In fact, the following theorem provides conditions which ensure that the proposed system (5) is a positive linear functional filter of system (4), by providing an output $\hat{u}(t)$ that is always nonnegative and converges to the functional $Kx(t)$ by minimizing the disturbance effect of disturbances $w(t)$ on estimation error $e(t)$.

Theorem 2. *The filter defined in (5) is a positive functional H_∞ filter-based controller of system (4) if the following conditions are satisfied:*

1) *N is Metzler.*
2) $JC + ECA - NEC \succ 0$
3) $H + ECB \succ 0$
4) $ECD_1 \succ 0$
5) $NK - NEC + JC + ECA - KA = 0$
6) $H + ECB - KB = 0$
7) $\dot{e}(t) = Ne(t) + (ECD_1 - KD_1)\, w(t)$ *is stable and satisfies the H_∞ performance requirement, i.e.* $\dfrac{\|e\|_2}{\|w\|_2} < \gamma_1$ *for a given* $\gamma_1 > 0$.

Proof. Conditions 1)−4) are obtained from Lemma 4. They guarantee that the estimate $\hat{u}(t)$, output of the proposed filter (5), be nonnegative all the time. In addition, by considering the expression (15) of the estimation error dynamics, one can conclude that it is unbiased (does not depend explicitly on state $x(t)$ and input $u(t)$) if and only if conditions 5) and 6) are satisfied. So, it is clear that if conditions of the proposed theorem are satisfied then $\hat{u}(t)$ is always nonnegative and it tends to $Kx(t)$ such the H_∞ performance satisfies $\dfrac{\|e\|_2}{\|w\|_2} < \gamma_1$ for a given $\gamma_1 > 0$.

We are now ready to state the third result of the paper, namely the design procedure of the proposed positive functional H_∞ filter-based controller. In fact, the following subsection is devoted to founding the functional filter matrices N, J, H and E such that conditions of Theorem 2 are fulfilled.

4.3. Filter-based controller design

To achieve unbiasedness of the proposed filter, condition 5) of Theorem 2 must hold:

$$NK + \psi C + ECA = KA \tag{17}$$

where

$$\psi = J - NE \tag{18}$$

with condition 6) of Theorem 2,

$$H = KB - ECB. \tag{19}$$

The equation (17), which has three unknowns (K is obtained in step 1), that are N, ψ and E can be transformed to

$$\begin{bmatrix} N & \psi & E \end{bmatrix} \begin{bmatrix} K \\ C \\ CA \end{bmatrix} = KA. \tag{20}$$

For the resolution of (20), let set

$$\begin{bmatrix} N & \psi & E \end{bmatrix} = X_1 \tag{21}$$

$$\begin{bmatrix} K \\ C \\ CA \end{bmatrix} = \Sigma \tag{22}$$

$$KA = \Theta \tag{23}$$

therefore (20) becomes

$$X_1 \Sigma = \Theta. \tag{24}$$

This equation has a solution X_1 if and only if

$$rank \begin{pmatrix} \Sigma \\ \Theta \end{pmatrix} = rank\ \Sigma. \tag{25}$$

In this case the general solution for (24), is given by

$$X_1 = \Theta\Sigma^- - Z(I_{p+2m} - \Sigma\Sigma^-) \tag{26}$$

where Σ^- is a generalized inverse of matrix Σ given by (22) and $Z \in \Re^{p\times(p+2m)}$ is an arbitrary matrix, that will be determined in the sequel. Once matrix X_1 is determined, it is easy to give the expressions of matrices N, ψ and E.

www.czasopisma.pan.pl PAN www.journals.pan.pl

In fact,

$$N = X_1 \begin{pmatrix} I_p \\ 0_{m\times p} \\ 0_{m\times p} \end{pmatrix} = A_{11} - ZB_{11} \tag{27}$$

where

$$A_{11} = \Theta\Sigma^{-} \begin{pmatrix} I_p \\ 0_{m\times p} \\ 0_{m\times p} \end{pmatrix} \tag{28}$$

$$B_{11} = (I_{p+2m} - \Sigma\Sigma^{-}) \begin{pmatrix} I_p \\ 0_{m\times p} \\ 0_{m\times p} \end{pmatrix} \tag{29}$$

$$\psi = X_1 \begin{pmatrix} 0_{p\times m} \\ I_m \\ 0_{m\times m} \end{pmatrix} = A_{22} - ZB_{22} \tag{30}$$

with

$$A_{22} = \Theta\Sigma^{-} \begin{pmatrix} 0_{p\times m} \\ I_m \\ 0_{m\times m} \end{pmatrix} \tag{31}$$

$$B_{22} = (I_{p+2m} - \Sigma\Sigma^{-}) \begin{pmatrix} 0_{p\times m} \\ I_m \\ 0_{m\times m} \end{pmatrix} \tag{32}$$

and

$$E = X_1 \begin{pmatrix} 0_{p\times m} \\ 0_{m\times m} \\ I_m \end{pmatrix} = A_{33} - ZB_{33} \tag{33}$$

with

$$A_{33} = \Theta\Sigma^{-} \begin{pmatrix} 0_{p\times m} \\ 0_{m\times m} \\ I_m \end{pmatrix}, \tag{34}$$

$$B_{33} = (I_{p+2m} - \Sigma\Sigma^{-}) \begin{pmatrix} 0_{p\times m} \\ 0_{m\times m} \\ I_m \end{pmatrix}. \tag{35}$$

