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On some important feature of the angular momentum of a rigid body in pure rotation about mass center

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In this paper we prove that the angular momentum of a rigid body performing a pure rotation about fixed mass center does not depend on the reference point, thus it may be considered as a natural measure of the amount of rotational motion. This implies also (somewhat forgotten) decomposition of the angular momentum of a rigid body in a general motion onto *spin* and *orbital* angular momenta. The canonical forms of definitions of three fundamental quantities: linear momentum, angular momentum and kinetic energy are presented in this context, emphasizing the specific role of the mass center. Various formulas for the angular momentum are discussed and their practical usefulness is analyzed. There is also shown some lack of clarity in interpretation of the physical meaning of this quantity, which may be a source of problems, especially in teaching the classical mechanics on a basic university level.

1. Introduction

We would like to focus attention on some interesting and important feature of the angular momentum of a rigid body, which consists in its independence on the reference point in case of pure rotation of a body about its mass center. We will assume in this paper, that the term *pure rotation* concerns rotation of a rigid body about its mass center, despite that this term is used frequently for denoting a rotation about an *arbitrary* fixed point of the body. Although the proof of this fact is quite simple, it is, to our knowledge, not widely recognized, which seems to be surprising if one realizes the implications for understanding the nature and the role of angular momentum as a fundamental quantity of mechanical systems.

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We will show that the proven feature allows for an elegant interpretation of the angular momentum about the mass center as the basic quantity describing the amount of rotation, not influenced by translation. Following Synge [1] we recall then the decomposition of angular momentum onto spin and orbital angular momenta, which exactly matches the nature of general motion of a rigid body consisting of translation and rotation.

In the light of presented interpretations the particular role of the mass center will be emphasized. Finally, we will show that linear momentum, angular momentum and kinetic energy of a rigid body can be defined in the simplest, canonical, forms.

Apart from the main result concerning the angular momentum, other results presented in this paper are not the new ones. We are convinced, however, that the proposed interpretations may be valuable in teaching the classical mechanics of a rigid body, giving a deeper insight into the nature of angular momentum, since the basic result on independency is not intuitively obvious, especially when the reference point is fixed to the body.

The paper is composed as follows. First, we prove the independence the angular momentum of a rigid body performing pure rotation about its mass center upon the reference point. Based on this result we recall decomposition of the angular momentum onto the spin and orbital momenta. Then we remind the basic forms of the three fundamental quantities: momentum, angular momentum and kinetic energy emphasizing the role of mass center. Finally, we discuss the significance of the obtained result for understanding the nature of angular momentum in teaching the classical mechanics.

Since the definitions of mechanical quantities are included in all positions of the bibliography, [1–14], we will not refer to the specific items except of the cases, when some important questions are to be emphasized. We use the standard notations for the basic mechanical quantities: $\mathbf{H}$ for the angular momentum, $\mathbf{v}$ for the linear velocity, $\boldsymbol{\omega}$ for the angular velocity, T for the kinetic energy and $\mathbf{r}, \boldsymbol{\rho}$ for the position vectors. We have chosen the symbol $\mathbf{p}$ for the linear momentum, although other notations are also used in the literature (e.g., $\mathbf{M}$ or $\mathbf{L}$).

2. Angular momentum of a rigid body

2.1. Pure rotation about mass center

Let us consider an arbitrary set $\mathcal{B}$ of material particles constituting a rigid body, i.e., fulfilling the condition that the distances between any of its points $A, B \in \mathcal{B}$ are constant. Without a loss of generality we will assume that $\mathcal{B}$ is a continuous bounded set of points, therefore $m = \int_m dm$ is the mass of the body, whereas integration is carried out over the body mass. The center of mass of the body

relative to an arbitrary point A (Fig. 1) is defined as

$$\mathbf{r}_{AC} = \frac{1}{m} \int_m \mathbf{r} \, dm. \quad (1)$$

We will first analyze a pure rotation of a rigid body around its mass center, fixed in some absolute reference system.

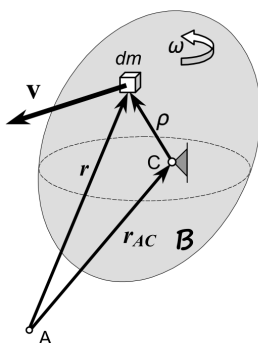


Fig. 1. Rigid body in pure rotation about mass center

The angular momentum of the body relative to an arbitrary point A (which will be also called the *reference point*) is defined as

$$\mathbf{H}_A = \int_m (\mathbf{r} \times \mathbf{v}) \, dm, \quad (2)$$

where $\mathbf{r}$ is the position vector of an elementary mass dm relative to A and $\mathbf{v}$ is the absolute velocity of dm , defined in some inertial reference frame. We emphasize that the reference point A is quite arbitrary: it can be either fixed or it can move relative to the inertial reference frame, as well as it can be fixed to the body.

