10.24425/acs.2025.157148

*Archives of Control Sciences*
*Volume* 35(LXXI), 2025
*No.* 4*, pages 793–833*

# Controllability of mild solutions for fractional neutral evolution equations with state-dependent delay

Selma BAGHLI-BENDIMERAD and Chahrazed BOUDEFLA

This paper examines the controllability of mild solutions for a specific class of neutral fractional evolution equations with finite state-dependent delays in Fréchet space, employing Caputo fractional derivatives. The study combines Avramescu's nonlinear alternatives in Fréchet spaces and semigroup theory, establishing sufficient conditions to the controllability of these solutions.

**Key words:** functional evolution equations, neutral problems, Caputo's fractional derivative, mild solution, controllability, state-dependent delay, fixed point, nonlinear alternative, semigroup theory, Fréchet spaces

## 1. Introduction

In this paper, we investigate the controllability of mild solutions for a class of neutral fractional evolution equations with finite state-dependent delay in a real Banach space $(E, |\cdot|)$.

In Section 3, we consider the following neutral functional evolution equation

$$^{c}D_0^{\alpha}\left[y(t) - g(t, y_{\rho(t,y_t)})\right] = A(t)y(t) + Cu(t) + f(t, y_{\rho(t,y_t)}),\quad \text{a.e. } t \in J := [0, +\infty), \tag{1}$$

$$y(t) = \varphi(t), \qquad t \in H := [-r, 0] \tag{2}$$

where $r > 0$, $\varphi \in C(H, E)$, $f,\ g : J \times C(H, E) \to E$ and $\rho : J \times C(H, E) \to [-r, +\infty)$ are given functions; $^{c}D_0^{\alpha}$ is the Caputo's fractional derivative for



S. Baghli-Bendimerad (corresponding author, e-mail: selmabaghli@gmail.com) and C. Boudefla (e-mail: chahrazed.boudefla@univ-sba.dz) are with Po. Box 89, Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbes 22000, Algeria.

This paper was supported by General Directorate for Scientific Research and Technological Development (DGRSDT), University-Training Research Projects: C00L03UN220120210002 PRFU 2021 project.

Received 28.10.2024 Revised 25.10.2025.

$\alpha \in (0,1)$; the control function $u(\cdot)$ is given in $L^2(J,E)$ is the Banach space of admissible control function; $C$ is a bounded linear operator from $E$ into $E$ and $\{A(t)\}_{t\in J}$ is a family of linear closed (not necessarily bounded) operators from $E$ into $E$.

For any continuous function $y$ defined on $[-r,+\infty)$ and any $t \in J$, we denote by $y_t$ the element of $C(H,E)$ defined by

$$y_t(\theta) = y(t+\theta) \quad \text{for } \theta \in H.$$

Here $y_t(\cdot)$ represents the history of the state from time $t-r$ up to the present time $t$.

Then in Section 4, we give an example illustrating the result of this paper.

Fractional calculus is a branch of mathematical analysis that extends integral and differential operations to non-integer orders. It finds applications in various fields such as applications in chemistry, physics, mathematical biology, fractal media, electromagnetism, statistical mechanics, etc. Technical applications results and inductance separation performs can be related as in [11] and [12]. The field has attracted attention for its ability to describe phenomena related to long-range dependence, memory, and genetic traits. Applications include modeling viscoelastic materials, describing anomalous diffusion, and analyzing circuits with component damage. See Kilbas *et al.* [22], Podlubny [29], and references therein for more details.

Control theory is an area of applied mathematics concerned with the analysis and design of control systems. The concept of controllability plays an important role in various fields of science and technology. More specifically, the controllability problem involves the existence of a control function that governs the solution of a system from an initial state to a final state, where these two states can vary spatially.

In recent years, considerable attention has been devoted to exploring the existence and controllability of first-order function evolution equations with both finite and infinite time delays. Baghli and Benchohra have played a pivotal role in advancing this area of research, as evident from their referenced papers [4, 9, 10]. Concurrently, the investigation of state-dependent time-delay functional differential equations has seen a notable rise in interest due to their significance across various scientific and engineering applications, including chemical engineering, life sciences, dynamical systems, and medicine. Numerous studies have contributed to this field, with notable works for first and second order evolution equations with state-dependent delay with local and nonlocal conditions such as those by Baghli *et al.* in [2, 3, 5, 7, 8, 13–15, 26].

The existence of solutions for various kinds of fractional evolution has been investigated by many authors. In recent years, El-Borai [16] discussed the existence and uniqueness of mild solutions to the Cauchy problem in a Banach space fractional evolution equation. Using the Laplace transform, he provided equivalent integral equations for a class of fractional evolution equations, expressed in terms of certain probability densities. Mesri and Benchohra utilized non-compactness measures to investigate fractional non-autonomous evolution equations in the Fréchet space in [27].

Very recently, Lachachi *et al.* in [23–25] have conducted studies on the existence and uniqueness of mild solutions to various perturbed and non-perturbed partial functional and neutral functional evolution problems with finite state-dependent delay in Banach spaces

$$\begin{aligned} {}^{c}D_0^{\alpha} y(t) - A(t)y(t) &= f(t, y_{\rho(t,y_t)}) + w(t, y_{\rho(t,y_t)}) \\ &\quad + {}^{c}D_0^{\alpha} g(t, y_{\rho(t,y_t)}), \quad \text{a.e. } t \in [0,b], \ \ b > 0, \\ y(t) &= \varphi(t), \qquad t \in H := [-r, 0], \ \ r > 0, \end{aligned}$$

where $f,\ w,\ g : J \times C(H,E) \to E$ and $\rho : J \times C(H,E) \to [-r,b]$ are given functions.

The main aim of this paper is to extend the controllability results previously established by Baghli *et al.* to encompass the fractional derivation problem (1)–(2). Utilizing Avramescu's nonlinear alternative method [6] for the sum of compact operators and contraction maps in Fréchet spaces, in conjunction with semigroup theory [28], we establish sufficient conditions for the controllability of mild solutions for neutral evolution equations with finite state-dependent delay involving Caputo's fractional derivative.

## 2. Preliminaries

In this section, we present some notations, definitions, lemmas and theorems that will be useful in this paper.

Let $C(H,E)$ be the Banach space of continuous functions from $H$ into $E$ with the norm

$$\|y\| = \sup\{|y(t)| : \ t \in H\}.$$

Let $B(E)$ be all bounded linear operators' space from $E$ into $E$, with the norm

$$\|N\|_{B(E)} = \sup\{|N(y)| : |y| = 1\}.$$

A measurable function $y : J \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. Let $L^1(J,E)$ be the Banach space of measurable functions

$y : J \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^{+\infty} |y(t)| \mathrm{d}t.$$

Let $X$ be a Fréchet space with a semi-norms family $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. We assume that the semi-norms family $\{\|\cdot\|_n\}$ verifies:

$$\|x\|_1 \leqslant \|x\|_2 \leqslant \|x\|_3 \leqslant \cdots \quad \text{for every } x \in X.$$

Let $Y \subset X$, we say that $Y$ is bounded if for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|y\|_n \leqslant \overline{M}_n \quad \text{for all } y \in Y.$$

**Definition 1.** *[19] A function $f : J \times C(H, E) \to E$ is said to be an $L^1_{loc}$-Carathéodory function if it satisfies:*

*(i) for each $t \in J$ the function $f(t, \cdot) : C(H, E) \to E$ is continuous;*

*(ii) for each $y \in C(H, E)$ the function $f(\cdot, y) : J \to E$ is measurable;*

*(iii) for every positive integer $q$ there exists $\vartheta_q \in L^1_{loc}(J, \mathbb{R}^+)$ such that*

$$|f(t, y)| \leqslant \vartheta_q(t) \quad \textit{for all } \|y\| \leqslant q \textit{ and almost each } t \in J.$$

Set

$$\mathcal{R}(\rho^-) = \{\rho(s, \varphi) : (s, \varphi) \in J \times C(H, E),\ \rho(s, \varphi) \leqslant 0\}.$$

We always assume that $\rho : J \times C(H, E) \to [-r, +\infty)$ is continuous. Additionally, we introduce the following hypothesis:

$(H_\varphi)$ The function $t \to \varphi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $C(H, E)$ and there exists a continuous and bounded function $\mathcal{L}^\varphi : \mathcal{R}(\rho^-) \to (0, +\infty)$ such that

$$\|\varphi_t\| \leqslant \mathcal{L}^\varphi(t)\|\varphi\| \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

**Remark 1.** *Continuous and bounded functions verified frequently the condition $(H_\varphi)$ (see [1, 21]).*

**Lemma 1.** *[21] If $y : [-r, +\infty) \to E$ is a function such that $y_0 = \varphi$, then*

$$\|y_s\| \leqslant \mathcal{L}^\varphi\|\varphi\| + \sup\{|y(\theta)|;\ \theta \in [0, \max\{0, s\}]\},\ s \in \mathcal{R}(\rho^-) \cup J$$

*where* $\mathcal{L}^\varphi = \sup\limits_{t\in\mathcal{R}(\rho^-)} \mathcal{L}^\varphi(t)$.

**Proposition 1.** *From $(H_\varphi)$, $(A_1)$ and Lemma 1, for all $t \in [0, n]$ and $n \in \mathbb{N}$ we have*

$$\|y_{\rho(t,y_t)}\| \leqslant |y(t)| + \mathcal{L}^\varphi\|\varphi\|.$$

**Definition 2.** *[22, 29] The Riemann-Liouville fractional integral operator of order $\alpha > 0$ for a function $f : \mathbb{R}^+ \to \mathbb{R}$ is defined as*

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s,$$

*where $\Gamma(\cdot)$ is the Euler's gamma function defined by the integral $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t}\,\mathrm{d}t$.*

For instance, $I^\alpha f$ exists for all $\alpha > 0$ when $f \in C(\mathbb{R}^+) \cap L^1_{loc}(\mathbb{R}^+)$. Note also that when $f \in C(\mathbb{R}^+)$ then $I^\alpha f \in C(\mathbb{R}^+)$ and moreover $I^\alpha f(0) = 0$.

**Definition 3.** *[22, 29] The Caputo fractional derivative of order $\alpha > 0$ for a function $f : \mathbb{R}^+ \to \mathbb{R}$ is defined by*

$${}^cD_0^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^{(m)}(s)\,\mathrm{d}s = I_0^{m-\alpha} f^{(m)}(t).$$

*Here $m = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of $\alpha$.*

In what follows, we assume that $\{A(t)\}_{t\in J}$ is a family of closed densely defined linear operators not necessarily bounded on the Banach space $E$ with domain $D(A(t))$ independent of $t$. Additionally, throughout this paper, we assume that the linear operator $A(t)$ satisfies the following conditions (see [16], for more details):

($A_1$) For any $\lambda$ with $Re(\lambda) \geqslant 0$, the operator $\lambda I - A(t)$ exists a bounded inverse operator $(\lambda I - A(t))^{-1}$ in $B(E)$ and

$$\left\|(\lambda I - A(t))^{-1}\right\| \leqslant \frac{K}{|\lambda| + 1},$$

where $K$ is a positive constant independent of both $t$ and $\lambda$.

($A_2$) For any $t, \tau, s \in I$ , there exists a constant $\gamma \in (0, 1]$ such that

$$\|[A(t) - A(\tau)]A^{-1}(s)\| \leqslant K|t-\tau|^\gamma,$$

where the constants $\gamma$ and $K > 0$ are independent of both $t, \tau$ and $s$.

**Remark 2.** *From Henry [20], Pazy [28] and Temam [31], we know that the assumption ($A_1$) means that for each $s \in I$, the operator $A(s)$ generates an analytic semigroup $e^{-tA(s)}$ ($t > 0$), and there exists a positive constant $K$ independent of both $t$ and $s$ such that*

$$\left\|-A(s)e^{tA(s)}\right\| \leqslant \frac{K}{t},$$

*where $t > 0$ and $s \in J$.*

**Definition 4.** *[16] Define the operators $\Psi(t,s)$, $\phi(t,s)$ and $U(t)$ by*

$$\Psi(t,s) = \alpha \int_0^\infty \theta t^{\alpha-1} \xi_\alpha(\theta) e^{t^\alpha \theta A(s)} \mathrm{d}\theta,$$

$$\phi(t,s) = \sum_{k=1}^{+\infty} \phi_k(t,s)$$

*and*

$$U(t) = -A(t)A^{-1}(0) - \int_0^t \phi(t,s)A(s)A^{-1}(0)\,\mathrm{d}s,$$

*where $\xi_\alpha$ is a probability density function defined on $[0,+\infty)$ such that its Laplace transform is given by*

$$\int_0^\infty \xi_\alpha(\theta) e^{\theta x} \mathrm{d}\theta = \sum_{i=1}^{\infty} \frac{(-x)^i}{\Gamma(1+\alpha i)} \quad 0 < \alpha \leqslant 1,\ x > 0,$$

$$\phi_1(t,s) = [-A(t) + A(s)]\Psi(t-s,s),$$

*and*

$$\phi_{k+1}(t,\eta) = \int_s^t \phi_k(t,\tau)\phi_1(\tau,s)\,\mathrm{d}\tau, \quad k = 1,2,\ldots.$$

For more details about the definition and property of the probability density function, one can see the paper [18].

The following properties about the operators $\Psi$, $\phi$ and $U$ will be needed in our argument.

**Lemma 2.** *[16] The operator-valued functions $\Psi(t-s,s)$ and $A(t)\Psi(t-s,s)$ are continuous in uniform topology about the variables $t$ and $s$, where $t \in J$, $0 \leqslant s \leqslant t - \epsilon$ for any $\epsilon > 0$, and*

$$\|\Psi(t-s,s)\| \leqslant K(t-s)^{\alpha-1}, \tag{3}$$

*where $K$ is a positive constant independent of both $t$ and $s$. Furthermore,*

$$\|\phi(t,s)\| \leqslant K(t-s)^{\gamma-1} \tag{4}$$

*and*

$$\|U(t)\| \leqslant K(1+t^\gamma). \tag{5}$$

Now, we present the following integrals which will be widely used in the proof of our main result

$$I_1 = \int_0^t (t-s)^{\alpha-1}(1+s^\gamma)\,\mathrm{d}s = t^\alpha \left(\alpha^{-1} + t^\gamma \beta(\alpha,\gamma+1)\right), \tag{6}$$

$$I_2 = \int_0^t (t-s)^{\alpha-1}\,\mathrm{d}s = t^\alpha \alpha^{-1} \tag{7}$$

and

$$I_3 = \int_0^t \int_0^s (t-s)^{\alpha-1}(s-\tau)^{\gamma-1}\,\mathrm{d}\tau\,\mathrm{d}s = t^{\alpha+\gamma}\gamma^{-1}\beta(\alpha,\gamma+1), \tag{8}$$

where $\beta(\cdot,\cdot)$ is the beta Euler's function defined by $\beta(\alpha,\gamma) = \int_0^1 t^{\alpha-1}(1-t)^{\gamma-1}\,\mathrm{d}t$.