Hence functional filter-based controller matrices N, E and J (can be computed from (18)) are determined if and only if the matrix Z is known.

Now, we propose a method to compute this matrix Z such that conditions 1)–4) and 7) of Theorem 2 are satisfied; Note that equations (24) and (19) correspond to conditions 5) and 6) of Theorem 2. Condition 3) of Theorem 2 is all time verified due to the nonnegativity of matrices B and K (see condition 6)).

The estimation error dynamics (15b) become, as mentioned in condition 7) of Theorem 2

$$\dot{e}(t) = Ne(t) + \alpha\, w(t) \tag{36}$$

with

$$\alpha = ECD_1 - KD_1 = \alpha_1 - Z\alpha_2\,, \tag{37}$$

where

$$\alpha_1 = \Theta\Sigma^- \begin{pmatrix} 0_{p\times q_1} \\ 0_{m\times q_1} \\ CD_1 \end{pmatrix} - KD_1 \tag{38}$$

$$\alpha_2 = (I_{p+2m} - \Sigma\Sigma^-) \begin{pmatrix} 0_{p\times q_1} \\ 0_{m\times q_1} \\ CD_1 \end{pmatrix}. \tag{39}$$

We intend in following lemma to recast remaining existence conditions, 1), 2), 4) and 7) of Theorem 2, of functional filter (5) for positive system (4) using Lemma 2.

Lemma 5. *Let us consider positive system (4). The functional filter-based controller defined in (5) is a positive linear functional H_∞ filter of system (4) and satisfies the H_∞ performance requirement; $J_{ew} < \gamma_1$ for a given $\gamma_1 > 0$ where $J_{ew} = \|(sI - N)^{-1}\alpha\|_\infty$, if rank condition (25) is satisfied and there exist a matrix Z and a diagonal positive definite matrix $P \in \Re^{p\times p} > 0$ such that the following LMI:*

$$\begin{pmatrix} \Lambda & P\alpha & I_p \\ \alpha^T P & -\gamma_1 I_{q_1} & 0_{q_1\times p} \\ I_p & 0_{p\times q_1} & -\gamma_1 I_p \end{pmatrix} < 0 \tag{40}$$

under following constraints,

1) $A_{11} - ZB_{11}$ *is Metzler,*

2) $(A_{33} - ZB_{33})CA + (A_{22} - ZB_{22})C \succ 0,$

3) $(A_{33} - ZB_{33})CD_1 \succ 0$

is satisfied, for a given $\gamma_1 > 0$ where,

$$\begin{aligned} \Lambda &= (A_{11}^T - (ZB_{11})^T)P + P(A_{11} - ZB_{11}) \\ \alpha &= \alpha_1 - Z\alpha_2. \end{aligned}$$

At this stage, we're ready to state the main result of the paper that permits us to obtain the filter-based controller matrices N, E and J such that conditions 1), 2),

4) and 7) of Theorem 2 are satisfied. Note that based on optimization problem, one can get the gain matrix Z which parameterizes the functional H_∞ filter matrices N, E and J (see (27), (33) and (30) using (18)). In fact, by following theorem, conditions (of Theorem 2) for the establishment of the positive functional H_∞ filter-based controller (5) for positive linear system (4) are formulated in terms of convex optimization problem.

Theorem 3. *Given positive system (4). The functional filter defined in (5) is a positive linear stable functional H_∞ filter-based controller of system (4) and satisfies the H_∞ performance requirement, $J_{ew} < \gamma_1$ for a given $\gamma_1 > 0$ where $J_{ew} = \|(sI - N)^{-1}\alpha\|_\infty$ if the following conditions hold:*

