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Asymptotic behavior of nonlinear systems with impulses: Application to Hopfield neural networks

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In this paper, we provide some sufficient conditions for the exponential stability of solutions of nonlinear impulsive differential systems by using some inequality of Gronwall-Bellman type. Practical exponential stability is also investigated for a class of perturbed impulsive systems. Several numerical examples are provided to demonstrate the effectiveness of the theoretical results. Furthermore, Hopfield neural networks system is discussed as an application.

Key words: nonlinear systems, impulsive perturbation, asymptotic stability, practical exponential stability, Gronwall-Bellman inequality.

1. Introduction

The impulsive differential equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states. It is now recognized that the theory of impulsive differential equations is not only richer than the corresponding theory of differential equations but also represents a more natural framework for mathematical modelling of many real-world phenomena. Hence it has been widely used in many fields such as engineering [10], theoretical physics, industrial robotics, control theory [16, 21], mechanical systems with impact, aircraft control, biological systems such as heartbeats, blood flow, neural networks [30, 36, 37], medical science, medicine [9, 12], population dynamics [29], chemical technology and so on. Many researchers have made great contributions to this topic. Various examples of problems of this sort can be found

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in the works by A. Halanay, D. Wexler [17], Myshkis and Samoilenko [33], Bainov and Simeonov [2, 3], Lakshimikantham et al. [26], Dishliev and Bainov [11] in 1989, Milman and Myshkis 1960 [31], Leela 1977 [27].

Stability is one of the most important issues in the study of differential systems with and without impulses [4, 5, 19]. Therefore, several types of stability of impulsive systems have been defined and studied in the past by applying various methods [1, 12, 18, 35]. Milman and Myshkis [31] investigated the stability of the zero solution of differential equations with impulses effects by using the second Lyapunov method. Bainov and Simeonov [2] investigated the exponential stability of the solutions for impulsive differential equations by using the comparison method and piecewise continuous auxiliary functions which are analogues to Lyapunov's functions. Also, Kulev and Bainov [25] introduced the notions of various types of uniform Lipschitz stability for impulsive differential equations and obtained sufficient conditions for these notions and their relations. A number of authors investigated the asymptotic stability and solution behaviors for specific classes of nonlinear systems by enforcing certain necessary requirements (see [22–24, 34]). Other results on the study of stability of impulsive differential equations are based on the application of the integral inequalities method for discontinuous functions. Moreover Gronwall-Bellman-Bihari inequalities [6, 13, 14] and their extensions play an important role in studing the qualitative behavior of solutions of impulsive differential equations such as existence uniqueness, boundedness, stability with various kinds of perturbations [4, 7, 8, 20].

In this paper, we investigate the problem of stability for impulsive differential equations by Gronwall integral inequalities. In Section 2, we describe impulsive differential systems and introduce some notations and definitions. In Section 3, we establish some exponential stability criteria. An application is given in Section 4, where we apply our results to Hopfield neural networks.

2. Statement of the problem

Throughout this paper, let $\mathbb{R}^n$ denote the n-dimensional Euclidean space endowed with a suitable norm $|\cdot|$, and let Ω be an open subset of $\mathbb{R}^n$ containing the origin. We consider the following class of impulsive differential systems:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + F(t, x(t)), & t \neq t_i, \\ \Delta x(t) = I_i(x(t)), & t = t_i,\ i = 1, 2, \ldots, \\ x(t_0) = x_0. \end{cases} \tag{1}$$

We impose the following assumptions:

($\mathcal{A}_1$) The sequence of impulse moments $\{t_i\}$ is strictly increasing and unbounded, satisfying

$$0 \leqslant t_0 < t_1 < t_2 < \cdots, \qquad \lim_{i\to\infty} t_i = +\infty.$$

($\mathcal{A}_2$) The function $F : \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}^n$ is piecewise continuous in t, with discontinuities of the first kind at $t = t_i$, left-continuous at each t_i, and locally Lipschitz continuous with respect to x on $(t_{i-1}, t_i]\times\Omega$ for $i = 1, 2, \ldots$. Moreover, $F(t, 0) = 0$ for all $t \in \mathbb{R}_+$.

($\mathcal{A}_3$) The matrix-valued function $A : \mathbb{R}_+ \longrightarrow M_n(\mathbb{R})$ is continuous except for possible discontinuities of the first kind at the impulse moments $t = t_i$.

($\mathcal{A}_4$) For each $i = 1, 2, \ldots$, the impulse function $I_i : \Omega \longrightarrow \mathbb{R}^n$ is continuous and locally Lipschitz continuous, and satisfies $I_i(0) = 0$.