**Definition 5.** *[17] A function $f\colon X \to X$ is said to be a contraction if there exists $\kappa \in [0,1)$ such that:*

$$\|f(x) - f(y)\| \leqslant \kappa\|x-y\| \quad \text{for all } x,y \in X.$$

Now we present the Avramescu nonlinear alternative in Fréchet space used in this paper, which is an extension of the Burton and Kirk alternative in Banach space. For more details, the reader is referred to [6] and the references therein.

**Theorem 1.** *Nonlinear Alternative of Avramescu*

*Let $X$ be a Fréchet space and let $A,B\colon X \to X$ be two operators satisfying:*

*(1) $A$ is a compact operator,*

*(2) $B$ is a contraction.*

*Then either one of the following statements holds:*

*(C1) The operator $A+B$ has a fixed point;*

*(C2) The set $\left\{x \in X,\ x = \lambda A(x) + \lambda B\left(\frac{x}{\lambda}\right),\ \text{for some } 0<\lambda<1\right\}$ is unbounded.*

## 3. Semilinear Evolution Equation

In this section, we give the controllability results for the problem (1)−(2). Before stating and proving this result, we introduce the definition of mild solutions for the neutral fractional evolution problem (1)–(2), which is obtained from El-Borai's work in [16].

**Definition 6.** *We say that the continuous function $y(\cdot) : [-r, +\infty) \to E$ is a mild solution of (1)–(2) if $y(t) = \varphi(t)$ for $t \in H$ and $y$ satisfies the following integral equation*

$$\begin{aligned} y(t) &= \varphi(0) - g(0,\varphi) + g(t, y_{\rho(t,y_t)}) \\ &\quad - \int_0^t \Psi(t-s,s)U(s)A(0)[\varphi(0) - g(0,\varphi)]\,\mathrm{d}s \\ &\quad - \int_0^t \Psi(t-s,s)A(0)g(s, y_{\rho(s,y_s)})\,\mathrm{d}s + \int_0^t \Psi(t-s,s)Cu(s)\,\mathrm{d}s \\ &\quad + \int_0^t\int_0^s \Psi(t-s,s)\phi(s,\tau)Cu(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + \int_0^t \Psi(t-s,s)f(s, y_{\rho(s,y_s)})\,\mathrm{d}s \\ &\quad + \int_0^t\int_0^s \Psi(t-s,s)\phi(s,\tau)f(\tau, y_{\rho(\tau,y_\tau)})\,\mathrm{d}\tau\,\mathrm{d}s, \qquad \textit{for each } t \in J. \end{aligned} \tag{9}$$

**Definition 7.** *The neutral evolution problem (1)–(2) is said to be controllable if for every initial function $\varphi \in C(H,E)$, $\widehat{y} \in E$ and $n \in \mathbb{N}$, there is some control $u \in L^2([0,n],E)$ such that the mild solution $y(\cdot)$ of (1)–(2) satisfies the terminal condition*

$$y(n) = y^\star. \tag{10}$$

We will need to introduce the following hypotheses which are assumed thereafter:

$(H1)$ The function $f$ is Carathéodory.

$(H2)$ There exists a continuous function $p : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t,u)\| \leqslant p(t)(1 + \|u\|); \text{ for a.e. } t \in \mathbb{R}^+, \text{ and each } u \in E.$$

For $n \in \mathbb{N}$, let $p_n^* = \sup\limits_{t\in[0,n]} p(t)$.

$(H3)$ For all $R > 0$, there exists $l_R \in L^1_{loc}(J, \mathbb{R}^+)$ such that:

$$|f(t,u) - f(t,v)| \leqslant l_R(t)\|u - v\|$$

for all $u, v \in C(H,E)$ with $\|u\| \leqslant R$ and $\|v\| \leqslant R$.
Set $l_n^* = \text{ess} \sup\limits_{t\in[0,n]} l_n(t)$.

$(H4)$ For each $n \in \mathbb{N}$, the linear operator $W : L^2([0,n],E) \longrightarrow E$ is defined by

$$Wu = \int_0^n \Psi(n-s,s)Cu(s)\,\mathrm{d}s + \int_0^n\int_0^s \Psi(n-s,s)\phi(s,\tau)Cu(\tau)\,\mathrm{d}\tau\,\mathrm{d}s,$$

has a pseudo invertible operator $\widetilde{W}^{-1}$ which takes values in $L^2([0,n],E)/\ker W$ and there exists positive constants $\widetilde{M}$ and $\widetilde{M}_1$ such that:

$$\|C\|_{B(E)} \leqslant \widetilde{M} \qquad \text{and} \qquad \|\widetilde{W}^{-1}\|_{B(E)} \leqslant \widetilde{M}_1.$$

$(H5)$ There exists a constant $\overline{M}_0 > 0$ such that

$$\|A^{-1}(t)\|_{B(E)} \leqslant \overline{M}_0 \qquad \text{for all } t \in J.$$

$(H6)$ There exists a constant $0 < L < \frac{1}{\overline{M}_0}$ such that

$$|A(t)g(t,\varphi)| \leqslant L\left(\|\varphi\|+1\right), \qquad \text{for all } t \in J \text{ and } \varphi \in C(H,E).$$

$(H7)$ The function $g$ is completely continuous and for any bounded set $Q \subset C(H,E)$ the set
$\{t \longrightarrow g(t, y_{\rho(t,y_t)}) : y \in Q\}$ is equicontinuous in $C(J,E)$.

$(H8)$ There exists a constant $\Upsilon_n = \alpha^{-1} + Kn^{\gamma}\gamma^{-1}\beta(\alpha,\gamma+1)$ for $n \in \mathbb{N}$ such that $l_n^* < K^{-1}n^{-\alpha}\Upsilon_n^{-1}$ and
$p_n^* < \Upsilon_n^{-1}\left[K^{-1}n^{-\alpha}\left(K\widetilde{M}\widetilde{M}_1 n^{\alpha}\Upsilon_n + 1\right)^{-1}\left(1-\overline{M}_0 L\right) - \overline{M}_0 L|A(0)|\alpha^{-1}\right].$

**Remark 3.** *For the construction of* $\widetilde{W}^{-1}$*, see the paper of Quinn and Carmichael [30].*

For every $n \in \mathbb{N}$, we define in $C([-r,+\infty),E)$ the semi-norms by:

$$\|y\|_n = \sup_{t\in[0,n]} |y(t)|, \qquad \text{for } n \in \mathbb{N}.$$

Then $C([-r,+\infty),E)$ is a Fréchet space with these family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$.

**Theorem 2.** *Assume that hypotheses* $(H_\varphi)$ *and* $(H1)$*-*$(H8)$ *are satisfied. Then, the fractional evolution equation with finite state-dependent delay* (1)–(2) *is controllable on* $[-r,+\infty)$.

**Proof.** We transform the problem (1)–(2) into a fixed-point problem. For that, let us consider the operator $N : C([-r,+\infty),E) \to C([-r,+\infty),E)$ defined by

$$N(y)(t) = \begin{cases} \varphi(t), & \text{if } t \in H; \\ \varphi(0) - g(0,\varphi) + g(t, y_{\rho(t,y_t)}) & \\ \quad - \displaystyle\int_0^t \Psi(t-s,s)U(s)A(0)[\varphi(0) - g(0,\varphi)]\,\mathrm{d}s & \\ \quad - \displaystyle\int_0^t \Psi(t-s,s)A(0)g(s, y_{\rho(s,y_s)})\,\mathrm{d}s & \\ \quad + \displaystyle\int_0^t \Psi(t-s,s)Cu_y(s)\,\mathrm{d}s & \\ \quad + \displaystyle\int_0^t\int_0^s \Psi(t-s,s)\phi(s,\tau)Cu_y(\tau)\,\mathrm{d}\tau\,\mathrm{d}s & \\ \quad + \displaystyle\int_0^t \Psi(t-s,s)f(s, y_{\rho(s,y_s)})\,\mathrm{d}s & \\ \quad + \displaystyle\int_0^t\int_0^s \Psi(t-s,s)\phi(s,\tau)f(\tau, y_{\rho(\tau,y_\tau)})\,\mathrm{d}\tau\,\mathrm{d}s & \text{if } t \in J. \end{cases}$$

Using assumption $(H4)$, for arbitrary function $y(\cdot)$, we define the control for each $t \in [0,n]$ and $n \in \mathbb{N}$

$$\begin{aligned} u_y(t) = \widetilde{W}^{-1}\Bigg[ & y^\star - \varphi(0) + g(0,\varphi) - g(n, y_{\rho(n,y_n)}) \\ & + \int_0^n \Psi(n-s,s)U(s)A(0)[\varphi(0) - g(0,\varphi)]\,\mathrm{d}s \\ & + \int_0^n \Psi(n-s,s)A(0)g(s, y_{\rho(s,y_s)})\,\mathrm{d}s - \int_0^n \Psi(n-s,s)f(s, y_{\rho(s,y_s)})\,\mathrm{d}s \\ & - \int_0^n\int_0^s \Psi(n-s,s)\phi(s,\tau)f(\tau, y_{\rho(\tau,y_\tau)})\,\mathrm{d}\tau\,\mathrm{d}s \Bigg](t). \end{aligned}$$

By the hypothesis $(H4)$ and inequalities (3), (4) and (5), we get for each $t \in [0, n]$ and $n \in \mathbb{N}$

$$
\begin{aligned}
|u_y(t)| \leqslant \|\widetilde{W}^{-1}\|_{B(E)} \Bigg[ & |y^\star| + |\varphi(0)| + |g(0,\varphi)| + |g(n, y_{\rho(n,y_n)})| \\
& + \int_0^n \|\Psi(n-s,s)\| \|U(s)\| |A(0)| \left(|\varphi(0)| + |g(0,\varphi)|\right) \mathrm{d}s \\
& + \int_0^n \|\Psi(n-s,s)\| |A(0)| |g(s, y_{\rho(s,y_s)})| \mathrm{d}s \\
& + \int_0^n \|\Psi(n-s,s)\| |f(s, y_{\rho(s,y_s)})| \mathrm{d}s \\
& + \int_0^n \int_0^s \|\Psi(n-s,s)\| \|\phi(s,\tau)\| |f(\tau, y_{\rho(\tau,y_\tau)})| \mathrm{d}\tau \mathrm{d}s \Bigg] \\
\leqslant \widetilde{M}_1 \Bigg[ & |y^\star| + \|\varphi\| + \left\|A^{-1}(0)\right\| |A(0) g(0,\varphi)| \\
& + \left\|A^{-1}(n)\right\| \left|A(n) g(n, y_{\rho(n,y_n)})\right| \\
& + K^2 |A(0)| \int_0^n (n-s)^{\alpha-1} (1+s^\gamma) \left( \|\varphi\| + \left\|A^{-1}(0)\right\| |A(0) g(0,\varphi)| \right) \mathrm{d}s \\
& + K |A(0)| \int_0^n (n-s)^{\alpha-1} \left\|A^{-1}(s)\right\| \left|A(s) g(s, y_{\rho(s,y_s)})\right| \mathrm{d}s \\
& + K \int_0^n (n-s)^{\alpha-1} |f(s, y_{\rho(s,y_s)})| \mathrm{d}s \\
& + K^2 \int_0^n \int_0^s (n-s)^{\alpha-1} (s-\tau)^{\gamma-1} |f(\tau, y_{\rho(\tau,y_\tau)})| \mathrm{d}\tau \mathrm{d}s \Bigg].
\end{aligned}
$$

Using $(H5)$ and $(H6)$, we obtain for each $t \in [0,n]$ and $n \in \mathbb{N}$

$$
\begin{aligned}
|u_y(t)| \leqslant \widetilde{M}_1 \Bigg[ & |y^\star| + \|\varphi\| + \overline{M}_0 L \left(\|\varphi\| + 1\right) + \overline{M}_0 L \left(\|y_{\rho(n,y_n)}\| + 1\right) \\
& + K^2 |A(0)| \int_0^n (n-s)^{\alpha-1}(1+s^\gamma) \Big(\|\varphi\| + \overline{M}_0 L \left(\|\varphi\| + 1\right)\Big) \, \mathrm{d}s \\
& + K |A(0)| \int_0^n (n-s)^{\alpha-1} \overline{M}_0 L \left(\|y_{\rho(s,y_s)}\| + 1\right) \mathrm{d}s \\
& + K \int_0^n (n-s)^{\alpha-1} |f(s, y_{\rho(s,y_s)})| \mathrm{d}s \\
& + K^2 \int_0^n \int_0^s (n-s)^{\alpha-1}(s-\tau)^{\gamma-1} |f(\tau, y_{\rho(\tau,y_\tau)})| \mathrm{d}\tau \, \mathrm{d}s \Bigg].
\end{aligned}
$$

Then, for each $t \in [0,n]$ and $n \in \mathbb{N}$

$$
\begin{aligned}
|u_y(t)| \leqslant \widetilde{M}_1 \Bigg[ & |y^\star| + \left(\overline{M}_0 L + 1\right) \|\varphi\| + 2\overline{M}_0 L + \overline{M}_0 L \left\|y_{\rho(n,y_n)}\right\| \\
& + K^2 |A(0)| \left(\left(\overline{M}_0 L + 1\right) \|\varphi\| + \overline{M}_0 L\right) \int_0^n (n-s)^{\alpha-1}(1+s^\gamma) \mathrm{d}s \\
& + K \overline{M}_0 L |A(0)| \int_0^n (n-s)^{\alpha-1} \mathrm{d}s \\
& + K \overline{M}_0 L |A(0)| \int_0^n (n-s)^{\alpha-1} \left\|y_{\rho(s,y_s)}\right\| \mathrm{d}s \\
& + K \int_0^n (n-s)^{\alpha-1} |f(s, y_{\rho(s,y_s)})| \mathrm{d}s \\
& + K^2 \int_0^n \int_0^s (n-s)^{\alpha-1}(s-\tau)^{\gamma-1} |f(\tau, y_{\rho(\tau,y_\tau)})| \mathrm{d}\tau \, \mathrm{d}s \Bigg]. \qquad (11)
\end{aligned}
$$