1. *Rank condition(25) is satisfied.*
2. *There exist a diagonal positive definite matrix $P > 0 \in \Re^{p\times p}$, $Y_1 \in \Re^{p\times(p+2m)}$ and a vector $\widetilde{v} \in \Re^{p\times 1}$ such that the following convex optimization problem is feasible:*

$$\begin{pmatrix} \Lambda & P\alpha_1 - Y_1\alpha_2 & I_p \\ \alpha_1^T P - \alpha_2^T Y_1^T & -\gamma_1 I_{q_1} & 0_{q_1\times p} \\ I_p & 0_{p\times q_1} & -\gamma_1 I_p \end{pmatrix} < 0 \tag{41}$$

under the inequalities,

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \\ \Lambda_{31} & \Lambda_{32} \end{bmatrix} \Upsilon \succ 0, \tag{42}$$

where,

$$\begin{aligned} \Lambda_{11} &= (A_{11} - V_1)^T, & \Lambda_{12} &= -B_{11}^T, \\ \Lambda_{21} &= (A_{33}CA + A_{22}C)^T, & \Lambda_{22} &= (-B_{33}CA - B_{22}C)^T, \\ \Lambda_{31} &= (A_{33}CD_1)^T, & \Lambda_{32} &= (-B_{33}CD_1)^T, \\ \Upsilon &= \begin{bmatrix} P & Y_1 \end{bmatrix}^T, & \Lambda &= A_{11}^T P - B_{11}^T Y_1^T + PA_{11} - Y_1 B_{11}, \\ V_1 &= diag(\widetilde{v}), & \widetilde{v} &= \left[\widetilde{v}_1,\ \widetilde{v}_2, \ldots .\widetilde{v}_p\right]^T. \end{aligned} \tag{43}$$

Then, positive functional H_∞ filter-based controller matrices N, E and J can be computed by (27), (33) and (30) using (18), with

$$Z = P^{-1}Y_1. \tag{44}$$

Proof. From Lemma 5, the functional filter defined in (5) is a positive functional H_∞ filter of system (4) and satisfies the H_∞ performance if the LMI (41), obtained by considering (40) with $Y_1 = PZ$, under the following inequalities is satisfied,

1. $A_{11} - ZB_{11}$ is required to be Metzler and that can be written as:

$$A_{11} - ZB_{11} \succ V_1\,, \tag{45}$$

where $V_1 = \text{diag}(\widetilde{v})$ for vector $\widetilde{v} = [\widetilde{v}_1, \widetilde{v}_2, \ldots.\widetilde{v}_p]^T$. Multiplying both sides of this inequality with the diagonal positive definite matrix $P > 0$ and taking into account a change of the variable $Y_1 = PZ$, one can verify that following inequality holds:

$$(A_{11} - V_1)^T\, P^T - B_{11}^T\, Y_1^T \succ 0. \tag{46}$$

2. Consider now condition 2) of Lemma 5:

$$(A_{33} - ZB_{33})CA + (A_{22} - ZB_{22})C \succ 0. \tag{47}$$

By multiplying (47) with diagonal positive definite matrix $P > 0$ with $Y_1 = PZ$ this inequality is thus equivalent to,

$$(A_{33}CA + A_{22}C)^T P^T + (-B_{33}CA - B_{22}C)^T Y_1^T \succ 0. \tag{48}$$

3. We consider now condition 3) of Lemma 5:

$$(A_{33} - ZB_{33})CD_1 \succ 0. \tag{49}$$

By multiplying (49) with diagonal positive definite matrix $P > 0$ with $Y_1 = PZ$ this inequality is thus equivalent to,

$$(A_{33}CD_1)^T P^T + (-B_{33}CD_1)^T Y_1^T \succ 0. \tag{50}$$

Consequently, with $\Upsilon = \begin{bmatrix} P & Y_1 \end{bmatrix}^T$ conditions (46), (48) and (50) can then be written as the convex condition (42). Therefore, (41) and (42) form the convex optimization problem. This completes the proof.

5. Positive functional H_∞ filter-based controller design steps summary

The different steps that must be achieved to design the proposed controller are given as follows:

1. If LMI (9) under inequality (10) is feasible, get matrices V and Y.
2. Compute the state feedback gain K by (11) for the control law design (7).
3. Compute Σ and Θ from (22) and (23).
4. Verify rank condition (25).
5. Compute A_{11}, B_{11}, A_{22}, B_{22}, A_{33} and B_{33} from relations (29), (32) and (35).
6. If optimization problem formed by LMI (41) under inequalities (42) is feasible, get matrices P and Y_1.

www.czasopisma.pan.pl PAN www.journals.pan.pl

7. Compute filter matrix gain Z from (44).
8. Get filter-based controller matrices N, E from (27) and (33). Deduce filter matrix H from (19).
9. Get matrix ψ from (30) and compute filter-based controller matrices J using (18).

Note that if rank conditions fail and/or the optimization problem turns out to be infeasible, then the filter-based controller does not exist and we need to significantly increase the filter dimension to have any hope of obtaining a functional filter. And if the algorithm runs well until the end, then the proposed positive functional H_∞ filter-based controller description (5) for positive linear system (4) is obtained.

The following section demonstrates the effectiveness of the proposed approach on a numerical example.