The position vector $\mathbf{r}$ may be expressed as

$$\mathbf{r} = \mathbf{r}_{AC} + \boldsymbol{\rho}, \quad (3)$$

where $\boldsymbol{\rho}$ is the position vector of the elementary mass dm relative to the mass center C .

Since an arbitrary motion of a rigid body can be decomposed onto the translation of some arbitrary point B of the body and rotation of the body around this point with the angular velocity $\boldsymbol{\omega}$, one may express the absolute velocity $\mathbf{v}$ of any point of the body in the form

$$\mathbf{v} = \mathbf{v}_B + \boldsymbol{\omega} \times \boldsymbol{\rho}. \quad (4)$$

In the considered case of rotation of the body around its mass center we set $B \equiv C$ and $\mathbf{v}_C \equiv \mathbf{0}$, thus

$$\mathbf{v} = \boldsymbol{\omega} \times \boldsymbol{\rho}. \quad (5)$$

Substituting (3) and (5) into (2) and using

$$\int_m \boldsymbol{\rho} \, dm = \mathbf{0}, \quad (6)$$

one obtains

$$\begin{aligned} \mathbf{H}_A &= \int_m (\mathbf{r}_{AC} + \boldsymbol{\rho}) \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) \, dm = \\ &= \mathbf{r}_{AC} \times (\boldsymbol{\omega} \times \int_m \boldsymbol{\rho} \, dm) + \int_m \boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) \, dm = \\ &= \int_m (\boldsymbol{\rho} \times \mathbf{v}) \, dm = \mathbf{H}_C. \end{aligned} \quad (7)$$

This follows immediately from (1) if we substitute (3).

This proves then an interesting feature of the angular momentum of a rigid body performing a pure rotation about the mass center.

Theorem *The angular momentum of a rigid body performing a pure rotation about its mass center does not depend on the reference point.*

We will now recall, that the angular momentum, although defined in (2) by means of linear velocity $\mathbf{v}$ is, in fact, independent on it when the rigid body performs a pure rotation. Using vector identity

$$\boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) = (\boldsymbol{\rho}^2 \mathbb{E} - \boldsymbol{\rho} \boldsymbol{\rho}) \boldsymbol{\omega}, \quad (8)$$

where $\mathbb{E}$ is an unit tensor and $\boldsymbol{\rho} \boldsymbol{\rho}$ is a dyadic product, one obtains from (7)

$$\begin{aligned} \mathbf{H}_C &= \int_m \boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) \, dm = \\ &= \int_m (\boldsymbol{\rho}^2 \mathbb{E} - \boldsymbol{\rho} \boldsymbol{\rho}) \, dm \boldsymbol{\omega} = \mathbb{I}_C \boldsymbol{\omega}. \end{aligned} \quad (9)$$

where $\mathbb{I}_C$ is the tensor of inertia of the body with respect to the mass center.

The final conclusion emerging from (7) and (9) is that *the angular momentum in rotation of the body around its mass center, being independent on translation of the body, is a natural measure of the amount of pure rotation of a rigid body.*

2.2. Angular momentum in general motion of a body

Keeping in mind the above conclusions we may consider the angular momentum in a general motion of a rigid body. If the mass center is not fixed, that is, if there is a translation of the body, ($\mathbf{v}_C \neq \mathbf{0}$), then from (2) and using (3), (4) and (6) one obtains

$$\begin{aligned} \mathbf{H}_A &= \int_m (\mathbf{r}_{AC} + \boldsymbol{\rho}) \times (\mathbf{v}_C + \boldsymbol{\omega} \times \boldsymbol{\rho}) \, dm = \\ &\quad \mathbf{r}_{AC} \times \mathbf{v}_C \int_m dm + \mathbb{I}_C \boldsymbol{\omega} + \\ &\quad \mathbf{r}_{AC} \times \left(\boldsymbol{\omega} \times \int_m \boldsymbol{\rho} \, dm \right) + \int_m \boldsymbol{\rho} \, dm \times \mathbf{v}_C = \\ &\quad m\mathbf{r}_{AC} \times \mathbf{v}_C + \mathbb{I}_C \boldsymbol{\omega}, \end{aligned} \quad (10)$$

which may be finally written in a concise form

$$\mathbf{H}_A = \mathbf{r}_{AC} \times \mathbf{p} + \mathbf{H}_C, \quad (11)$$

where $\mathbf{p} = m\mathbf{v}_C$ is the linear momentum of the body.

In general motion the angular momentum depends thus on the reference point A , but only due to the linear momentum of the body, that is, the quantity which is naturally associated with the translation. Note, that for the first term in (11) the term *moment of momentum* is even more adequate than *angular momentum*.