Applying $(H2)$ and using inequalities (6) and (7), we have for each $t \in [0, n]$ and $n \in \mathbb{N}$

$$\begin{aligned}
|u_y(t)| \leqslant \widetilde{M}_1 \Bigg[ |y^\star| &+ \left(\overline{M}_0 L + 1\right) \left(1 + K^2 |A(0)| n^\alpha \left(\alpha^{-1} + n^\gamma \beta(\alpha, \gamma+1)\right)\right) \|\varphi\| \\
&+ \overline{M}_0 L \Big(2 + K^2 |A(0)| n^\alpha \left(\alpha^{-1} + n^\gamma \beta(\alpha, \gamma+1)\right)\Big) + \overline{M}_0 L \left\|y_{\rho(n, y_n)}\right\| \\
&+ K \overline{M}_0 L n^\alpha \alpha^{-1} |A(0)| + K \overline{M}_0 L |A(0)| \int_0^n (n-s)^{\alpha-1} \left\|y_{\rho(s, y_s)}\right\| \mathrm{d}s \\
&+ K \int_0^n (n-s)^{\alpha-1} p(s) (1 + \|y_{\rho(s, y_s)}\|) \mathrm{d}s \\
&+ K^2 \int_0^n \int_0^s (n-s)^{\alpha-1} (s-\tau)^{\gamma-1} p(\tau) (1 + \|y_{\rho(\tau, y_\tau)}\|) \mathrm{d}\tau \mathrm{d}s \Bigg] \\
\leqslant \widetilde{M}_1 \Bigg[ |y^\star| &+ \left(\overline{M}_0 L + 1\right) \left(1 + K^2 |A(0)| n^\alpha \left(\alpha^{-1} + n^\gamma \beta(\alpha, \gamma+1)\right)\right) \|\varphi\| \\
&+ \overline{M}_0 L \Big(2 + K^2 |A(0)| n^\alpha \left(\alpha^{-1} + n^\gamma \beta(\alpha, \gamma+1)\right)\Big) + \overline{M}_0 L \left\|y_{\rho(n, y_n)}\right\| \\
&+ K \overline{M}_0 L n^\alpha \alpha^{-1} |A(0)| + K \overline{M}_0 L |A(0)| \int_0^n (n-s)^{\alpha-1} \left\|y_{\rho(s, y_s)}\right\| \mathrm{d}s \\
&+ K p_n^* \int_0^n (n-s)^{\alpha-1} \mathrm{d}s + K^2 p_n^* \int_0^n \int_0^s (n-s)^{\alpha-1} (s-\tau)^{\gamma-1} \mathrm{d}\tau \mathrm{d}s \\
&+ K p_n^* \int_0^n (n-s)^{\alpha-1} \|y_{\rho(s, y_s)}\| \mathrm{d}s \\
&+ K^2 p_n^* \int_0^n \int_0^s (n-s)^{\alpha-1} (s-\tau)^{\gamma-1} \|y_{\rho(\tau, y_\tau)}\| \mathrm{d}\tau \mathrm{d}s \Bigg].
\end{aligned}$$

Using inequalities (7) and (8), we obtain for each $t \in [0, n]$ and $n \in \mathbb{N}$

$$\begin{aligned}
|u_y(t)| \leqslant \widetilde{M}_1 \Bigg[ |y^\star| &+ \left(\overline{M}_0 L + 1\right) \left(1 + K^2 |A(0)| n^\alpha \Theta_n\right) \|\varphi\| \\
&+ \overline{M}_0 L \Big(2 + K |A(0)| n^\alpha \left(\alpha^{-1} + K \Theta_n\right)\Big) + \overline{M}_0 L \left\|y_{\rho(n, y_n)}\right\|
\end{aligned}$$

$$
\begin{aligned}
&+K\overline{M}_0L|A(0)|\int_0^n(n-s)^{\alpha-1}\left\|y_{\rho(s,y_s)}\right\|\mathrm{d}s\\
&+Kp_n^*n^\alpha\left(\alpha^{-1}+Kn^\gamma\gamma^{-1}\beta(\alpha,\gamma+1)\right)+Kp_n^*\int_0^n(n-s)^{\alpha-1}\|y_{\rho(s,y_s)}\|\mathrm{d}s\\
&+K^2p_n^*\int_0^n\int_0^s(n-s)^{\alpha-1}(s-\tau)^{\gamma-1}\|y_{\rho(\tau,y_\tau)}\|\mathrm{d}\tau\mathrm{d}s\Bigg].
\end{aligned}
$$

So, for each $t\in[0,n]$ and $n\in\mathbb{N}$, we have

$$
\begin{aligned}
|u_y(t)|\leqslant \widetilde{M}_1\Bigg[&|y^\star|+\left(\overline{M}_0L+1\right)\left(1+K^2|A(0)|n^\alpha\Theta_n\right)\|\varphi\|\\
&+\overline{M}_0L\Big(2+K|A(0)|n^\alpha\left(\alpha^{-1}+K\Theta_n\right)\Big)+Kp_n^*n^\alpha\Upsilon_n+\overline{M}_0L\left\|y_{\rho(n,y_n)}\right\|\\
&+K\overline{M}_0L|A(0)|\int_0^n(n-s)^{\alpha-1}\left\|y_{\rho(s,y_s)}\right\|\mathrm{d}s\\
&+Kp_n^*\int_0^n(n-s)^{\alpha-1}\|y_{\rho(s,y_s)}\|\mathrm{d}s\\
&+K^2p_n^*\int_0^n\int_0^s(n-s)^{\alpha-1}(s-\tau)^{\gamma-1}\|y_{\rho(\tau,y_\tau)}\|\mathrm{d}\tau\mathrm{d}s\Bigg].
\end{aligned}
\tag{12}
$$

Using this control, we shall show that the operator $N$ has a fixed point $y(\cdot)$ which is the mild solution of the evolution equation (1)–(2).

Define the operators $F,\ G: C([-r,+\infty),E)\to C([-r,+\infty),E)$ by

$$
F(y)(t)=\begin{cases}
\varphi(t), & \text{if } t\in H;\\
\varphi(0)-g(0,\varphi)+g(t,y_{\rho(t,y_t)})\\
-\int_0^t\Psi(t-s,s)U(s)A(0)[\varphi(0)-g(0,\varphi)]\mathrm{d}s\\
-\int_0^t\Psi(t-s,s)A(0)g(s,y_{\rho(s,y_s)})\mathrm{d}s+\int_0^t\Psi(t-s,s)Cu_y(s)\mathrm{d}s\\
+\int_0^t\int_0^s\Psi(t-s,s)\phi(s,\tau)Cu_y(\tau)\mathrm{d}\tau\mathrm{d}s, & \text{if } t\in J.
\end{cases}
$$

and

$$G(y)(t)=\begin{cases}0, & \text{if } t\in H;\\ \displaystyle\int_0^t \Psi(t-s,s)f(s,y_{\rho(s,y_s)})\,\mathrm{d}s \\ \displaystyle+\int_0^t\int_0^s \Psi(t-s,s)\phi(s,\tau)f(\tau,y_{\rho(\tau,y_\tau)})\,\mathrm{d}\tau\,\mathrm{d}s, & \text{if } t\in J.\end{cases}$$

Clearly the operator $N$ has a fixed point is equivalent to $F+G$ has one, so it turns to prove that $F+G$ has a fixed point. The proof will be given in the following steps.

**Step 1:** $F$ is continuous.

Let $(y_k)_{k\in\mathbb{N}}$ be a sequence in $C([-r,+\infty),E)$ such that $y_k \longrightarrow y$ in $C([-r,+\infty),E)$. Then

$$\begin{aligned}|F(y_k)(t)-F(y)(t)| &\leqslant \left|g(t,y_{k\rho(t,y_{kt})})-g(t,y_{\rho(t,y_t)})\right| \\ &+\int_0^t \|\psi(t-s,s)\||A(0)|\left|g(s,y_{k\rho(s,y_{ks})})-g(s,y_{\rho(s,y_s)})\right|\mathrm{d}s \\ &+\int_0^t \|\Psi(t-s,s)\|\|C\|_{B(E)}|u_{y_k}(s)-u_y(s)|\mathrm{d}s \\ &+\int_0^t\int_0^s \|\Psi(t-s,s)\|\|\phi(s,\tau)\|\|C\|_{B(E)}|u_{y_k}(\tau)-u_y(\tau)|\mathrm{d}\tau\,\mathrm{d}s.\end{aligned}$$

Using the hypothesis $(H4)$ and inequalities (3), (4) and (11), we obtain for each $t\in[0,n]$ and $n\in\mathbb{N}$

$$\begin{aligned}|F(y_k)(t)-F(y)(t)| &\leqslant \left|g(t,y_{k\rho(t,y_{kt})})-g(t,y_{\rho(t,y_t)})\right| \\ &+K|A(0)|\int_0^t (t-s)^{\alpha-1}\left|g(s,y_{k\rho(s,y_{ks})})-g(s,y_{\rho(s,y_s)})\right|\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+ K\widetilde{M}\int_0^t (t-s)^{\alpha-1}\widetilde{M}_1\Bigg[\left|g(n,y_{k\rho(n,y_{kn})}) - g(n,y_{\rho(n,y_n)})\right| \\
&+ K|A(0)|\int_0^n (n-\tau)^{\alpha-1}\left|g(\tau,y_{k\rho(\tau,y_{k\tau})}) - g(\tau,y_{\rho(\tau,y_\tau)})\right|\mathrm{d}\tau \\
&+ K\int_0^n (n-\tau)^{\alpha-1}|f(\tau,y_{k\rho(\tau,y_{k\tau})}) - f(\tau,y_{\rho(\tau,y_\tau)})|\mathrm{d}\tau \\
&+ K^2\int_0^n\int_0^\tau (n-\tau)^{\alpha-1}(\tau-\eta)^{\gamma-1}|f(\eta,y_{k\rho(\eta,y_{k\eta})}) - f(\eta,y_{\rho(\eta,y_\eta)})|\mathrm{d}\eta\,\mathrm{d}\tau\Bigg]\mathrm{d}s \\
&+ K^2\widetilde{M}\int_0^t\int_0^s (t-s)^{\alpha-1}(s-\tau)^{\gamma-1}\widetilde{M}_1\Bigg[\left|g(n,y_{k\rho(n,y_{kn})}) - g(n,y_{\rho(n,y_n)})\right| \\
&+ K|A(0)|\int_0^n (n-\eta)^{\alpha-1}\left|g(\eta,y_{k\rho(\eta,y_{k\eta})}) - g(\eta,y_{\rho(\eta,y_\eta)})\right|\mathrm{d}\eta \\
&+ K\int_0^n (n-\eta)^{\alpha-1}|f(\eta,y_{k\rho(\eta,y_{k\eta})}) - f(\eta,y_{\rho(\eta,y_\eta)})|\mathrm{d}\eta \\
&+ K^2\int_0^n\int_0^\eta (n-\eta)^{\alpha-1}(\eta-w)^{\gamma-1}\Big|f(w,y_{k\rho(w,y_{kw})}) \\
&- f(w,y_{\rho(w,y_w)})\Big|\mathrm{d}w\,\mathrm{d}\eta\Bigg]\mathrm{d}\tau\,\mathrm{d}s \\
\leqslant{} & \left|g(t,y_{k\rho(t,y_{kt})}) - g(t,y_{\rho(t,y_t)})\right| \\
&+ K|A(0)|\int_0^t (t-s)^{\alpha-1}\left|g(s,y_{k\rho(s,y_{ks})}) - g(s,y_{\rho(s,y_s)})\right|\mathrm{d}s \\
&+ K\widetilde{M}\widetilde{M}_1 n^\alpha\left[\alpha^{-1} + Kn^\gamma\gamma^{-1}\beta(\alpha,\gamma+1)\right]\Bigg[\left|g(n,y_{k\rho(n,y_{kn})}) - g(n,y_{\rho(n,y_n)})\right| \\
&+ K|A(0)|\int_0^n (n-s)^{\alpha-1}\left|g(s,y_{k\rho(s,y_{ks})}) - g(s,y_{\rho(s,y_s)})\right|\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+ K \int_0^n (n-s)^{\alpha-1} |f(s, y_{k\rho(s,y_{ks})}) - f(s, y_{\rho(s,y_s)})| \mathrm{d}s \\
&+ K^2 \int_0^n \int_0^s (n-s)^{\alpha-1}(s-\tau)^{\gamma-1} |f(\tau, y_{k\rho(\tau,y_{k\tau})}) - f(\tau, y_{\rho(\tau,y_\tau)})| \mathrm{d}\tau \mathrm{d}s \Big] \\
\leqslant{} & \left|g(t, y_{k\rho(t,y_{kt})}) - g(t, y_{\rho(t,y_t)})\right| \\
&+ K|A(0)| \int_0^t (t-s)^{\alpha-1} \left|g(s, y_{k\rho(s,y_{ks})}) - g(s, y_{\rho(s,y_s)})\right| \mathrm{d}s \\
&+ K\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n \left|g(n, y_{k\rho(n,y_{kn})}) - g(n, y_{\rho(n,y_n)})\right| \\
&+ K^2\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n |A(0)| \int_0^n (n-s)^{\alpha-1} \left|g(s, y_{k\rho(s,y_{ks})}) - g(s, y_{\rho(s,y_s)})\right| \mathrm{d}s \\
&+ K^2\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n \int_0^n (n-s)^{\alpha-1} |f(s, y_{k\rho(s,y_{ks})}) - f(s, y_{\rho(s,y_s)})| \mathrm{d}s \\
&+ K^3\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n \int_0^n \int_0^s (n-s)^{\alpha-1}(s-\tau)^{\gamma-1} \big|f(\tau, y_{k\rho(\tau,y_{k\tau})}) \\
&- f(\tau, y_{\rho(\tau,y_\tau)})\big| \mathrm{d}\tau \mathrm{d}s.
\end{aligned}
$$

Then, for each $t \in [0, n]$ and $n \in \mathbb{N}$, we have

$$
\begin{aligned}
&\|F(y_k) - F(y)\|_n \\
&\leqslant \left\|g(\cdot, y_{k\rho(\cdot,y_{k\cdot})}) - g(\cdot, y_{\rho(\cdot,y_\cdot)})\right\|_n \\
&\quad + K|A(0)| \int_0^t (t-s)^{\alpha-1} \mathrm{d}s \left\|g(\cdot, y_{k\rho(\cdot,y_{k\cdot})}) - g(\cdot, y_{\rho(\cdot,y_\cdot)})\right\|_n \\
&\quad + K\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n \left\|g(\cdot, y_{k\rho(\cdot,y_{k\cdot})}) - g(\cdot, y_{\rho(\cdot,y_\cdot)})\right\|_n \\
&\quad + K^2\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n |A(0)| \int_0^n (n-s)^{\alpha-1} \mathrm{d}s \left\|g(\cdot, y_{k\rho(\cdot,y_{k\cdot})}) - g(\cdot, y_{\rho(\cdot,y_\cdot)})\right\|_n \\
&\quad + K^2\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n \int_0^n (n-s)^{\alpha-1} \mathrm{d}s \left\|f(\cdot, y_{k\rho(\cdot,y_{k\cdot})}) - f(\cdot, y_{\rho(\cdot,y_\cdot)})\right\|_n
\end{aligned}
$$

$$+ K^3\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n \int_0^n\int_0^s (n-s)^{\alpha-1}(s-\tau)^{\gamma-1}\,\mathrm{d}\tau\,\mathrm{d}s\,\left\|f(\cdot, y_{k\rho(\cdot,y_{k\cdot})}) - f(\cdot, y_{\rho(\cdot,y_\cdot)})\right\|_n$$

$$\leqslant \left(K\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n + 1\right)\left(1 + K|A(0)|n^\alpha \alpha^{-1}\right)\left\|g(\cdot, y_{k\rho(\cdot,y_{k\cdot})}) - g(\cdot, y_{\rho(\cdot,y_\cdot)})\right\|_n$$
$$+ K^2\widetilde{M}\widetilde{M}_1 n^{2\alpha}\Upsilon_n^2 \|f(\cdot, y_{k\rho(\cdot,y_{k\cdot})}) - f(\cdot, y_{\rho(\cdot,y_\cdot)})\|_n.$$

From the hypothesis $(H7)$ and since $f$ is Carathéodory function, by the Lebesgue dominated convergence theorem, we obtain

$$\|F(y_k) - F(y)\|_n \to 0 \quad \text{as } k \to +\infty.$$

Thus $F$ is continuous.