6. Numerical results

Consider the system presented in Section 3, where

$$A = \begin{bmatrix} -9 & 0 & 0.5 \\ 0.2 & -8 & 1 \\ 0.3 & 1.3 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0.6 & 0.5 \\ 0.1 & 0.8 \\ 2 & 0.1 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0.2 & 0.5 & 0.7 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 1 \\ 1 \\ 0.2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 5 \\ 0 & 1 & 4 \end{bmatrix}.$$

With the computational approach summarized in previous section 5, which is based on constrained LMIs and an optimization problem, we have obtained the following results:

Positive state-feedback gain synthesis

The proposed LMI (9) under inequality (10) is feasible for $\gamma = 4.7$. One such feasible solution provides:

1. $V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $Y = \begin{bmatrix} 0.2018 & 0.4873 & 0.0274 \\ 2.9846 & 5.0809 & 0.4701 \end{bmatrix}$.
2. Then, the state feedback gain is given by:

$$K = \begin{bmatrix} 0.2018 & 0.0975 & 0.0274 \\ 2.9846 & 1.0162 & 0.4701 \end{bmatrix}.$$

One can verify that the closed-loop system is positive, stable and satisfies the H_∞ performance requirement, where the infinity norm of the closed loop is equal to $0.85618 < 4.7$.

Positive filter-based controller synthesis

For this design, after checking conditions (25), the proposed optimization problem (41)–(42) is feasible, for $\gamma_1 = 1.5$. One such feasible solution provides:

1. $Z = \begin{bmatrix} 10.3408 & 9.7210 & 2.1797 & 0.8590 & 2.5690 & 1.2851 \\ 7.9819 & 137.3429 & 49.9984 & 3.7245 & 14.4897 & 5.9390 \end{bmatrix}.$
2. Then, filter-based controller matrices N, E and H are given as follows:

$$N = \begin{bmatrix} -9.6776 & 0.0236 \\ 0.7289 & -8.9493 \end{bmatrix}, \quad E = \begin{bmatrix} -0.0393 & 0.1093 \\ -2.2903 & 2.4866 \end{bmatrix},$$
$$H = \begin{bmatrix} -0.3066 & 0.0701 \\ 5.5942 & 0.5135 \end{bmatrix}.$$

3. $\psi = \begin{bmatrix} -0.4357 & 0.7371 \\ -18.6314 & 23.3539 \end{bmatrix}$, then we obtain $J = \begin{bmatrix} -0.1090 & -0.2624 \\ 1.8362 & 1.1807 \end{bmatrix}.$

So, the proposed design of the positive filter-based controller for positive linear system subject to bounded energy disturbances is obtained. Notice that $J_{ew} = \|(sI - N)^{-1}\alpha\|_\infty = 0.2152 < 1.5$.

Simulation results are illustrated by Figures 1–4, where we present in Figure 1 the behavior of the used disturbances $w(t)$. Note that Figure 2 demonstrates the positivity of the functional state $Kx(t)$ components, Figure 3 shows the responses