($\mathcal{A}_5$) The solution $x(t, t_0, x_0)$ of system (1), satisfying the initial condition $x(t_0^+, t_0, x_0) = x_0$, is defined on the interval $[t_0, +\infty)$.

($\mathcal{A}_6$) The impulse moments t_i satisfy the following conditions:

(i) For $t \neq t_i$, the solution $x(t, t_0, x_0)$ evolves according to the ordinary differential equation

$$\dot{x}(t) = A(t)x(t) + F(t, x(t)).$$

(ii) At each impulse moment $t = t_i$, the solution undergoes a jump discontinuity such that

$$x(t_i^-) = x(t_i), \qquad x(t_i^+) = x(t_i) + \Delta x(t_i) = I_i(x(t_i)).$$

2.1. Definitions

To study the stability of the perturbed system, first, we recall the following definitions.

Definition 1. *The equilibrium point $x^* = 0$ is said*

(i) uniformly exponentially stable (U.E.S) if there exist $\delta > 0$, $k > 0$ and $\alpha > 0$ such that $\forall\, t_0 \geqslant 0$, $\forall\, \|x_0\| \leqslant \delta$,

$$\|x(t)\| \leqslant k\|x_0\|e^{-\alpha(t-t_0)} \quad \forall\, t \geqslant t_0.$$

(ii) globally uniformly exponentially stable (G.U.E.S) if there exist $k > 0$ and $\alpha > 0$ such that $\forall\, t_0 \geqslant 0$ and $\forall\, x_0 \in \mathbb{R}^n$,

$$\|x(t)\| \leqslant k\|x_0\|e^{-\alpha(t-t_0)}, \quad \forall\, t \geqslant t_0.$$

Definition 2. *Let $r \geqslant 0$ and $\mathcal{B}_r = \{x \in \mathbb{R}^n \mid \|x\| \leqslant r\}$.*

(i) The ball $\mathcal{B}_r$ is globally uniformly exponentially stable if there exist $\alpha > 0$ and $k \geqslant 0$ such that $\forall\, t_0 \in \mathbb{R}_+$ and $\forall\, x_0 \in \mathbb{R}^n$,

$$\|x(t)\| \leqslant k\|x_0\|\, e^{-\alpha(t-t_0)} + r, \quad \forall\, t \geqslant t_0.$$

(ii) The system (1) is globally practically uniformly exponentially stable if there exists $r > 0$ such that $\mathcal{B}_r$ is globally uniformly exponentially stable.

2.2. Integral inequalities

We need the following lemmas which play a major role in the proof of the theorems.

Lemma 1. *[32] Let us consider that a nonnegative piecewise continuous function $u(t)$ at $t \geqslant t_0 \geqslant 0$, with discontinuities of first kind in the points t_i ($t_0 < t_1 < t_2 <, \ldots, \lim\limits_{i\to\infty} t_i = \infty$) and suppose that $p \in C(\mathbb{R}_+, \mathbb{R}_+)$ and for $i = 1, 2, \ldots$*

$$u(t) \leqslant c + \int\limits_{t_0}^{t} p(s)u(s)\,\mathrm{d}s + \sum_{t_0<t_i<t} \beta_i u(t_i), \quad \forall\, t \geqslant t_0,$$

where $\beta_i \geqslant 0$, and c are constants. Then,

$$u(t) \leqslant c \prod_{t_0<t_i<t} (1+\beta_i) \exp\left(\int\limits_{t_0}^{t} p(\sigma)\,\mathrm{d}\sigma\right), \quad \forall\, t \geqslant t_0.$$

Corollary 1. *[14] Assume that for all $x \geqslant x_0$ the following integro–sum inequality holds:*

$$u(x) \leqslant \varphi(x) + q(x) \int\limits_{x_0}^{x} f(\tau)\, W\big(u(p(\tau))\big)\,\mathrm{d}\tau + \sum_{x_0<x_i<x} \beta_i\, u^m(x_i - 0), \tag{2}$$

where $q(x) \geqslant 1$, $\varphi(x)$ is a positive and nondecreasing function, $\beta_i \geqslant 0$ are constants, $f : \mathbb{R}_+ \to \mathbb{R}_+$, and $m > 0$ is a constant. The function $u(x)$ is nonnegative and piecewise continuous, with jump discontinuities of the first kind at the points

$$x_0 < x_1 < x_2 < \cdots, \qquad \lim_{n\to\infty} x_n = \infty,$$

and $p : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

$$p(t) \leqslant t, \qquad \lim_{|t|\to\infty} p(t) = \infty.$$

Then the following estimates hold:

(A) If $q(x) = 1$, $W(u) = u$, and $p(t) = t$, then for $m \in (0,1]$ and all $x \geqslant x_0$,

$$u(x) \leqslant \varphi(x) \prod_{x_0<x_i<x} \left(1 + \beta_i \, \varphi^{m-1}(x_i)\right) \exp\left(\int_{x_0}^{x} f(\tau)\,\mathrm{d}\tau \right).$$

(B) If $W(u) = u^l$ with $l \neq 1$ and $m = 1$, then for $0 < l < 1$ and all $x \geqslant x_0$,

$$u(x) \leqslant \prod_{x_0<x_i<x} (1+\beta_i) \left[\varphi(x)^{1-l} + (1-l) \int_{x_0}^{x} f(\tau)\,\mathrm{d}\tau \right]^{\frac{1}{1-l}}.$$

Lemma 2. (*see [15]*) *Suppose that $a, b \in \mathbb{R}$, $p > 0$. Then*

$$(|a| + |b|)^p \leqslant C_p \left(|a|^p + |b|^p\right),$$

where $C_p = 1$ for $0 < p \leqslant 1$, and $C_p = 2^{p-1}$ for $p > 1$.

3. Main results

The linear time varying system

$$\dot{x} = A(t)x, \tag{3}$$

is said uniformly asymptotically stable if and only if there exist constants $k > 0$ and $\alpha > 0$ such that $\forall\, t_0 \geqslant 0$,

$$\|R(t,t_0)\| \leqslant k e^{-\alpha(t-t_0)}, \ \forall\, t \geqslant t_0, \tag{4}$$

where $R(t,t_0)$ denotes the state transition matrix of the system (3) see [26] for instance). Hence, in the linear case, uniform asymptotic stability of an equilibrium point is equivalent to its global uniform exponential stability. To investigate the global exponential stability of system (1), we assume throughout the paper that system (3) represents the nominal system, corresponding to the linear part of the perturbed system (1).

Theorem 1. *Assume that the nominal system (3) is uniformly asymptotically stable and let the following conditions be fulfilled*
(i) there exists a function $\lambda \in L^p(\mathbb{R}_+, \mathbb{R}_+)$ for some $p \in [1, +\infty)$ verifying

$$\|F(t,x)\| \leqslant \lambda(t)\|x\|,$$

(ii) $\|I_i\| \leqslant \beta_i \|x\|$, $\quad \beta_i = const. > 0$,
(iii) there exists a positive constant $m > 0$ such that

$$\prod_{t_0<t_i<t} (1 + k\beta_i) \leqslant m < \infty.$$

Then the trivial solution of system (1) *is globally uniformly exponentially stable.*

Proof. Consider $x(t) = x(t,t_0,x_0)$ the solution of the Cauchy problem for system (1). Then

$$x(t,t_0,x_0) = R(t,t_0)x_0 + \int_{t_0}^{t} R(t,s)F(s,x(s,t_0,x_0))\,\mathrm{d}s + \sum_{t_0<t_i<t} R(t,t_i)I_i(x(t_i-0,t_0,x_0)).$$

Using (4) and the assumptions of Theorem 1, we obtain

$$\|x(t,t_0,x_0)\| \leqslant ke^{-\alpha(t-t_0)}\|x_0\| + k\int_{t_0}^{t} e^{-\alpha(t-s)}\lambda(s)\|x(s,t_0,x_0)\|\,\mathrm{d}s + \sum_{t_0<t_i<t} ke^{-\alpha(t-t_i)}\beta_i\|x(t_i-0,t_0,x_0)\|. \tag{5}$$

Multiplying both sides by $e^{\alpha(t-t_0)}$ and denoting $V(t) = \|x(t,t_0,x_0)\|e^{\alpha(t-t_0)}$ yields

$$V(t) \leqslant k\|x_0\| + k\int_{t_0}^{t} \lambda(s)V(s)\,\mathrm{d}s + \sum_{t_0<t_i<t} k\beta_i V(t_i-0).$$

It follows from Lemma 1, that

$$V(t) \leqslant k\|x_0\| \prod_{t_0<t_i<t} (1+k\beta_i)e^{k\int_{t_0}^{t}\lambda(s)\,\mathrm{d}s}.$$

Since $\|x(t,t_0,x_0)\| = e^{-\alpha(t-t_0)}V(t)$, we obtain

$$\|x(t,t_0,x_0)\| \leqslant k\|x_0\| \prod_{t_0<t_i<t} (1+k\beta_i)e^{k\int_{t_0}^{t}\lambda(s)\,\mathrm{d}s-\alpha(t-t_0)}.$$

From condition (iii), we get

$$\|x(t,t_0,x_0)\| \leqslant km\|x_0\|e^{k\int_{t_0}^{t}\lambda(s)\,\mathrm{d}s-\alpha(t-t_0)}. \tag{6}$$

Now, we discuss three cases.