**Step 2:** $F$ maps bounded sets of $C([-r, +\infty), E)$ into bounded sets.

Indeed, it is enough to show that for any $d > 0$, there exists a positive constant $\ell$ such that for each $y \in B_d = \{y \in C([-r, +\infty), E) : \|y\|_n \leqslant d\}$ one has $\|F(y)\|_n \leqslant \ell$.

Let $y \in B_d$. By the hypothesis $(H4)$ and inequalities (3), (4) and (5), we have

$$|F(y)(t)| \leqslant |\varphi(0)| + |g(0, \varphi)| + |g(t, y_{\rho(t,y_t)})|$$
$$+ \int_0^t \|\Psi(t-s, s)\|\|U(s)\||A(0)|\left(|\varphi(0)| + |g(0, \varphi)|\right)\mathrm{d}s$$
$$+ \int_0^t \|\Psi(t-s, s)\||A(0)||g(s, y_{\rho(s,y_s)})|\,\mathrm{d}s$$
$$+ \int_0^t \|\Psi(t-s, s)\|\|C\|_{B(E)}|u_y(s)|\,\mathrm{d}s$$
$$+ \int_0^t\int_0^s \|\Psi(t-s, s)\|\|\phi(s, \tau)\|\|C\|_{B(E)}|u_y(\tau)|\,\mathrm{d}\tau\,\mathrm{d}s$$
$$\leqslant \|\varphi\| + \|A^{-1}(0)\||A(0)g(0, \varphi)| + \|A^{-1}(t)\||A(t)g(t, y_{\rho(t,y_t)})|$$
$$+ K^2|A(0)|\int_0^t (t-s)^{\alpha-1}(1 + s^\gamma)\left(\|\varphi\| + \|A^{-1}(0)\||A(0)g(0, \varphi)|\right)\mathrm{d}s$$

$$+ K|A(0)| \int_0^t (t-s)^{\alpha-1} \|A^{-1}(s)\| |A(s)g(s, y_{\rho(s,y_s)})| \mathrm{d}s$$

$$+ K\widetilde{M} \int_0^t (t-s)^{\alpha-1} |u_y(s)| \mathrm{d}s$$

$$+ K^2\widetilde{M} \int_0^t \int_0^s (t-s)^{\alpha-1} (s-\tau)^{\gamma-1} |u_y(\tau)| \mathrm{d}\tau \mathrm{d}s.$$

Using inequality (12) and hypotheses $(H5)$ and $(H6)$, we obtain

$$|y(t)| \leqslant \|\varphi\| + \overline{M}_0 L\left(\|\varphi\| + 1\right) + \overline{M}_0 L\left(\|y_{\rho(t,y_t)}\| + 1\right)$$

$$+ K^2|A(0)| \left(\|\varphi\| + \overline{M}_0 L\left(\|\varphi\| + 1\right)\right) \int_0^t (t-s)^{\alpha-1} (1+s^\gamma) \mathrm{d}s$$

$$+ K\overline{M}_0 L |A(0)| \int_0^t (t-s)^{\alpha-1} \left(\|y_{\rho(s,y_s)}\| + 1\right) \mathrm{d}s$$

$$+ K\widetilde{M} \int_0^t (t-s)^{\alpha-1} \widetilde{M}_1 \Bigg[ |y^\star| + \left(\overline{M}_0 L + 1\right) \left(1 + K^2|A(0)| n^\alpha \Theta_n\right) \|\varphi\|$$

$$+ \overline{M}_0 L \left(2 + K|A(0)| n^\alpha \left(\alpha^{-1} + K\Theta_n\right)\right) + K p_n^* n^\alpha \Upsilon_n + \overline{M}_0 L \left\|y_{\rho(n,y_n)}\right\|$$

$$+ K\overline{M}_0 L |A(0)| \int_0^n (n-\tau)^{\alpha-1} \left\|y_{\rho(\tau,y_\tau)}\right\| \mathrm{d}\tau$$

$$+ K p_n^* \int_0^n (n-\tau)^{\alpha-1} \|y_{\rho(\tau,y_\tau)}\| \mathrm{d}\tau$$

$$+ K^2 p_n^* \int_0^n \int_0^\tau (n-\tau)^{\alpha-1} (\tau-\eta)^{\gamma-1} \|y_{\rho(\eta,y_\eta)}\| \mathrm{d}\eta \mathrm{d}\tau \Bigg] \mathrm{d}s$$

$$+ K^2\widetilde{M} \int_0^t \int_0^s (t-s)^{\alpha-1} (s-\tau)^{\gamma-1} \widetilde{M}_1$$

$$\cdot \Bigg[ |y^\star| + \left(\overline{M}_0 L + 1\right)\left(1 + K^2|A(0)|n^\alpha \Theta_n\right)\|\varphi\| + \overline{M}_0 L\Big(2 + K|A(0)|n^\alpha\left(\alpha^{-1} + K\Theta_n\right)\Big) + K p_n^* n^\alpha \Upsilon_n + \overline{M}_0 L \left\|y_{\rho(n,y_n)}\right\| + K\overline{M}_0 L|A(0)|\int_0^n (n-\eta)^{\alpha-1}\left\|y_{\rho(\eta,y_\eta)}\right\| \mathrm{d}\eta + K p_n^* \int_0^n (n-\eta)^{\alpha-1}\|y_{\rho(\eta,y_\eta)}\|\mathrm{d}\eta + K^2 p_n^* \int_0^n\int_0^\eta (n-\eta)^{\alpha-1}(\eta-w)^{\gamma-1}\|y_{\rho(w,y_w)}\|\mathrm{d}w\,\mathrm{d}\eta \Bigg]\mathrm{d}\tau\,\mathrm{d}s.$$

So, for each $t \in [0,n]$ and $n \in \mathbb{N}$, we get

$$|y(t)| \leqslant \left(\overline{M}_0 L + 1\right)\|\varphi\| + 2\overline{M}_0 L + \overline{M}_0 L\|y_{\rho(t,y_t)}\| + K^2\overline{M}_0 L|A(0)|\int_0^t (t-s)^{\alpha-1}(1+s^\gamma)\mathrm{d}s + K^2|A(0)|\left(\overline{M}_0 L + 1\right)\|\varphi\|\int_0^t (t-s)^{\alpha-1}(1+s^\gamma)\mathrm{d}s + K\overline{M}_0 L|A(0)|\int_0^t (t-s)^{\alpha-1}\mathrm{d}s + K\overline{M}_0 L|A(0)|\int_0^t (t-s)^{\alpha-1}\|y_{\rho(s,y_s)}\|\mathrm{d}s + K\widetilde{M}\widetilde{M}_1 t^\alpha\left(\alpha^{-1} + K t^\gamma \gamma^{-1}\beta(\alpha,\gamma+1)\right) \cdot \Bigg[ |y^\star| + \left(\overline{M}_0 L + 1\right)\left(1 + K^2|A(0)|n^\alpha\Theta_n\right)\|\varphi\| + \overline{M}_0 L\Big(2 + K|A(0)|n^\alpha\left(\alpha^{-1} + K\Theta_n\right)\Big) + K p_n^* n^\alpha\Upsilon_n + \overline{M}_0 L\left\|y_{\rho(n,y_n)}\right\|$$

$$+ K\overline{M}_0 L|A(0)| \int_0^n (n-s)^{\alpha-1} \left\| y_{\rho(s,y_s)} \right\| \mathrm{d}s$$

$$+ Kp_n^* \int_0^n (n-s)^{\alpha-1} \| y_{\rho(s,y_s)} \| \mathrm{d}s$$

$$+ K^2 p_n^* \int_0^n \int_0^s (n-s)^{\alpha-1} (s-\tau)^{\gamma-1} \| y_{\rho(\tau,y_\tau)} \| \mathrm{d}\tau \,\mathrm{d}s \Bigg].$$

From inequalities (6), (7) and (8), we obtain for each $t \in [0,n]$ and $n \in \mathbb{N}$

$$\begin{aligned}
|y(t)| \leqslant{}& (\overline{M}_0 L + 1)\|\varphi\| + 2\overline{M}_0 L + \overline{M}_0 L \|y_{\rho(t,y_t)}\| \\
&+ K^2 \overline{M}_0 L |A(0)| n^\alpha \left( \alpha^{-1} + n^\gamma \beta(\alpha, \gamma+1) \right) \\
&+ K^2 |A(0)| \left( \overline{M}_0 L + 1 \right) n^\alpha \left( \alpha^{-1} + n^\gamma \beta(\alpha,\gamma+1) \right) \|\varphi\| \\
&+ K\overline{M}_0 L |A(0)| n^\alpha \alpha^{-1} + K\overline{M}_0 L |A(0)| \int_0^t (t-s)^{\alpha-1} \|y_{\rho(s,y_s)}\| \mathrm{d}s \\
&+ K\widetilde{M}\widetilde{M}_1 n^\alpha \left( \alpha^{-1} + K n^\gamma \gamma^{-1} \beta(\alpha,\gamma+1) \right) \\
&\cdot \Bigg[ |y^\star| + \left( \overline{M}_0 L + 1 \right) \left( 1 + K^2 |A(0)| n^\alpha \Theta_n \right) \|\varphi\| \\
&+ \overline{M}_0 L \left( 2 + K|A(0)| n^\alpha \left( \alpha^{-1} + K\Theta_n \right) \right) + K p_n^* n^\alpha \Upsilon_n + \overline{M}_0 L \left\| y_{\rho(n,y_n)} \right\| \\
&+ K\overline{M}_0 L |A(0)| \int_0^n (n-s)^{\alpha-1} \left\| y_{\rho(s,y_s)} \right\| \mathrm{d}s \\
&+ K p_n^* \int_0^n (n-s)^{\alpha-1} \|y_{\rho(s,y_s)}\| \mathrm{d}s \\
&+ K^2 p_n^* \int_0^n \int_0^s (n-s)^{\alpha-1} (s-\tau)^{\gamma-1} \|y_{\rho(\tau,y_\tau)}\| \mathrm{d}\tau\,\mathrm{d}s \Bigg] \\
\leqslant{}& \left( \overline{M}_0 L + 1 \right) \left( 1 + K^2 |A(0)| n^\alpha \Theta_n \right) \|\varphi\| \\
&+ \overline{M}_0 L \left( 2 + K|A(0)| n^\alpha \left( \alpha^{-1} + K\Theta_n \right) \right)
\end{aligned}$$

$$
\begin{aligned}
&+ \overline{M}_0 L \|y_{\rho(t,y_t)}\| + K\overline{M}_0 L |A(0)| \int_0^t (t-s)^{\alpha-1} \|y_{\rho(s,y_s)}\| \mathrm{d}s \\
&+ K\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n \Big[ |y^\star| + \left(\overline{M}_0 L + 1\right) \left(1 + K^2 |A(0)| n^\alpha \Theta_n\right) \|\varphi\| \\
&+ \overline{M}_0 L \Big(2 + K|A(0)| n^\alpha \left(\alpha^{-1} + K\Theta_n\right)\Big) + K p_n^* n^\alpha \Upsilon_n + \overline{M}_0 L \left\|y_{\rho(n,y_n)}\right\| \\
&+ K\overline{M}_0 L |A(0)| \int_0^n (n-s)^{\alpha-1} \left\|y_{\rho(s,y_s)}\right\| \mathrm{d}s \\
&+ K p_n^* \int_0^n (n-s)^{\alpha-1} \|y_{\rho(s,y_s)}\| \mathrm{d}s \\
&+ K^2 p_n^* \int_0^n \int_0^s (n-s)^{\alpha-1} (s-\tau)^{\gamma-1} \|y_{\rho(\tau,y_\tau)}\| \mathrm{d}\tau \mathrm{d}s \Big].
\end{aligned}
$$

Then

$$
\begin{aligned}
|y(t)| \leqslant{} & K\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n |y^\star| \\
&+ \left(\overline{M}_0 L + 1\right) \left(K\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n + 1\right) \left(1 + K^2 |A(0)| n^\alpha \Theta_n\right) \|\varphi\| \\
&+ \overline{M}_0 L \left(K\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n + 1\right) \left(2 + K|A(0)| n^\alpha (\alpha^{-1} + K\Theta_n)\right) \\
&+ K^2 \widetilde{M}\widetilde{M}_1 n^{2\alpha} p_n^* \Upsilon_n^2 + \overline{M}_0 L \|y_{\rho(t,y_t)}\| \\
&+ K\overline{M}_0 L |A(0)| \int_0^t (t-s)^{\alpha-1} \|y_{\rho(s,y_s)}\| \mathrm{d}s \\
&+ K\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n \overline{M}_0 L \left\|y_{\rho(n,y_n)}\right\| \\
&+ K^2 \widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n \overline{M}_0 L |A(0)| \int_0^n (n-s)^{\alpha-1} \left\|y_{\rho(s,y_s)}\right\| \mathrm{d}s \\
&+ K^2 \widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n p_n^* \int_0^n (n-s)^{\alpha-1} \|y_{\rho(s,y_s)}\| \mathrm{d}s \\
&+ K^3 \widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n p_n^* \int_0^n \int_0^s (n-s)^{\alpha-1} (s-\tau)^{\gamma-1} \|y_{\rho(\tau,y_\tau)}\| \mathrm{d}\tau \mathrm{d}s.
\end{aligned}
$$

Using Proposition 1, we get

$$\left\|y_{\rho(n,y_n)}\right\| \leqslant |y^\star| + \mathcal{L}^\varphi\|\varphi\| \tag{13}$$

and

$$\left\|y_{\rho(t,y_t)}\right\| \leqslant |y(t)| + \mathcal{L}^\varphi\|\varphi\|. \tag{14}$$