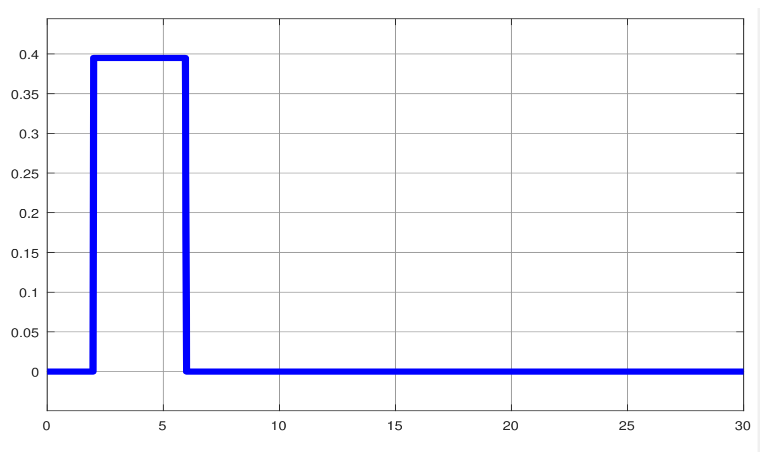

Figure 1: Disturbances $w(t)$ behavior

www.czasopisma.pan.pl PAN POLSKA AKADEMIA NAUK www.journals.pan.pl

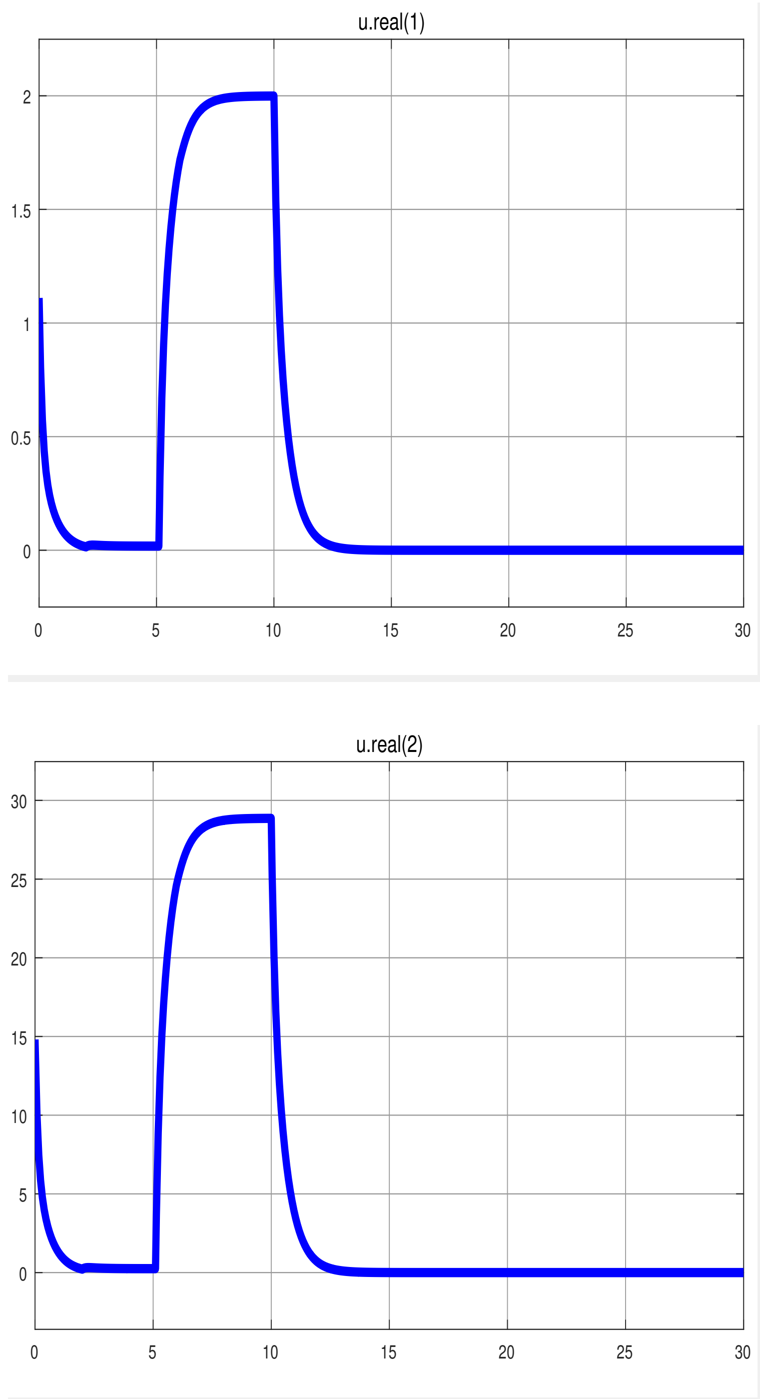

Figure 2: Evolution of $u.real1$ and $u.real2$ components of functional $Kx(t)$