First case: if $\lambda \in L^1(\mathbb{R}_+,\mathbb{R}_+)$, then

$$\|x(t,t_0,x_0)\| \leqslant C\|x_0\|e^{-\alpha(t-t_0)},$$

with $C = kme^{k\|\lambda\|_1}$. Hence the system (1) is globally uniformly exponentially stable.

Second case: if $\lambda \in L^p(\mathbb{R}_+, \mathbb{R}_+)$ with $1 < p < \infty$, we obtain

$$\|x(t, t_0, x_0)\| \leqslant km\|x_0\|e^{k\|\lambda\|_p(t-t_0)^{\frac{1}{q}} - \alpha(t-t_0)}. \tag{7}$$

Using that $x \mapsto x^\alpha$ is concave for $\alpha \in (0, 1)$, we deduce that $x^\alpha \leqslant y^{\alpha-1}(\alpha x + (1 - \alpha)y)$ for any $y > 0$. Hence, $\forall t \geqslant t_0$

$$(t - t_0)^{\frac{1}{q}} \leqslant y^{-\frac{1}{p}} \left(\frac{1}{q}(t - t_0) + \frac{y}{p}\right) \qquad \text{for any } y > 0. \tag{8}$$

Choose $y = \left(\frac{2k\|\lambda\|_p}{q\alpha}\right)^p$ and denote $C = kme^{\frac{\alpha y}{2(p-1)}}$, we obtain from (7) and (8)

$$\|x(t, t_0, x_0)\| \leqslant C\|x_0\|e^{-\frac{\alpha}{2}(t-t_0)}$$

and the system (1) is globally uniformly exponentially stable.

Third case: if $\lambda \in L^\infty(\mathbb{R}_+, \mathbb{R}_+)$ then (6) implies

$$\|x(t, t_0, x_0)\| \leqslant km\|x_0\|e^{-(\alpha - k\|\lambda\|_\infty)(t-t_0)}.$$

This shows that the system (1) is globally uniformly exponentially stable under the condition $\|\lambda\|_\infty < \alpha/k$.

Theorem 2. *Assume that all conditions of Theorem 1 are satisfied except that condition (i) is replaced by*

$$\|F(t, x)\| \leqslant \lambda(t)\|x\|^l, \quad \textit{with } l \in (0, 1),$$

Then the trivial solution of system (1) is globally uniformly practically exponentially stable.

Proof. Using (4), (5) and the assumptions of Theorem 2, we have for any $t \geqslant t_0 \geqslant 0$

$$\|x(t, t_0, x_0)\| \leqslant k\|x_0\|e^{-\alpha(t-t_0)} + k\int_{t_0}^{t} e^{-\alpha(t-s)}\lambda(s)\|x(s, t_0, x_0)\|^l \mathrm{d}s$$
$$+ \sum_{t_0<t_i<t} ke^{-\alpha(t-t_i)}\beta_i\|x(t_i - 0)\|,$$

or equivalently

$$V(t) \leqslant k\|x_0\| + k\int_{t_0}^{t} e^{(1-l)\alpha(s-t_0)}\lambda(s)V(s)^l \mathrm{d}s + \sum_{t_0<t_i<t} k\beta_i V(t_i - 0).$$

where $V(t) = \|x(t,t_0,x_0)\|e^{\alpha(t-t_0)}$. Then, applying Corollary 1 to this inequality we obtain

$$V(t) \leqslant \prod_{t_0<t_i<t}(1+k\beta_i)\left[(k\|x_0\|)^{1-l} + (1-l)k\int_{t_0}^{t} e^{(1-m)\alpha(s-t_0)}\lambda(s)\right]^{\frac{1}{1-l}}.$$

It follows from Lemma 2 and Hölder inequality that

$$V(t) \leqslant \prod_{t_0<t_i<t}(1+k\beta_i)\left[2^{\frac{l}{1-l}}k\|x_0\| + \left(2^l k(1-l)\|\lambda\|_p\right)^{\frac{1}{1-l}}\left(\int_{t_0}^{t} e^{(1-l)q\alpha(s-t_0)}\right)^{\frac{1}{q(1-l)}}\right].$$