Since $y \in B_d$, then we have

$$\left\|y_{\rho(t,y_t)}\right\| \leqslant d + \mathcal{L}^\varphi\|\varphi\| \leqslant \delta. \tag{15}$$

By the inequalities (14) and (13), we obtain

$$\begin{aligned}
|y(t)| \leqslant\ & K\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n |y^\star| \\
& + \left(\overline{M}_0 L + 1\right)\left(K\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n + 1\right)\left(1 + K^2|A(0)|n^\alpha \Theta_n\right)\|\varphi\| \\
& + \overline{M}_0 L\left(K\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n + 1\right)\left(2 + K|A(0)|n^\alpha(\alpha^{-1} + K\Theta_n)\right) \\
& + K^2\widetilde{M}\widetilde{M}_1 n^{2\alpha} p_n^* \Upsilon_n^2 + \overline{M}_0 L\left(|y(t)| + \mathcal{L}^\varphi\|\varphi\|\right) \\
& + K\overline{M}_0 L|A(0)|\int_0^t (t-s)^{\alpha-1}\left(|y(s)| + \mathcal{L}^\varphi\|\varphi\|\right)\mathrm{d}s \\
& + K\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n \overline{M}_0 L\left(|y^\star| + \mathcal{L}^\varphi\|\varphi\|\right) \\
& + K^2\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n \overline{M}_0 L|A(0)|\int_0^n (n-s)^{\alpha-1}\left(|y(s)| + \mathcal{L}^\varphi\|\varphi\|\right)\mathrm{d}s \\
& + K^2\widetilde{M}\widetilde{M}_1 n^\alpha p_n^* \Upsilon_n \int_0^n (n-s)^{\alpha-1}\left(|y(s)| + \mathcal{L}^\varphi\|\varphi\|\right)\mathrm{d}s \\
& + K^3\widetilde{M}\widetilde{M}_1 n^\alpha p_n^* \Upsilon_n \int_0^n\int_0^s (n-s)^{\alpha-1}(s-\tau)^{\gamma-1}\left(|y(\tau)| + \mathcal{L}^\varphi\|\varphi\|\right)\mathrm{d}\tau\,\mathrm{d}s \\
\leqslant\ & K\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n(\overline{M}_0 L + 1)|y^\star| + K\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n \overline{M}_0 L\mathcal{L}^\varphi\|\varphi\| \\
& + \left(\overline{M}_0 L + 1\right)\left(K\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n + 1\right)\left(1 + K^2|A(0)|n^\alpha \Theta_n\right)\|\varphi\| \\
& + \overline{M}_0 L\left(K\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n + 1\right)\left(2 + K|A(0)|n^\alpha(\alpha^{-1} + K\Theta_n)\right) \\
& + \overline{M}_0 L\left(1 + K|A(0)|n^\alpha \alpha^{-1}\left(K\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n + 1\right)\right)\delta \\
& + K^2\widetilde{M}\widetilde{M}_1 n^{2\alpha} p_n^* \Upsilon_n^2(\delta + 1) := \ell_n.
\end{aligned}$$

Thus, there exists a positive number $\ell_n$ such that

$$\|F(z)\|_n \leqslant \ell_n.$$

Therefor, $F(B_d) \subset B_{\ell_n}$.

**Step 3:** $F$ maps bounded sets into equicontinuous sets of $C([-r,+\infty),E)$.

We consider $B_d$ as in Step 2 and we show that $F(B_d)$ is equicontinuous. Let $t_1,\ t_2 \in J$ with $t_2 > t_1$ and $y \in B_d$. Then

$$\begin{aligned}
|F(y)(t_2) - F(y)(t_1)| \leqslant{}& \left|g(t_2, y_{\rho(t_2,y_{t_2})}) - g(t_1, y_{\rho(t_1,y_{t_1})})\right| \\
&+ \int_0^{t_1} \|\psi(t_2 - s, s) - \psi(t_1 - s, s)\| \|U(s)\| |A(0)| (\|\varphi\| + |g(0,\varphi)|)\, \mathrm{d}s \\
&+ \int_{t_1}^{t_2} \|\psi(t_2 - s, s)\| \|U(s)\| |A(0)| (\|\varphi\| + |g(0,\varphi)|)\, \mathrm{d}s \\
&+ \int_0^{t_1} \|\psi(t_2 - s, s) - \psi(t_1 - s, s)\| |A(0)| |g(s, y_{\rho(s,y_s)})|\, \mathrm{d}s \\
&+ \int_{t_1}^{t_2} \|\psi(t_2 - s, s)\| |A(0)| |g(s, y_{\rho(s,y_s)})|\, \mathrm{d}s \\
&+ \int_0^{t_1} \|\psi(t_2 - s, s) - \psi(t_1 - s, s)\|\, \|C\| |u_y(s)|\, \mathrm{d}s \\
&+ \int_{t_1}^{t_2} \|\psi(t_2 - s, s)\| \|C\| |u_y(s)|\, \mathrm{d}s \\
&+ \int_0^{t_1} \int_0^{s} \|\psi(t_2 - s, s) - \psi(t_1 - s, s)\|\, \|\phi(s,\tau)\| \|C\| |u_y(\tau)|\, \mathrm{d}\tau\, \mathrm{d}s \\
&+ \int_{t_1}^{t_2} \int_0^{s} \|\psi(t_2 - s, s)\|\, \|\phi(s,\tau)\| \|C\| |u_y(\tau)|\, \mathrm{d}\tau\, \mathrm{d}s.
\end{aligned}$$

By the inequalities (12) and (15), we have

$$
\begin{aligned}
|u_y(t)| \leqslant \widetilde{M}_1 \Bigg[ &(\overline{M}_0 L + 1)|y^\star| + \Big(\overline{M}_0 L \mathcal{L}^\varphi + (\overline{M}_0 L + 1) \left(1 + K^2 |A(0)| n^\alpha \Theta_n\right)\Big) \|\varphi\| \\
&+ \overline{M}_0 L \Big(2 + K|A(0)| n^\alpha \left(\alpha^{-1} + K\Theta_n\right)\Big) + K\overline{M}_0 L |A(0)| n^\alpha \alpha^{-1} \\
&+ K p_n^* n^\alpha \Upsilon_n (\delta + 1) \Bigg] := \omega_n. \qquad (16)
\end{aligned}
$$

Using hypotheses $(H4)$, $(H5)$ and $(H6)$ and inequality (15) and the previous one, we get

$$
\begin{aligned}
&|F(y)(t_2) - F(y)(t_1)| \\
&\quad \leqslant \left| g(t_2, y_{\rho(t_2, y_{t_2})}) - g(t_1, y_{\rho(t_1, y_{t_1})}) \right| \\
&\quad + (\|\varphi\| + \overline{M}_0 L (\|\varphi\| + 1)) |A(0)| \int_0^{t_1} \|\psi(t_2 - s, s) - \psi(t_1 - s, s)\| \|U(s)\| \mathrm{d}s \\
&\quad + (\|\varphi\| + \overline{M}_0 L (\|\varphi\| + 1)) |A(0)| \int_{t_1}^{t_2} \|\psi(t_2 - s, s)\| \|U(s)\| \mathrm{d}s \\
&\quad + \overline{M}_0 L (\delta + 1) |A(0)| \int_0^{t_1} \|\psi(t_2 - s, s) - \psi(t_1 - s, s)\| \mathrm{d}s \\
&\quad + \overline{M}_0 L (\delta + 1) |A(0)| \int_{t_1}^{t_2} \|\psi(t_2 - s, s)\| \mathrm{d}s \\
&\quad + \widetilde{M}\omega_n \int_0^{t_1} \|\psi(t_2 - s, s) - \psi(t_1 - s, s)\| \,\mathrm{d}s + \widetilde{M}\omega_n \int_{t_1}^{t_2} \|\psi(t_2 - s, s)\| \mathrm{d}s \\
&\quad + \widetilde{M}\omega_n \int_0^{t_1} \int_0^{s} \|\psi(t_2 - s, s) - \psi(t_1 - s, s)\| \, \|\phi(s, \tau)\| \mathrm{d}\tau \mathrm{d}s \\
&\quad + \widetilde{M}\omega_n \int_{t_1}^{t_2} \int_0^{s} \|\psi(t_2 - s, s)\| \, \|\phi(s, \tau)\| \mathrm{d}\tau \mathrm{d}s.
\end{aligned}
$$

From hypothesis $(H7)$ and since $\Psi$ is a continuous function in uniform topology. The right-hand side of the above inequality tends to zero as $t_2 - t_1 \to 0$.

As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem it suffices to show that the operator $F$ maps $B_d$ into a precompact set in $E$.

Let $t \in J$ be fixed and let $\epsilon$ be a real number satisfying $0 < \epsilon < t$. For $y \in B_d$ we define

$$\begin{aligned}F_\epsilon(y)(t) = \varphi(0) - g(0,\varphi) + g(t, y_{\rho(t,y_t)}) \\ - \int_0^{t-\epsilon} \Psi(t-s,s)U(s)A(0)[\varphi(0) - g(0,\varphi)]\,\mathrm{d}s \\ - \int_0^{t-\epsilon} \Psi(t-s,s)A(0)g(s, y_{\rho(s,y_s)})\,\mathrm{d}s + \int_0^{t-\epsilon} \Psi(t-s,s)Cu_y(s)\,\mathrm{d}s \\ + \int_0^{t-\epsilon}\int_0^{s} \Psi(t-s,s)\phi(s,\tau)Cu_y(\tau)\,\mathrm{d}\tau\,\mathrm{d}s.\end{aligned}$$

Note that the set

$$\begin{aligned}\Bigg\{\varphi(0) - g(0,\varphi) + g(t, y_{\rho(t,y_t)}) - \int_0^{t-\epsilon} \Psi(t-s,s)U(s)A(0)[\varphi(0) - g(0,\varphi)]\,\mathrm{d}s \\ - \int_0^{t-\epsilon} \Psi(t-s,s)A(0)g(s, y_{\rho(s,y_s)})\,\mathrm{d}s + \int_0^{t-\epsilon} \Psi(t-s,s)Cu_y(s)\,\mathrm{d}s \\ + \int_0^{t-\epsilon}\int_0^{s} \Psi(t-s,s)\phi(s,\tau)Cu_y(\tau)\,\mathrm{d}\tau\,\mathrm{d}s : y \in B_d\Bigg\}\end{aligned}$$

is bounded.

Then for $t > 0$, the set $Y_\epsilon(t) = \{F_\epsilon(y)(t) : y \in B_d\}$ is precompact in $E$ for every $\epsilon$ sufficiently small, $0 < \epsilon < t$. Moreover using (16), we have

$$\begin{aligned}|F(y)(t) - F_\epsilon(y)(t)| \\ \leqslant \int_{t-\epsilon}^{t} \|\Psi(t-s,s)\|\|U(s)\||A(0)|\left(\|\varphi\| + \|A^{-1}(0)\||A(0)g(0,\varphi)|\right)\mathrm{d}s\end{aligned}$$

$$+\int_{t-\epsilon}^{t}\|\Psi(t-s,s)\|\|A(0)\|\|A^{-1}(s)\||A(s)g(s,y_{\rho(s,y_s)})|\mathrm{d}s$$
$$+\int_{t-\epsilon}^{t}\|\Psi(t-s,s)\|\|C\|_{B(E)}|u_y(s)|\mathrm{d}\tau\mathrm{d}s$$
$$+\int_{t-\epsilon}^{t}\int_{0}^{s}\|\Psi(t-s,s)\|\|\phi(s,\tau)\|\|C\|_{B(E)}|u_y(s)|\mathrm{d}s$$
$$\leqslant\Big(\|\varphi\|+\overline{M}_0L(\|\varphi\|+1)\Big)|A(0)|\int_{t-\epsilon}^{t}\|\Psi(t-s,s)\|\|U(s)\|\mathrm{d}s$$
$$+\overline{M}_0L(\delta+1)|A(0)|\int_{t-\epsilon}^{t}\|\Psi(t-s,s)\|\mathrm{d}s+\widetilde{M}\omega_n\int_{t-\epsilon}^{t}\|\Psi(t-s,s)\|\mathrm{d}s$$
$$+\widetilde{M}\omega_n\int_{t-\epsilon}^{t}\int_{0}^{s}\|\Psi(t-s,s)\|\|\phi(s,\tau)\|\mathrm{d}\tau\mathrm{d}s.$$

Then
$$|F(y)(t)-F_\epsilon(y)(t)|\to 0\quad as\ \epsilon\to 0.$$

Therefore, the set $\{F(y)(t) : y\in B_d\}$ is precompact in $E$. So we deduce from Steps 1, 2 and 3 that $F$ is a continuous compact operator.

**Step 4:** $G$ is a contraction.

Indeed, consider $y,\ \overline{y}\in C([-r,+\infty),E)$. By the inequalities (3) and (4) and the hypothesis $(H3)$, we get

$$|G(y)(t)-G(\overline{y})(t)|$$
$$\leqslant\int_{0}^{t}\|\psi(t-s,s)\|\left|f(s,y_{\rho(s,y_s)})-f(s,\overline{y}_{\rho(s,\overline{y}_s)})\right|\mathrm{d}s$$
$$+\int_{0}^{t}\int_{0}^{s}\|\psi(t-s,s)\|\|\phi(s,\tau)\|\left|f(\tau,y_{\rho(\tau,y_\tau)})-f(\tau,\overline{y}_{\rho(\tau,\overline{y}_\tau)})\right|\mathrm{d}s$$
$$\leqslant K\int_{0}^{t}(t-s)^{\alpha-1}l_n(s)\|y_{\rho(s,y_s)}-\overline{y}_{\rho(s,\overline{y}_s)}\|\ \mathrm{d}s$$

$$+ K^2 \int_0^t \int_0^s (t-s)^{\alpha-1}(s-\tau)^{\gamma-1} l_n(\tau) \| y_{\rho(\tau,y_\tau)} - \overline{y}_{\rho(\tau,\overline{y}_\tau)} \| \mathrm{d}\tau \mathrm{d}s.$$

Using inequality (14), we obtain for each $t \in [0, n]$ and $n \in \mathbb{N}$

$$\begin{aligned} |G(y)(t) - G(\overline{y})(t)| &\leqslant K \int_0^t (t-s)^{\alpha-1} l_n(s) |y(s) - \overline{y}(s)| \mathrm{d}s \\ &\quad + K^2 \int_0^t \int_0^s (t-s)^{\alpha-1}(s-\tau)^{\gamma-1} l_n(\tau) |y(\tau) - \overline{y}(\tau)| \mathrm{d}\tau \mathrm{d}s \\ &\leqslant K l_n^* \int_0^t (t-s)^{\alpha-1} \mathrm{d}s \|y - \overline{y}\|_n \\ &\quad + K^2 l_n^* \int_0^t \int_0^s (t-s)^{\alpha-1}(s-\tau)^{\gamma-1} \mathrm{d}\tau \mathrm{d}s \|y - \overline{y}\|_n \\ &\leqslant K l_n^* \left[ t^\alpha \alpha^{-1} + K t^{\alpha+\gamma} \gamma^{-1} \beta(\alpha, \gamma+1) \right] \|y - \overline{y}\|_n \\ &\leqslant K l_n^* n^\alpha \left[ \alpha^{-1} + K n^\gamma \gamma^{-1} \beta(\alpha, \gamma+1) \right] \|y - \overline{y}\|_n. \end{aligned}$$

Therefore,

$$\|G(y) - G(\overline{y})\|_n \leqslant K l_n^* n^\alpha \Upsilon_n \|y - \overline{y}\|_n.$$

So, for $K l_n^* n^\alpha \Upsilon_n < 1$, the operator $G$ is a contraction for all $n \in \mathbb{N}$.