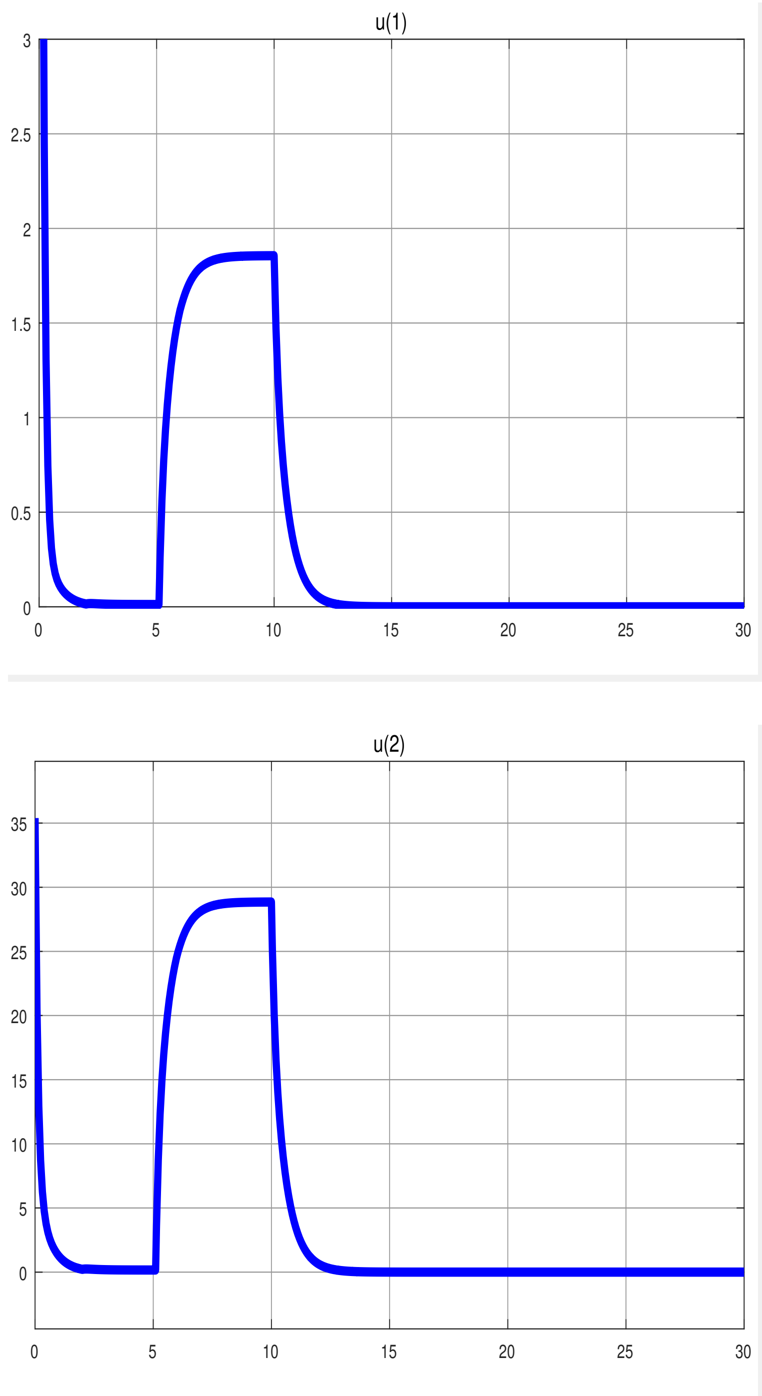

Figure 3: Evolution of $u1$ and $u2$ components of the output of the controller $\hat{u}(t)$

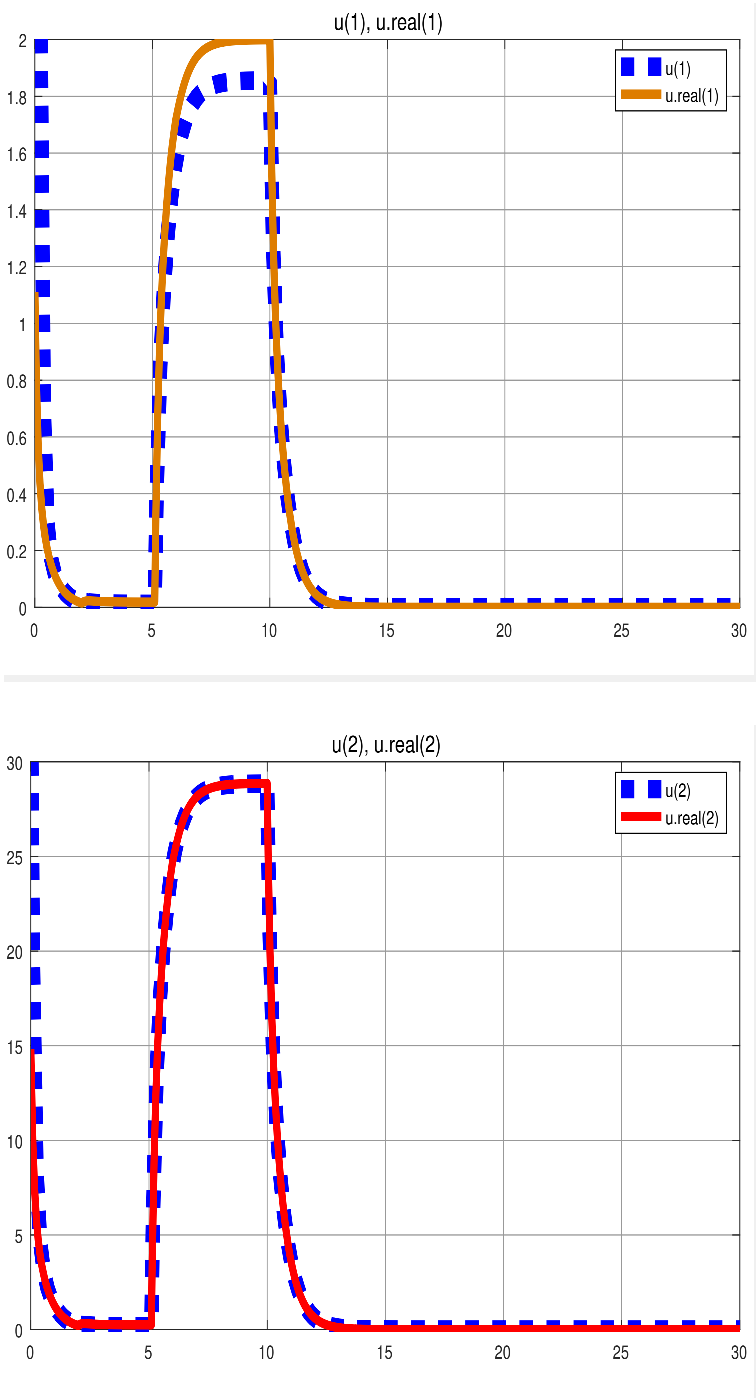

Figure 4: Evolution of $u.real1$ and $u.real1$ with their estimates $u(1)$ and $u(2)$