The formula (11) reflects the natural decomposition of the amount of motion (in the sense defined by Descartes or Newton) onto two parts: the linear momentum (that corresponds to translation) and the pure angular momentum (that corresponds to pure rotation). Such a structure of the angular momentum is consistent with the basic decomposition of general motion of a rigid body onto the translation and rotation.

The general form of angular momentum (11) has a clever interpretation given by Synge [1, p.35]. By an analogy to the quantum chemistry the two parts of the angular momentum in (11) may be called the *spin* $\mathbf{H}_A^s$ and the *orbital angular momentum* $\mathbf{H}_A^o$, respectively. One should remember, however, that in quantum mechanics the spin and orbital angular momentum are independent state variables, whereas in classical mechanics these two quantities constitute a single state variable. One may thus conclude, that the total angular momentum of a general motion of a rigid body relative to an arbitrary point A is a sum of the spin and the orbital angular momentum

$$\mathbf{H}_A = \mathbf{H}_A^s + \mathbf{H}_A^o. \quad (12)$$

3. Interpretation the angular momentum

The main result of Section 2 concerning the independence of an angular momentum on the reference point in the case of pure rotation of a rigid body, is not obvious by far, especially when one looks at the basic definition (2). To clarify this, let us consider three types of reference points: fixed in an inertial reference frame (i), moving in inertial reference frame (ii) and moving with the body (or fixed in the reference frame moving with the body) (iii).

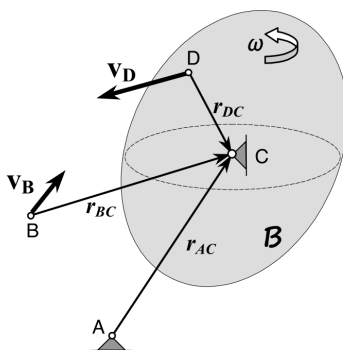


Fig. 2. General definition of momentum

Looking at Fig. 2 one may conclude, that in the first case (i), when point A is fixed, equality $\mathbf{H}_A = \mathbf{H}_C$ is intuitively clear, whereas in the second case (ii) $\mathbf{H}_B = \mathbf{H}_C$ is somewhat less obvious. The last case (iii), where point D is moving with the body, the equality $\mathbf{H}_D = \mathbf{H}_C$ is not evident without a proof. In our opinion, this non-obviousness may introduce some difficulty in proper interpretation of the angular momentum as a quality associated with the rotation.

The non-obviousness in interpretation of angular momentum is even more evident in the case of general motion of a rigid body if one wants to express the angular momentum with respect to a fixed point A using an auxiliary point of the body D, different from the center of mass C (see Fig. 3).

In such a case

$$\begin{aligned}\mathbf{r} &= \mathbf{r}_{AD} + \boldsymbol{\rho}_D, \\ \boldsymbol{\rho}_D &= \mathbf{r}_{DC} + \boldsymbol{\rho},\end{aligned}\tag{13}$$

and then

$$\begin{aligned}\mathbf{H}_A &= \int_m (\mathbf{r}_{AD} + \boldsymbol{\rho}_D) \times \mathbf{v} \, dm = \\ &= \mathbf{r}_{AD} \times \int_m \mathbf{v} \, dm + \int_m (\boldsymbol{\rho}_D \times \mathbf{v}) \, dm,\end{aligned}\tag{14}$$

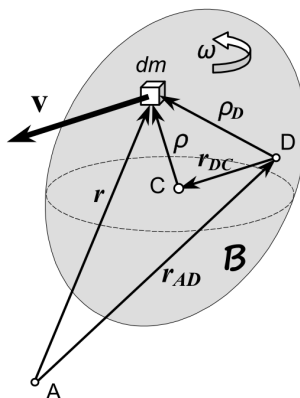


Fig. 3. Various types of reference points

which finally reads

$$\mathbf{H}_A = \mathbf{r}_{AD} \times \mathbf{p} + \mathbf{H}_D. \quad (15)$$

Although the formulas (11) and (15) coincide when $D \equiv C$, one can state that physical interpretation of (15) is *not* straightforward, because $\mathbf{r}_{AD} \times \mathbf{p}$ is not an orbital moment of momentum: $\mathbf{r}_{AD}$ points to D , whereas $\mathbf{p}$ is defined by the velocity of the mass center $\mathbf{v}_C$, not the velocity of point D . Moreover, $\mathbf{H}_D$ is not a spin, because it contains also some orbital momentum; if we express $\mathbf{H}_D$ in terms of $\mathbf{H}_C$ and $\mathbf{r}_{DC}$, that is, by explicit use of (11) with $A \equiv D$, we obtain

$$\mathbf{H}_D = \mathbf{r}_{DC} \times \mathbf{p} + \mathbf{H}_C. \quad (16)$$

Introducing (16) into (15) leads straightforward to the formula (11), since $\mathbf{r}_{AD} \times \mathbf{p} + \mathbf{r}_{DC} \times \mathbf{p} = \mathbf{r}_{AC} \times \mathbf{p}$.