**Step 5:** To apply Theorem 1, we must check (C2): i.e. it remains to show that the following set is bounded

$$\varGamma = \left\{ y \in C([-r, +\infty), E) : y = \lambda F(y) + \lambda G\left(\frac{y}{\lambda}\right) \text{ for some } 0 < \lambda < 1 \right\}.$$

Let $y \in \varGamma$. By the hypothesis $(H4)$ and inequalities (3), (4) and (5), we get

$$\begin{aligned} |y(t)| &\leqslant \lambda |\varphi(0)| + \lambda |g(0, \varphi)| + \lambda |g(t, y_{\rho(t,y_t)})| \\ &\quad + \lambda \int_0^t \|\Psi(t-s, s)\| \|U(s)\| |A(0)| \left( |\varphi(0)| + |g(0, \varphi)| \right) \mathrm{d}s \\ &\quad + \lambda \int_0^t \|\Psi(t-s, s)\| |A(0)| |g(s, y_{\rho(s,y_s)})| \mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&+\lambda \int_0^t \|\Psi(t-s,s)\| \|C\|_{B(E)} |u_y(s)| \mathrm{d}s \\
&+\lambda \int_0^t \int_0^s \|\Psi(t-s,s)\| \|\phi(s,\tau)\| \|C\|_{B(E)} |u_y(\tau)| \mathrm{d}\tau \mathrm{d}s \\
&+\lambda \int_0^t \|\Psi(t-s,s)\| \left| f\left(s, \frac{y_{\rho(s,\frac{y_s}{\lambda})}}{\lambda}\right)\right| \mathrm{d}s \\
&+\lambda \int_0^t \int_0^s \|\Psi(t-s,s)\| \|\phi(s,\tau)\| \left| f\left(\tau, \frac{y_{\rho(\tau,\frac{y_\tau}{\lambda})}}{\lambda}\right)\right| \mathrm{d}\tau \mathrm{d}s \\
\leqslant{}& \lambda\|\varphi\| + \lambda\|A^{-1}(0)\| |A(0)g(0,\varphi)| + \lambda\|A^{-1}(t)\| |A(t)g(t, y_{\rho(t,y_t)})| \\
&+\lambda K^2 |A(0)| \int_0^t (t-s)^{\alpha-1}(1+s^\gamma) \left(\|\varphi\| + \|A^{-1}(0)\| |A(0)g(0,\varphi)|\right) \mathrm{d}s \\
&+\lambda K |A(0)| \int_0^t (t-s)^{\alpha-1} \|A^{-1}(s)\| |A(s)g(s, y_{\rho(s,y_s)})| \mathrm{d}s \\
&+\lambda K \widetilde{M} \int_0^t (t-s)^{\alpha-1} |u_y(s)| \mathrm{d}s \\
&+\lambda K^2 \widetilde{M} \int_0^t \int_0^s (t-s)^{\alpha-1}(s-\tau)^{\gamma-1} |u_y(\tau)| \mathrm{d}\tau \mathrm{d}s \\
&+\lambda K \int_0^t (t-s)^{\alpha-1} \left| f\left(s, \frac{y_{\rho(s,\frac{y_s}{\lambda})}}{\lambda}\right)\right| \mathrm{d}s \\
&+\lambda K^2 \int_0^t \int_0^s (t-s)^{\alpha-1}(s-\tau)^{\gamma-1} \left| f\left(\tau, \frac{y_{\rho(\tau,\frac{y_\tau}{\lambda})}}{\lambda}\right)\right| \mathrm{d}\tau \mathrm{d}s.
\end{aligned}
$$

By hypotheses $(H5)$ and $(H6)$ and inequality (12), we obtain for each $t \in [0,n]$ and $n \in \mathbb{N}$

$$
|y(t)| \leqslant \lambda\|\varphi\| + \lambda\overline{M}_0 L\left(\|\varphi\|+1\right) + \lambda\overline{M}_0 L\left(\|y_{\rho(t,y_t)}\|+1\right)
$$

$$+ \lambda K^2 |A(0)| \left( \|\varphi\| + \overline{M}_0 L \left( \|\varphi\| + 1 \right) \right) \int_0^t (t-s)^{\alpha-1} (1 + s^\gamma) \mathrm{d}s$$

$$+ \lambda K |A(0)| \int_0^t (t-s)^{\alpha-1} \overline{M}_0 L \left( \|y_{\rho(s,y_s)}\| + 1 \right) \mathrm{d}s$$

$$+ \lambda K \widetilde{M} \int_0^t (t-s)^{\alpha-1} \widetilde{M}_1 \Bigg[ |y^\star| + \left( \overline{M}_0 L + 1 \right) \left( 1 + K^2 |A(0)| n^\alpha \Theta_n \right) \|\varphi\|$$

$$+ \overline{M}_0 L \Big( 2 + K |A(0)| n^\alpha \left( \alpha^{-1} + K \Theta_n \right) \Big) + K p_n^* n^\alpha \Upsilon_n + \overline{M}_0 L \left\| y_{\rho(n,y_n)} \right\|$$

$$+ K \overline{M}_0 L |A(0)| \int_0^n (n-\tau)^{\alpha-1} \left\| y_{\rho(\tau,y_\tau)} \right\| \mathrm{d}\tau$$

$$+ K p_n^* \int_0^n (n-\tau)^{\alpha-1} \|y_{\rho(\tau,y_\tau)}\| \mathrm{d}\tau$$

$$+ K^2 p_n^* \int_0^n \int_0^\tau (n-\tau)^{\alpha-1} (\tau-\eta)^{\gamma-1} \|y_{\rho(\eta,y_\eta)}\| \mathrm{d}\eta \mathrm{d}\tau \Bigg] \mathrm{d}s$$

$$+ \lambda K^2 \widetilde{M} \int_0^t \int_0^s (t-s)^{\alpha-1} (s-\tau)^{\gamma-1} \widetilde{M}_1$$

$$\cdot \Bigg[ |y^\star| + \left( \overline{M}_0 L + 1 \right) \left( 1 + K^2 |A(0)| n^\alpha \Theta_n \right) \|\varphi\|$$

$$+ \overline{M}_0 L \Big( 2 + K |A(0)| n^\alpha \left( \alpha^{-1} + K \Theta_n \right) \Big) + K p_n^* n^\alpha \Upsilon_n + \overline{M}_0 L \left\| y_{\rho(n,y_n)} \right\|$$

$$+ K \overline{M}_0 L |A(0)| \int_0^n (n-\eta)^{\alpha-1} \left\| y_{\rho(\eta,y_\eta)} \right\| \mathrm{d}\eta$$

$$+ K p_n^* \int_0^n (n-\eta)^{\alpha-1} \|y_{\rho(\eta,y_\eta)}\| \mathrm{d}\eta$$

$$+ K^2 p_n^* \int_0^n \int_0^\eta (n-\eta)^{\alpha-1} (\eta-w)^{\gamma-1} \|y_{\rho(w,y_w)}\| \mathrm{d}w \mathrm{d}\eta \Bigg] \mathrm{d}\tau \mathrm{d}s$$

$$
\begin{aligned}
&+ \lambda K \int_0^t (t-s)^{\alpha-1} \left| f\left(s, \frac{y_{\rho(s,\frac{y_s}{\lambda})}}{\lambda}\right)\right| \mathrm{d}s \\
&+ \lambda K^2 \int_0^t \int_0^s (t-s)^{\alpha-1}(s-\tau)^{\gamma-1} \left| f\left(\tau, \frac{y_{\rho(\tau,\frac{y_\tau}{\lambda})}}{\lambda}\right)\right| \mathrm{d}\tau\,\mathrm{d}s \\
\leqslant\ & \lambda(\overline{M}_0 L + 1)\|\varphi\| + 2\lambda \overline{M}_0 L + \lambda \overline{M}_0 L \|y_{\rho(t,y_t)}\| \\
&+ \lambda K^2 |A(0)| \Big(\|\varphi\| + \overline{M}_0 L\,(\|\varphi\|+1)\Big) \int_0^t (t-s)^{\alpha-1}(1+s^\gamma)\,\mathrm{d}s \\
&+ \lambda K |A(0)| \int_0^t (t-s)^{\alpha-1} \overline{M}_0 L \left(\|y_{\rho(s,y_s)}\| + 1\right) \mathrm{d}s \\
&+ \lambda K \widetilde{M}\widetilde{M}_1 t^\alpha \left(\alpha^{-1} + K t^\gamma \gamma^{-1} \beta(\alpha, \gamma+1)\right) \\
&\cdot \Bigg[ |y^\star| + \left(\overline{M}_0 L + 1\right)\left(1 + K^2 |A(0)| n^\alpha \Theta_n\right) \|\varphi\| \\
&+ \overline{M}_0 L \Big(2 + K|A(0)| n^\alpha \left(\alpha^{-1} + K\Theta_n\right)\Big) + K p_n^* n^\alpha \Upsilon_n + \overline{M}_0 L \left\|y_{\rho(n,y_n)}\right\| \\
&+ K \overline{M}_0 L |A(0)| \int_0^n (n-s)^{\alpha-1} \left\|y_{\rho(s,y_s)}\right\| \mathrm{d}s \\
&+ K p_n^* \int_0^n (n-s)^{\alpha-1} \|y_{\rho(s,y_s)}\| \mathrm{d}s \\
&+ K^2 p_n^* \int_0^n \int_0^s (n-s)^{\alpha-1}(s-\tau)^{\gamma-1} \|y_{\rho(\tau,y_\tau)}\| \mathrm{d}\tau\,\mathrm{d}s \Bigg] \\
&+ \lambda K \int_0^t (t-s)^{\alpha-1} \left| f\left(s, \frac{y_{\rho(s,\frac{y_s}{\lambda})}}{\lambda}\right)\right| \mathrm{d}s \\
&+ \lambda K^2 \int_0^t \int_0^s (t-s)^{\alpha-1}(s-\tau)^{\gamma-1} \left| f\left(\tau, \frac{y_{\rho(\tau,\frac{y_\tau}{\lambda})}}{\lambda}\right)\right| \mathrm{d}\tau\,\mathrm{d}s.
\end{aligned}
$$

Applying $(H2)$, we have for each $t \in [0, n]$ and $n \in \mathbb{N}$

$$
\begin{aligned}
|y(t)| \leqslant{} & \lambda(\overline{M}_0 L + 1)\|\varphi\| + 2\lambda\overline{M}_0 L + \lambda\overline{M}_0 L\|y_{\rho(t,y_t)}\| \\
& + \lambda K^2|A(0)| \left(\|\varphi\| + \overline{M}_0 L\left(\|\varphi\| + 1\right)\right) \int_0^t (t-s)^{\alpha-1}(1+s^\gamma)\,\mathrm{d}s \\
& + \lambda K|A(0)| \int_0^t (t-s)^{\alpha-1}\overline{M}_0 L\left(\|y_{\rho(s,y_s)}\| + 1\right)\mathrm{d}s \\
& + \lambda K\widetilde{M}\widetilde{M}_1 t^\alpha \left(\alpha^{-1} + Kt^\gamma\gamma^{-1}\beta(\alpha,\gamma+1)\right) \\
& \cdot \Bigg[|y^\star| + \left(\overline{M}_0 L + 1\right)\left(1 + K^2|A(0)|n^\alpha\Theta_n\right)\|\varphi\| \\
& + \overline{M}_0 L\Big(2 + K|A(0)|n^\alpha\left(\alpha^{-1} + K\Theta_n\right)\Big) + Kp_n^* n^\alpha\Upsilon_n + \overline{M}_0 L\left\|y_{\rho(n,y_n)}\right\| \\
& + K\overline{M}_0 L|A(0)| \int_0^n (n-s)^{\alpha-1}\left\|y_{\rho(s,y_s)}\right\|\mathrm{d}s + Kp_n^* \int_0^n (n-s)^{\alpha-1}\|y_{\rho(s,y_s)}\|\,\mathrm{d}s \\
& + K^2 p_n^* \int_0^n\int_0^s (n-s)^{\alpha-1}(s-\tau)^{\gamma-1}\|y_{\rho(\tau,y_\tau)}\|\,\mathrm{d}\tau\,\mathrm{d}s\Bigg] \\
& + \lambda K \int_0^t (t-s)^{\alpha-1} p(s)\left(1 + \left\|\frac{y_{\rho(s,\frac{y_s}{\lambda})}}{\lambda}\right\|\right)\mathrm{d}s \\
& + \lambda K^2 \int_0^t\int_0^s (t-s)^{\alpha-1}(s-\tau)^{\gamma-1} p(\tau)\left(1 + \left\|\frac{y_{\rho(\tau,\frac{y_\tau}{\lambda})}}{\lambda}\right\|\right)\mathrm{d}\tau\,\mathrm{d}s.
\end{aligned}
$$

Using Proposition 1, we get

$$
\left\|\frac{y_{\rho(s,\frac{y_s}{\lambda})}}{\lambda}\right\| \leqslant \frac{|y(s)|}{\lambda} + \mathcal{L}^\varphi\|\varphi\|. \tag{17}
$$