www.czasopisma.pan.pl PAN www.journals.pan.pl
POLSKA AKADEMIA NAUK

of the proposed observer-based controller (components of the output $\hat{u}(t)$). In Figure 4 we plot together the components of the functional state $Kx(t)$ and their estimated components. Note that the simulations are performed with the following positive initial conditions:

$$x(0) = \begin{bmatrix} 2 & 5 & 8 \end{bmatrix}^T .$$

The estimates are nonnegative and the designed controller based-filter estimated the functional state $Kx(t)$ as expected. Finally, simulation results show the behavior of the proposed positive controller based-H_∞ functional filter for positive linear systems and so, the effectiveness of our approach.

7. Conclusion

A new design of a positive functional filter-based controller for linear multivariable positive linear systems subject to bounded energy disturbances is proposed in this paper. The order of this controller is equal to the dimension of the functional to be estimated. The positive filter-based controller, that is always nonnegative at any time, is obtained into two steps. First, we search for a positive state feedback gain for the design of a control law through H_∞ techniques such that the closed-loop is positive, stable and ensures an H_∞ performance requirement. Then, we search in a second step for a positive functional filter-based controller permitting us to reconstruct this control law, and so to estimate only a functional of the state useful for control purposes. The proposed procedure is based on the positivity of an augmented system composed of dynamics of both considered system and proposed filter-based controller and also, on the unbiasedness of the estimation error by the resolution of Sylvester equation. Then we derive conditions for the establishment of such filter-based controller in terms of an optimization problem, that can be solved via constrained LMIs. The positive functional filter used has the advantages that it estimates the control law without estimating all the state of the system contrary to full order observer-based control approach. An algorithm that summarizes the different steps of the proposed positive controller based on positive functional filters for positive linear systems design is given.

References

[1] M. Ait Rami and F. Tadeo: Controller synthesis for positive linear systems with bounded controls. *IEEE Transactions on Circuits and Systems–II: Express Briefs*, **54**(2), (2007), 151–155. DOI: 10.1109/TCSII.2006.886888

[2] A. Berman, M. Neumann and J.R. Stern: Nonnegative Matrices in Dynamic Systems. New York, Wiley. 1989.

[3] L. Caccetta, R.L. Foulds and G.V. Rumchev: A positive linear discrete-time model of capacity planning and its controllability properties. *Mathematical and Computer Modelling*, **40**(1-2), (2004), 217–226. DOI: 10.1016/j.mcm.2003.03.010

[4] E.R. Carson, C. Cobelli and L. Finkelstein: Modeling and identification of metabolic systems. *American Journal of Physiology-Regulatory, Integrative and Comparative Physiology*, **240**(3), (1981), 120–129. DOI: 10.1152/ajpregu.1981.240.3.R120

[5] M. Darouach, M. Zasadzinski and H. Souley Ali: Robust reduced order unbiased filtering via LMI. *Proceedings of the 6th European Control Conference*, (2001). Porto, Portugal. DOI: 10.23919/ECC.2001.7076162

[6] M. Darouach: Existence and design of functional observers for linear systems. *IEEE Transaction on Automatic Control Processing*, **45**(5), (2000), 940–943. DOI: 10.1109/9.855556

[7] N. Dautrebande and G. Bastin: Positive linear observers for positive linear systems. *Proceedings of the European Control Conference*, Karlsruhe, Germany, 1999. DOI: 10.23919/ECC.1999.7099454

[8] P. De Leenheer and D. Aeyels: Stabilization of positive linear systems. *Systems and Control Letters*, **44**(4), (2001), 259–271. DOI: 10.1016/S0167-6911(01)00146-3

[9] M. Ezzine, M. Darouach, H. Souley Ali and H. Messaoud: Time and frequency domain design of functional filters. *American Control Conference*, Marriott Waterfront, Baltimore, MD, USA. (2010). DOI: 10.1109/ACC.2010.5531072

[10] M. Ezzine, M. Darouach, H. Souley Ali and H. Messaoud: A new positive linear functional filters design for positive linear systems. *Proceedings of the 22nd Mediterranean Conference on Control and Automation*, Palermo, Italy. (2014), 407–411. DOI: 10.1109/MED.2014.6961406