In practical applications one requires usually such a formula for the angular momentum that contains *explicitly* the inertia tensor relative to the auxiliary point D . In such a case, from (13) and (6) one obtains

$$\begin{aligned}
 \mathbf{H}_D &= \int_m \boldsymbol{\rho}_D \times (\mathbf{v}_D + \boldsymbol{\omega} \times \boldsymbol{\rho}_D) \, dm = \\
 &= \int_m (\mathbf{r}_{DC} + \boldsymbol{\rho}) \times \mathbf{v}_D \, dm + \int_m \boldsymbol{\rho}_D \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_D) \, dm = \\
 &= m\mathbf{r}_{DC} \times \mathbf{v}_D + \int_m \boldsymbol{\rho} \, dm \times \mathbf{v}_D + \mathbb{I}_D \boldsymbol{\omega} = \\
 &= \mathbf{r}_{DC} \times m\mathbf{v}_D + \mathbb{I}_D \boldsymbol{\omega}.
 \end{aligned} \quad (17)$$

The final formula for the angular momentum with explicit inertia tensor $\mathbb{I}_D$ reads

$$\mathbf{H}_A = \mathbf{r}_{AD} \times \mathbf{p} + \mathbf{r}_{DC} \times m\mathbf{v}_D + \mathbb{I}_D \boldsymbol{\omega}. \quad (18)$$

This formula is also not very intuitive: in the first term $\mathbf{r}_{AD} \times \mathbf{p}$ is not a pure moment of momentum and in the second term $m\mathbf{v}_D$ is not a pure momentum.

Despite this interpretation problems one should note, that formula (18) is used commonly in practical applications (c.f., [3, 14]). In fact, the use of mass center for definition of angular momentum is not necessary, since usually it is much convenient to chose as the origin of the body coordinate system, some geometric point of a structure (rocket, aircraft, car, ship, etc.) instead of the mass center, which may change in moving of the body (due to fuel consumption, jettisoning of a cargo, deformation of the structure, etc.).

All the examples presented above show that the interpretation and understanding of angular momentum is not necessarily a straightforward task, especially for the beginners. The non-obviousness in the concept of angular momentum may result in some doubts in understanding its nature, especially on the basic level of university education, all the more since the questions discussed here are omitted in a large majority of books (In fact, Authors do not know any book or paper which discusses this question.). In particular, textbooks on classical mechanics do not consider these problems at all. This remark concerns both the books published in the West (e.g., [5–11]), and those used in the East [4, 12–14]. The lack of these results in purely scientific books (e.g., [1–4]), may be explained, to some extent, by the fact that they seem less important from the scientific point of view.

4. Canonical forms of dynamics quantities

In the light of the above results it may be stated, that the definition (11) may be considered as the basic (canonical) form of the angular momentum of a rigid body. The term *canonical* is used here in the meaning "basic", "simplest" or "most elementary", and should not be confused with its use for canonical equations of mechanics. This is evident if we remind that in the definitions of other quantities: linear momentum and kinetic energy, the two forms of motion are decoupled and the mass center is distinguished. Choosing the mass center C as a reference point for the angular momentum, the canonical definitions of all three fundamental quantities of classical mechanics take the form

$$\left. \begin{aligned} \mathbf{p} &= m\mathbf{v}_C \\ \mathbf{H}_C &= \mathbb{I}_C \boldsymbol{\omega} \\ T &= \frac{1}{2}m\mathbf{v}_C^2 + \frac{1}{2}\boldsymbol{\omega} \cdot \mathbb{I}_C \boldsymbol{\omega} \end{aligned} \right\} \quad (19)$$

Here, the linear momentum $\mathbf{p}$ depends only on the linear velocity $\mathbf{v}_C$, the angular momentum $\mathbf{H}_C$ depends only on the angular velocity $\boldsymbol{\omega}$, whereas the kinetic energy,

as a "composed" quantity, contains contributions of both forms of motion. Note also, that the definition of kinetic energy of the rigid body, expressed by the K oenig's theorem [5], has just the canonical form.

5. Conclusions

The results of this paper concerning angular momentum of a rigid body in pure rotation and general motion focus the attention on a true nature of this quantity. They give a deeper insight into the structure of a general movement of a rigid body and resolve non-obviousness of the working formulas. This may be helpful in better understanding the role of angular momentum in classical mechanics.

The presented results and interpretations are, in our opinion, consistent and elegant. They may be used in didactics of mechanics, especially on basic university level; therefore including them in the textbooks might be valuable.

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