By the inequalities (13), (14) and the previous one, we obtain for each $t \in [0, n]$ and $n \in \mathbb{N}$

$$
\begin{aligned}
|y(t)| \leqslant{} & \lambda K\widetilde{M}\widetilde{M}_1 n^\alpha\Upsilon_n|y^\star| \\
& + \lambda\left(\overline{M}_0 L + 1\right)\left(K\widetilde{M}\widetilde{M}_1 n^\alpha\Upsilon_n + 1\right)\left(1 + K^2|A(0)|n^\alpha\Theta_n\right)\|\varphi\| \\
& + \lambda K^2\widetilde{M}\widetilde{M}_1 n^{2\alpha} p_n^*\Upsilon_n^2
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda\overline{M}_0L\left(K\widetilde{M}\widetilde{M}_1n^\alpha\Upsilon_n+1\right)\left(2+K|A(0)|n^\alpha(\alpha^{-1}+K\Theta_n)\right)\\
&+\lambda\overline{M}_0L\left(|y(t)|+\mathcal{L}^\varphi\|\varphi\|\right)\\
&+\lambda K\overline{M}_0L|A(0)|\int_0^t(t-s)^{\alpha-1}\left(|y(s)|+\mathcal{L}^\varphi\|\varphi\|\right)\mathrm{d}s\\
&+\lambda K\widetilde{M}\widetilde{M}_1n^\alpha\Upsilon_n\overline{M}_0L\left(|y^\star|+\mathcal{L}^\varphi\|\varphi\|\right)\\
&+\lambda K^2\widetilde{M}\widetilde{M}_1n^\alpha\Upsilon_n\overline{M}_0L|A(0)|\int_0^n(n-s)^{\alpha-1}\left(|y(s)|+\mathcal{L}^\varphi\|\varphi\|\right)\mathrm{d}s\\
&+\lambda K^2\widetilde{M}\widetilde{M}_1n^\alpha p_n^*\Upsilon_n\int_0^n(n-s)^{\alpha-1}\left(|y(s)|+\mathcal{L}^\varphi\|\varphi\|\right)\mathrm{d}s\\
&+\lambda K^3\widetilde{M}\widetilde{M}_1n^\alpha p_n^*\Upsilon_n\int_0^n\int_0^s(n-s)^{\alpha-1}(s-\tau)^{\gamma-1}\left(|y(\tau)|+\mathcal{L}^\varphi\|\varphi\|\right)\mathrm{d}\tau\mathrm{d}s\\
&+\lambda Kp_n^*\frac{n^\alpha}{\alpha}+K^2p_n^*n^{\alpha+\gamma}\gamma^{-1}\beta(\alpha,\gamma+1)\\
&+\lambda Kp_n^*\int_0^t(t-s)^{\alpha-1}\left(\frac{|y(s)|}{\lambda}+\mathcal{L}^\varphi\|\varphi\|\right)\mathrm{d}s\\
&+\lambda K^2p_n^*\int_0^t\int_0^s(t-s)^{\alpha-1}(s-\tau)^{\gamma-1}\left(\frac{|y(\tau)|}{\lambda}+\mathcal{L}^\varphi\|\varphi\|\right)\mathrm{d}\tau\mathrm{d}s.
\end{aligned}
$$

Consider the function $\widetilde{u}(t):=\sup\limits_{\theta\in[0,t]}|y(\theta)|$. Then, we get for each $t\in[0,n]$ and $n\in\mathbb{N}$

$$
\begin{aligned}
&\left(1-\overline{M}_0L\right)\widetilde{u}(t)\\
&\qquad\leqslant\lambda K\widetilde{M}\widetilde{M}_1n^\alpha\Upsilon_n\left(\overline{M}_0L+1\right)|y^\star|\\
&\qquad\quad+\lambda Kp_n^*n^\alpha\Upsilon_n\left(K\widetilde{M}\widetilde{M}_1n^\alpha\Upsilon_n+1\right)\\
&\qquad\quad+\lambda(\overline{M}_0L+1)\left(K\widetilde{M}\widetilde{M}_1n^\alpha\Upsilon_n+1\right)\left(1+K^2|A(0)|n^\alpha\Theta_n\right)\|\varphi\|\\
&\qquad\quad+\lambda\overline{M}_0L\left(K\widetilde{M}\widetilde{M}_1n^\alpha\Upsilon_n+1\right)\left(2+K|A(0)|n^\alpha(\alpha^{-1}+K\Theta_n)\right)
\end{aligned}
$$

$$+\lambda\overline{M}_0L\left(K\widetilde{M}\widetilde{M}_1n^{\alpha}\Upsilon_n+1\right)\mathcal{L}^{\varphi}\|\varphi\|$$
$$+\lambda K\overline{M}_0L|A(0)|\int_0^t(t-s)^{\alpha-1}\left(\widetilde{u}(s)+\mathcal{L}^{\varphi}\|\varphi\|\right)\mathrm{d}s$$
$$+\lambda K^2\widetilde{M}\widetilde{M}_1n^{\alpha}\Upsilon_n\overline{M}_0L|A(0)|\int_0^n(n-s)^{\alpha-1}\left(\widetilde{u}(s)+\mathcal{L}^{\varphi}\|\varphi\|\right)\mathrm{d}s$$
$$+\lambda K^2\widetilde{M}\widetilde{M}_1n^{\alpha}p_n^*\Upsilon_n\int_0^n(n-s)^{\alpha-1}\left(\widetilde{u}(s)+\mathcal{L}^{\varphi}\|\varphi\|\right)\mathrm{d}s$$
$$+\lambda K^3\widetilde{M}\widetilde{M}_1n^{\alpha}p_n^*\Upsilon_n\int_0^n\int_0^s(n-s)^{\alpha-1}(s-\tau)^{\gamma-1}\left(\widetilde{u}(\tau)+\mathcal{L}^{\varphi}\|\varphi\|\right)\mathrm{d}\tau\,\mathrm{d}s$$
$$+\lambda Kp_n^*\int_0^t(t-s)^{\alpha-1}\left(\frac{\widetilde{u}(s)}{\lambda}+\mathcal{L}^{\varphi}\|\varphi\|\right)\mathrm{d}s$$
$$+\lambda K^2p_n^*\int_0^t\int_0^s(t-s)^{\alpha-1}(s-\tau)^{\gamma-1}\left(\frac{\widetilde{u}(\tau)}{\lambda}+\mathcal{L}^{\varphi}\|\varphi\|\right)\mathrm{d}\tau\,\mathrm{d}s.$$

We set

$$\begin{aligned}\varpi_n := {} & K\widetilde{M}\widetilde{M}_1n^{\alpha}\Upsilon_n(\overline{M}_0L+1)|y^{\star}|+Kp_n^*n^{\alpha}\Upsilon_n\left(K\widetilde{M}\widetilde{M}_1n^{\alpha}\Upsilon_n+1\right)\\ &+(\overline{M}_0L+1)\left(K\widetilde{M}\widetilde{M}_1n^{\alpha}\Upsilon_n+1\right)\left(1+K^2|A(0)|n^{\alpha}\Theta_n\right)\|\varphi\|\\ &+\overline{M}_0L\left(K\widetilde{M}\widetilde{M}_1n^{\alpha}\Upsilon_n+1\right)\left(2+K|A(0)|n^{\alpha}(\alpha^{-1}+K\Theta_n)\right)\\ &+\overline{M}_0L\left(K\widetilde{M}\widetilde{M}_1n^{\alpha}\Upsilon_n+1\right)\mathcal{L}^{\varphi}\|\varphi\|.\end{aligned}$$

So, for each $t\in[0,n]$ and $n\in\mathbb{N}$, we have

$$\left(1-\overline{M}_0L\right)\widetilde{u}(t)$$
$$\leqslant\lambda\varpi_n+\lambda K\overline{M}_0L|A(0)|\int_0^t(t-s)^{\alpha-1}\left(\widetilde{u}(s)+\mathcal{L}^{\varphi}\|\varphi\|\right)\mathrm{d}s$$

$$
\begin{aligned}
&+\lambda K^2\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n \overline{M}_0 L|A(0)|\int_0^n (n-s)^{\alpha-1}\left(\widetilde{u}(s)+\mathcal{L}^\varphi\|\varphi\|\right)\mathrm{d}s\\
&+\lambda K^2\widetilde{M}\widetilde{M}_1 n^\alpha p_n^*\Upsilon_n \int_0^n (n-s)^{\alpha-1}\left(\widetilde{u}(s)+\mathcal{L}^\varphi\|\varphi\|\right)\mathrm{d}s\\
&+\lambda K^3\widetilde{M}\widetilde{M}_1 n^\alpha p_n^*\Upsilon_n \int_0^n\int_0^s (n-s)^{\alpha-1}(s-\tau)^{\gamma-1}\left(\widetilde{u}(\tau)+\mathcal{L}^\varphi\|\varphi\|\right)\mathrm{d}\tau\,\mathrm{d}s\\
&+\lambda K p_n^*\int_0^t (t-s)^{\alpha-1}\left(\frac{\widetilde{u}(s)}{\lambda}+\mathcal{L}^\varphi\|\varphi\|\right)\mathrm{d}s\\
&+\lambda K^2 p_n^*\int_0^t\int_0^s (t-s)^{\alpha-1}(s-\tau)^{\gamma-1}\left(\frac{\widetilde{u}(\tau)}{\lambda}+\mathcal{L}^\varphi\|\varphi\|\right)\mathrm{d}\tau\,\mathrm{d}s.
\end{aligned}
$$

Thus, for each $t\in[0,n]$ and $n\in\mathbb{N}$, we obtain

$$
\begin{aligned}
\frac{\widetilde{u}(t)}{\lambda}+\mathcal{L}^\varphi\|\varphi\| \leqslant{}& \mathcal{L}^\varphi\|\varphi\|+\frac{\varpi_n}{1-\overline{M}_0 L}\\
&+\frac{1}{1-\overline{M}_0 L}\Bigg[K\overline{M}_0 L|A(0)|\int_0^t (t-s)^{\alpha-1}\left(\widetilde{u}(s)+\mathcal{L}^\varphi\|\varphi\|\right)\mathrm{d}s\\
&+K^2\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n \overline{M}_0 L|A(0)|\int_0^n (n-s)^{\alpha-1}\left(\widetilde{u}(s)+\mathcal{L}^\varphi\|\varphi\|\right)\mathrm{d}s\\
&+K^2\widetilde{M}\widetilde{M}_1 n^\alpha p_n^*\Upsilon_n \int_0^n (n-s)^{\alpha-1}\left(\widetilde{u}(s)+\mathcal{L}^\varphi\|\varphi\|\right)\mathrm{d}s\\
&+K^3\widetilde{M}\widetilde{M}_1 n^\alpha p_n^*\Upsilon_n \int_0^n\int_0^s (n-s)^{\alpha-1}(s-\tau)^{\gamma-1}\left(\widetilde{u}(\tau)+\mathcal{L}^\varphi\|\varphi\|\right)\mathrm{d}\tau\,\mathrm{d}s\\
&+K p_n^*\int_0^t (t-s)^{\alpha-1}\left(\frac{\widetilde{u}(s)}{\lambda}+\mathcal{L}^\varphi\|\varphi\|\right)\mathrm{d}s
\end{aligned}
$$

$$+ K^2 p_n^* \int_0^t \int_0^s (t-s)^{\alpha-1}(s-\tau)^{\gamma-1}\left(\frac{\widetilde{u}(\tau)}{\lambda} + \mathcal{L}^{\varphi}\|\varphi\|\right) \mathrm{d}\tau \, \mathrm{d}s \Bigg].$$

Set $\Lambda_n := \mathcal{L}^{\varphi}\|\varphi\| + \dfrac{\varpi_n}{1-\overline{M}_0 L}$. For $\lambda < 1$, we obtain for each $t \in [0,n]$ and $n \in \mathbb{N}$

$$\begin{aligned}
\frac{\widetilde{u}(t)}{\lambda} + \mathcal{L}^{\varphi}\|\varphi\| \leqslant \Lambda_n + \frac{K}{1-\overline{M}_0 L}\Bigg[ & \overline{M}_0 L |A(0)| \int_0^t (t-s)^{\alpha-1}\left(\frac{\widetilde{u}(s)}{\lambda} + \mathcal{L}^{\varphi}\|\varphi\|\right) \mathrm{d}s \\
& + K\widetilde{M}\widetilde{M}_1 n^{\alpha} \Upsilon_n \overline{M}_0 L |A(0)| \int_0^n (n-s)^{\alpha-1}\left(\frac{\widetilde{u}(s)}{\lambda} + \mathcal{L}^{\varphi}\|\varphi\|\right) \mathrm{d}s \\
& + K\widetilde{M}\widetilde{M}_1 n^{\alpha} p_n^* \Upsilon_n \int_0^n (n-s)^{\alpha-1}\left(\frac{\widetilde{u}(s)}{\lambda} + \mathcal{L}^{\varphi}\|\varphi\|\right) \mathrm{d}s \\
& + K^2\widetilde{M}\widetilde{M}_1 n^{\alpha} p_n^* \Upsilon_n \int_0^n \int_0^s (n-s)^{\alpha-1}(s-\tau)^{\gamma-1}\left(\frac{\widetilde{u}(\tau)}{\lambda} + \mathcal{L}^{\varphi}\|\varphi\|\right) \mathrm{d}\tau \, \mathrm{d}s \\
& + p_n^* \int_0^t (t-s)^{\alpha-1}\left(\frac{\widetilde{u}(s)}{\lambda} + \mathcal{L}^{\varphi}\|\varphi\|\right) \mathrm{d}s \\
& + K p_n^* \int_0^t \int_0^s (t-s)^{\alpha-1}(s-\tau)^{\gamma-1}\left(\frac{\widetilde{u}(\tau)}{\lambda} + \mathcal{L}^{\varphi}\|\varphi\|\right) \mathrm{d}\tau \, \mathrm{d}s \Bigg].
\end{aligned}$$

We consider the function $\mu$ defined by

$$\mu(t) := \sup_{0 \leqslant s \leqslant t} \frac{\widetilde{u}(s)}{\lambda} + \mathcal{L}^{\varphi}\|\varphi\|, \qquad \text{for } t \in J.$$