[11] M. Ezzine, H. Souley Ali, M. Darouach and H. Messaoud: Positive unknown inputs filters design for positive linear systems. *American Control Conference*, Denver, CO, USA. (2020), 3369–3374. DOI: 10.23919/ACC45564.2020.9147795

[12] M. Ezzine, M. Darouach, H. Souley Ali and H. Messaoud: Positive functional filters synthesis for positive linear systems. *IMA Journal of Mathematical Control and Information*, **41**(3), (2024), 478–494. DOI: 10.1093/imamci/dnae020

[13] M. Ezzine, H. Souley Ali, M. Darouach and H. Messaoud: A controller design based on functional H_∞ filter for descriptor systems: the time and frequency domains cases. *Automatica*, **48** (2012), 542–549. DOI: 10.1016/j.automatica.2011.08.060

[14] L. Farina and S. Rinaldi: *Positive Linear Systems: Theory and Applications*. *Wiley*, New York. 2000.

[15] P. Gahinet and P. Apkarian: A linear matrix inequality approach to H_∞ control. *International Journal of Robust and Nonlinear Control*, **4**(4), (1994), 421–448. DOI: 10.1002/rnc.4590040403

[16] H. Gao, J. Lam C. Wang and S. Xu: Control for stability and positivity: equivalent conditions and computation. *IEEE Transactions on Circuits and Systems II: Express Briefs*, **52**(9), (2005), 540–544. DOI: 10.1109/TCSII.2005.850525

[17] W.M. Haddad and V.S. Chellabonia: Stability and dissipativity theory for nonnegative dynamical systems: a unified analysis framework for biological and physio-

www.czasopisma.pan.pl PAN www.journals.pan.pl

logical systems. *Nonlinear Anal: Real World Application*, **6**(1), (2005), 35–65. DOI: 10.1016/j.nonrwa.2004.01.006

[18] W.P.M. H Heemels, S.J.L Van Eijndhoven and A. Stoorvogel: Linear quadratic regulator problem with positive controls. *International Journal of Control*, **70**(4), (1998), 551-578. DOI: 10.1080/002071798222208

[19] T. Kaczorek: Positive 1D and 2D Systems. *Springer*, London, 2001.

[20] T. Kaczorek: Stabilization of positive linear systems by state feedback. *Pomiary, Automatyka, Kontrola*, **3** (1999), 2–5.

[21] L.J. Liu and X. Zhao: Design of multiple-mode observer and multiple-mode controller for switched positive linear systems. *IET Control Theory and Applications*, (2018), 1320–1328. DOI: 10.1049/iet-cta.2018.5625

[22] P. Liu, Q. Zhang, X. Yang and L. Yang: Passivity and optimal control of descriptor biological complex systems. *IEEE Transactions on Automatic Control*, **53** (2008), 122–125. DOI: 10.1109/TAC.2007.911341

[23] D.G. Luenberger: An introduction to observers. *IEEE Transactions on Automatic Control*, **16**(6), (1971), 596–602. DOI: 10.1109/TAC.1971.1099826

[24] D.G. Luenberger: Introduction to Dynamic Systems: Theory, Models and Applications. *John Wiley and Sons*, New York, 1979.

[25] Y. Okamoto, J.I. Imura and M. Okada-Hatakeyama: Observer design of positive quadratic systems. *Proceedings of the European Control Conference*, Alborg, Denmark.(2016), 843–848. DOI: 10.1109/ECC.2016.7810394

[26] C. Tsui: A new algorithm for the design of multi-functional observers. *IEEE Transaction on Automatic Control*, **30**(1), (1985), 89–93. DOI: 10.1109/TAC.1985.1103795

[27] N. Yi, Q. Zhang, K. Mao, D. Yang and Q. Li: Analysis and control of an SEIR epidemic system with nonlinear transmission rate. *Mathematical and Computer Modelling*, **50**(9-10), (2009), 1498–1513. DOI: 10.1016/j.mcm.2009.07.014

[28] X.Yang, M. Huang, Y. Wu and S. Feng: Observer-based PID control protocol of positive multi-agent systems. *Mathematics*, **11**(2), (2023), 419. DOI: 10.3390/math11020419

[29] I. Iben Ammar, H. Gassara, A. El Hajjaji, F. Tadeo and M. Chaabane: Observer-based controller for positive polynomial systems with time delay. *Optimal Control Applications and Methods*, **41**(1), (2020), 278–291. DOI: 10.1002/oca.2542

[30] I. Zaid, M. Chaabane, T. Fernando and A. Benzaouia: Static state-feedback controller and observer design for interval positive systems with time delay. *IEEE Transactions on Circuits and Systems–II: Express Briefs*, **62**(5), (2015), 506–510. DOI: 10.1109/TCSII.2014.2385391