Let $t^{\star} \in [-r,t]$ be such that $\mu(t^{\star}) = \dfrac{\widetilde{u}(t^{\star})}{\lambda} + \mathcal{L}^{\varphi}\|\varphi\|$. If $t^{\star} \in [0,t]$, by the previous inequality, we have for each $t \in [0,n]$ and $n \in \mathbb{N}$

$$\mu(t) \leqslant \Lambda_n + \frac{K}{1-\overline{M}_0 L}\Bigg[\overline{M}_0 L |A(0)| \int_0^t (t-s)^{\alpha-1}\mu(s)\, \mathrm{d}s$$

$$
\begin{aligned}
&+ K\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n \overline{M}_0 L|A(0)| \int_0^n (n-s)^{\alpha-1}\mu(s)\,\mathrm{d}s \\
&+ K\widetilde{M}\widetilde{M}_1 n^\alpha p_n^* \Upsilon_n \int_0^n (n-s)^{\alpha-1}\mu(s)\,\mathrm{d}s \\
&+ K^2\widetilde{M}\widetilde{M}_1 n^\alpha p_n^* \Upsilon_n \int_0^n\int_0^s (n-s)^{\alpha-1}(s-\tau)^{\gamma-1}\mu(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&+ p_n^* \int_0^t (t-s)^{\alpha-1}\mu(s)\,\mathrm{d}s + Kp_n^* \int_0^t\int_0^s (t-s)^{\alpha-1}(s-\tau)^{\gamma-1}\mu(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\Bigg] \\
\leqslant{}& \Lambda_n + \frac{K}{1-\overline{M}_0 L}\Bigg[\overline{M}_0 L|A(0)|\left(K\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n + 1\right)\int_0^n (n-s)^{\alpha-1}\mu(s)\,\mathrm{d}s \\
&+ p_n^*\left(K\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n + 1\right)\int_0^n (n-s)^{\alpha-1}\mu(s)\,\mathrm{d}s \\
&+ Kp_n^*\left(K\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n + 1\right)\int_0^n\int_0^s (n-s)^{\alpha-1}(s-\tau)^{\gamma-1}\mu(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\Bigg] \\
\leqslant{}& \Lambda_n + \frac{K}{1-\overline{M}_0 L}\Bigg[\overline{M}_0 L|A(0)| n^\alpha \alpha^{-1}\left(K\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n + 1\right)\|\mu\| \\
&+ p_n^* n^\alpha \alpha^{-1}\left(K\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n + 1\right)\|\mu\| \\
&+ Kp_n^*\left(K\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n + 1\right) n^{\alpha+\gamma}\gamma^{-1}\beta(\alpha,\gamma+1)\|\mu\|\Bigg] \\
\leqslant{}& \Lambda_n + \frac{K}{1-\overline{M}_0 L}\Bigg[\overline{M}_0 L|A(0)| n^\alpha \alpha^{-1}\left(K\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n + 1\right)\|\mu\| \\
&+ p_n^* n^\alpha \Upsilon_n\left(K\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n + 1\right)\|\mu\|\Bigg] \\
\leqslant{}& \Lambda_n + \frac{Kn^\alpha}{1-\overline{M}_0 L}\left(K\widetilde{M}\widetilde{M}_1 n^\alpha \Upsilon_n + 1\right)\left(\overline{M}_0 L|A(0)|\alpha^{-1} + p_n^*\Upsilon_n\right)\|\mu\|.
\end{aligned}
$$

Consequently,

$$\|y\|_n \leqslant \frac{\Lambda_n}{1 - Kn^{\alpha}\dfrac{1 + K\widetilde{M}\widetilde{M}_1 n^{\alpha}\Upsilon_n}{1 - \overline{M}_0 L}\left(\overline{M}_0 L|A(0)|\alpha^{-1} + p_n^*\Upsilon_n\right)},$$

which shows that the set $\Gamma$ is bounded, i.e. the statement (C2) in Theorem 1 does not hold. Then the nonlinear alternative of Avramescu [6] implies that (C1) holds: i.e. the operator $F + G$ has a fixed-point $y^{\star}$. Then, there exists at least $y^{\star}(t)$, $t \in [-r, +\infty)$ which is a fixed point of the operator $N$, which is a mild solution of the problem (1)–(2). Thus the evolution system (1)–(2) is controllable on $[-r, +\infty)$.

## 4. Example

We consider the following control problem given by the neutral functional differential equation

$$\begin{cases} {}^cD_0^{\alpha}\left[v(t,\xi) - \displaystyle\int_{-r}^{0} a_3(s-t)v\left(s - \rho_1(t)\rho_2\left(\int_0^{\pi} a_2(\eta)|v(t,\eta)|^2\,\mathrm{d}\eta\right),\xi\right)\mathrm{d}s\right] \\ = a_0(t,\xi)\dfrac{\partial^2 v}{\partial \xi^2}(t,\xi) + d(\xi)u(t) \\ \quad + \displaystyle\int_{-r}^{0} a_1(s-t)v\left[s - \rho_1(t)\rho_2\left(\int_0^{\pi} a_2(\eta)|v(t,\eta)|^2\,\mathrm{d}\eta\right),\xi\right]\mathrm{d}s, \\ \hfill t \geqslant 0,\ \xi \in [0,\pi], \\ v(t,0) = v(t,\pi) = 0, \hfill t \geqslant 0, \\ v(\theta,\xi) = v_0(\theta,\xi), \hfill -r < \theta \leqslant 0,\ \xi \in [0,\pi], \end{cases} \tag{18}$$

where $0 < \alpha < 1$, $a_0 : \mathbb{R}^+ \times [0,\pi] \to \mathbb{R}$ is a given function such that $a_0(\cdot,\xi)$ is continuous and $a_0(t,\cdot)$ is uniformly Hölder continuous in $t$; $a_1,\ a_3 : [-r,0] \to \mathbb{R}$; $\rho_1 : \mathbb{R}^+ \to \mathbb{R}$ ; $\rho_2 : \mathbb{R} \to \mathbb{R}$; $a_2 : [0,\pi] \to \mathbb{R}$ and $v_0 : [-r,0] \times [0,\pi] \to \mathbb{R}$ are continuous functions.

Let $E = L^2([0,\pi],\mathbb{R})$, $u(\cdot) : \mathbb{R}^+ \to E$ is a given control and $d : [0,\pi] \to E$ is a continuous function.

Consider the operator $A : D(A) \subset E \to E$ given by $Aw = w''$ with domain

$$D(A) := \{w \in E : w'' \in E, w(0) = w(\pi) = 0\}.$$

A is the infinitesimal generator of an analytic semigroup $e^{tA(s)}$ on $E$ which satisfies the assumptions $(A_1)$ and $(A_2)$.

**Theorem 3.** *Let $\varphi \in C(H,E)$. Assume that condition $(H_\varphi)$ holds and the functions $d : [0,\pi] \to E$, $a_1$, $a_3 : [-r,0] \to \mathbb{R}$; $\rho_1 : \mathbb{R}^+ \to \mathbb{R}$ ; $\rho_2 : \mathbb{R} \to \mathbb{R}$; $a_2 : [0,\pi] \to \mathbb{R}$ and $v_0 : [-r,0] \times [0,\pi] \to \mathbb{R}$ are continuous. Then the system (18) is controllable on $[-r,+\infty)$.*

**Proof.** From the assumptions, we have

$$y(t)(\xi) = v(t,\xi), \qquad t\in[-r,+\infty),\ \xi\in[0,\pi],$$

$$f(t,\psi)(\xi) = \int_{-r}^{0} a_1(s)\psi(s,\xi)\,\mathrm{d}s, \qquad t\geqslant 0,\ \xi\in[0,\pi],$$

$$g(t,\psi)(\xi) = \int_{-r}^{0} a_3(s)\psi(s,\xi)\,\mathrm{d}s, \qquad t\geqslant 0,\ \xi\in[0,\pi],$$

$$\rho(t,\psi)(\xi) = t-\rho_1(t)\rho_2\left(\int_0^{\pi} a_2(\eta)|\psi(0,\xi)|^2\,\mathrm{d}\eta\right), \quad t\geqslant 0,\ \xi\in[0,\pi],$$

$$Cu(t)(\xi) = d(\xi)u(t), \qquad t\geqslant 0,\ \xi\in[0,\pi],\ u\in\mathbb{R},\ d(\xi)\in E,$$

and

$$\varphi(t)(\xi) = v_0(t,\xi), \qquad -r<t\leqslant 0,\ \xi\in[0,\pi],$$

are well defined which permits to transform system (18) into the abstract system (1)–(2). Moreover, the function $f$ and $g$ are bounded linear operators. Then, the controllability of mild solutions can be deduced from a direct application of Theorem 2.

From Remark 1, we have the following result.

**Corollary 1.** *Let $\varphi \in C(H,E)$ be continuous and bounded. Then the system (18) is controllable on $[-r,+\infty)$.*

## References


[1] N. Abada, R.P. Agarwal, M. Benchohra and H. Hammouche: Existence results for nondensely defined impulsive semilinear functional differential equations with state-dependent delay. *Asian-European Journal of Mathematics*, **1**(4), (2008), 449–468. DOI: 10.1142/s1793557108000382

[2] I. Abibssi and S. Baghli-Bendimerad: On nonlocal second order perturbed pseudo integro-differential evolution equations with infinite state-dependent delay. *Mathematica Applicanda (Journals of the Polish Mathematical Society)*, **52**(2), (2024), 271–298. DOI: 10.14708/ma.v52i2.7257

[3] I. Abibssi and S. Baghli-Bendimerad: On second order perturbed evolution equations with infinite state-dependent delay. *Applicationes Mathematicae*, **52**(1), (2025), 77–99. DOI: 10.4064/am2500-2-2025

[4] R.P. Agarwal, S. Baghli and M. Benchohra: Controllability for semilinear functional and neutral functional evolution equations with infinite delay in Fréchet spaces. *Applied Mathematics and Optimization*, **60** (2009), 253–274. DOI: 10.1007/s00245-009-9073-1

[5] D. Aoued and S. Baghli-Bendimerad: Mild solution for perturbed evolution equations with infinite state-dependent delay. *Electronic Journal of Qualitative Theory of Differential Equations*, **59** (2013), 1–24. DOI: 10.14232/ejqtde.2013.1.59

[6] C. Avramescu: Some remarks on a fixed point theorem of Krasnoselskii. *Electronic Journal of Qualitative Theory of Differential Equations*, **5** (2003), 1–15. DOI: 10.14232/ejqtde.2003.1.5

[7] S. Baghli-Bendimerad: Global mild solution for functional evolution inclusions with state-dependent delay. *Journal of Advanced Research in Dynamical and Control Systems*, **5**(4), (2013), 1-19.

[8] S. Baghli-Bendimerad and I. Abibssi: Existence of mild solutions for nonlocal perturbed evolution equations with infinite state-dependent delay in Fréchet spaces. *Archives of Control Sciences*, **35**(2), (2025), 321–349. DOI: 10.24425/acs.2025.155398

[9] S. Baghli and M. Benchohra: Global uniqueness results for partial functional and neutral functional evolution equations with infinite delay. *Differential Integral Equations*, **23**(1/2), (2010), 31–50. DOI: 10.57262/die/1356019385

[10] S. Baghli, M. Benchohra and K. Ezzinbi: Controllability results for semilinear functional and neutral functional evolution equations with infinite delay. *Surveys in Mathematics and its Applications*, **4**(2), (2009), 15–39.

[11] S.E. Bendimerad, A. Ayad, A. Bettache, S. Baghli and A.N.E.I. Ayad: A novel formula for calculating repulsive forces in horizontally aligned nonferrous cylindrical particles within an Eddy current separator. *Powder Technology*, **120300** (2024). DOI: 10.1016/j.powtec.2024.120300

[12] S.E. Bendimerad, S. Baghli, A. Ayad and A. Tilmatine: Mutual inductance of two coaxial rectangular PCB coils incorporating magnetic layer. *Journal of Electrical Engineering: The Journal of Slovak University of Technology*, **72**(5), (2021), 28–296. DOI: 10.2478/jee-2021-0041

[13] C. Boudefla and S. Baghli-Bendimerad: Nonlocal controllability of mild solutions for evolution equations with state-dependent delay in Fréchet spaces. *International Journal of Applied Mathematics*, **37**(1), (2024), 53–71. DOI: 10.12732/ijam.v37i1.6

[14] C. Boudefla and S. Baghli-Bendimerad: Nonlocal controllability of mild solutions for neutral evolution equations with state-dependent delay in Fréchet spaces. *Archives of Control Sciences*, **34**(1), (2024), 117–148. DOI: 10.24425/acs.2024.149655

[15] C. Boudefla, F. Sahraoui and S. Baghli-Bendimerad: Controllability of mild solutions for second-order neutral evolution equations with state-dependent delay. *Complex Analysis and Operator Theory*, **18** (2024). DOI: 10.1007/s11785-024-01524-2

[16] M.M. El-Borai: The fundamental solutions for fractional evolution equations of parabolic type. *Journal of Applied Mathematics and Stochastic Analysis*, **3** (2004), 197–211. DOI: 10.1155/S1048953304311020

[17] M. Frigon and A. Granas: Résultats de type Leray-Schauder pour des contractions sur des espaces de Fréchet. *Annales mathématiques du Québec*, **22**(2), (1998), 161–168.

[18] R. Gorenflo and F. Mainardi: Fractional calculus and stable probability distributions. *Archives of Mechanics*, **50**(3), (1998), 377–388.

[19] A. Granas and J. Dugundji: *Fixed Point Theory*. Springer-Verlag, New York, 2003. DOI: 10.1007/978-0-387-21593-8

[20] D. Henry: *Geometric Theory of Semilinear Parabolic Equations*. Lecture Notes in Mathematics, **840**, Springer-verlag, New York, 1981.

[21] E. Hernandez, R. Sakthivel and S. Tanaka Aki: Existence results for impulsive evolution differential equations with state-dependent delay. *Electronic Journal of Differential Equations*, **28** (2008), 1–11.

[22] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo: *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, Amsterdam, 2006.

[23] N. Lachachi-Merad, S. Baghli-Bendimerad and M. Benchohra: Unique mild solutions for Caputo's fractional perturbed evolution equations with state-dependent delay. *Evolution Equations and Control Theory*, **13**(1), (2024), 160–172. DOI: 10.3934/eect.2023041

[24] N. Lachachi-Merad, S. Baghli-Bendimerad, M. Benchohra and E. Karapinar: Nonlocal partial fractional evolution equations with state-dependent delay. *Proyecciones Journal of Mathematics*, **42**(5), (2023), 1191–1210. DOI: 10.22199/issn.0717-6279-5317

[25] N. Lachachi-Merad, S. Baghli-Bendimerad, M. Benchohra and D. Aouad: Unique mild solution for fractional partial and neutral evolution equations with state-dependent delay. *Rocky Mountain Journal of Mathematics*, **55**(5), (2025), 1351-1359. DOI: 10.1216/rmj.2025.55.1351

[26] A. Mebarki and S. Baghli-Bendimerad: Neutral multivalued integro-differential evolution equations with infinite state-dependent delay. *Turkish Journal of Mathematics*, **44** (2020), 2312–2329. DOI: 10.3906/mat-2007-66

[27] F. Mesri, J. E. Lazreg, M. Benchohra and J. Henderson: Fractional non autonomous evolution equations in Fréchet spaces. *Communications on Applied Nonlinear Analysis*, **28** (2021), 35-45.

[28] A. Pazy: *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag, New York, 1983. DOI: 10.1007/978-1-4612-5561-1

[29] I. Podlubny: *Fractional Differential Equation*. Academic Press, San Diego, 1999.

[30] M.D. Quinn and N. Carmichael: An approach to nonlinear control problems using fixed point methods, degree theory and pseudo-inverses. *Numerical Functional Analysis and Optimization*, **7**(2-3), (1985), 197–219. DOI: 10.1080/01630568508816189

[31] R. Temam: *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*. 2nd edn. Springer, New York, 1997.