Then, we obtain

$$V(t) \leqslant \prod_{t_0<t_i<t}(1+k\beta_i)\left[2^{\frac{l}{1-l}}\left(k\|x_0\| + \left(\frac{k^q(1-l)^q\|\lambda\|_p^q}{(1-l)\alpha q}\right)^{\frac{1}{(1-l)q}}\right.\right.$$
$$\left.\left.\times\left(e^{(1-l)\alpha q(t-t_0)}-1\right)^{\frac{1}{q(1-l)}}\right)\right],$$

by condition (iii), we get

$$V(t) \leqslant m\left(k\|x_0\| + \left(k\|\lambda\|_p\alpha^{\frac{-1}{q}}\right)^{\frac{1}{1-l}} e^{\alpha(t-t_0)}\right),$$

where $q = \dfrac{p}{p-1}$ is the conjugate of p. It follows that $\forall\, t \geqslant t_0 \geqslant 0$

$$\|x(t,t_0,x_0)\| \leqslant 2^{\frac{l}{1-l}}km\|x_0\|e^{-\alpha(t-t_0)} + m\left(2^l k\|\lambda\|_p\alpha^{\frac{1-p}{p}}\right)^{\frac{1}{1-l}}.$$

Therefore, we finally obtain

$$\|x(t,t_0,x_0)\| \leqslant C\|x_0\|e^{-\alpha(t-t_0)} + r,$$

with $C = 2^{\frac{l}{1-l}}mk$ and $r = m\left(2^l k\|\lambda\|_p\alpha^{\frac{1-p}{p}}\right)^{\frac{1}{1-l}}$, which means that the system (1) is globally practically uniformly exponentially stable and the solution is defined on $[t_0,+\infty)$.

The following theorem gives a practical stability result for system (1). Thus, the hypothesis $g(t,0) = 0$ for all time t is not required.

Theorem 3. *Assume that the nominal system (3) is uniformly asymptotically stable and suppose that*

$$\|F(t,x)\| \leqslant \lambda(t)\|x\| + \sigma(t),$$

where
(i) $\lambda \in L^1(\mathbb{R}_+, \mathbb{R}_+)$,
(ii) $\sigma \in L^p(\mathbb{R}_+, \mathbb{R}_+)$ *with* $p \in [1, \infty]$ *or* $\lim_{t\to+\infty} \sigma(t) = 0$
(iii) $\|I_i\| \leqslant \beta_i \|x\|$.

Then, the system (1) is globally uniformly practically exponentially stable.

Proof. Combining (4) and (5) with the hypotheses of Theorem 3, we obtain, for any $t \geqslant t_0 \geqslant 0$,

$$\|x(t,t_0,x_0)\| \leqslant ke^{-\alpha(t-t_0)}\|x_0\| + k\int_{t_0}^{t} e^{-\alpha(t-s)} \left(\lambda(s)\|x(s,t_0,x_0)\| + \sigma(s)\right) \mathrm{d}s \\ + \sum_{t_0<t_i<t} ke^{-\alpha(t-t_i)}\beta_i\|x(t_i-0,t_0,x_0)\|,$$

which implies

$$\|x(t,t_0,x_0)\|e^{\alpha(t-t_0)} \leqslant k\|x_0\| + k\int_{t_0}^{t} e^{\alpha(s-t_0)} \left(\lambda(s)\|x(s,t_0,x_0)\| + \sigma(s)\right) \mathrm{d}s \\ + \sum_{t_0<t_i<t} ke^{\alpha(t_i-t_0)}\beta_i\|x(t_i-0,t_0,x_0)\|,$$

or equivalently

$$V(t) \leqslant g(t) + k\int_{t_0}^{t} \lambda(s)V(s)\,\mathrm{d}s + \sum_{t_0<t_i<t} k\beta_i V(t_i-0),$$

where $V(t) = \|x(t,t_0,x_0)\|e^{\alpha(t-t_0)}$ and $g(t) = k\|x_0\| + k\int_{t_0}^{t} e^{\alpha(s-t_0)}\sigma(s)\,\mathrm{d}s$. By Corollary 1, we deduce that

$$V(t) \leqslant \phi(t)\prod_{t_0<t_i<t}(1+k\beta_i)e^{k\int_{t_0}^{t}\lambda(s)\,\mathrm{d}s}.$$

Thus

$$\|x(t,t_0,x_0)\| \leqslant \phi(t)\prod_{t_0<t_i<t}(1+k\beta_i)e^{k\int_{t_0}^{t}\lambda(s)\,\mathrm{d}s-\alpha(t-t_0)}, \tag{9}$$

since $\lambda \in L^1(\mathbb{R}_+, \mathbb{R}_+)$ and $\prod_{t_0<t_i<t} (1 + k\beta_i) < m$, we obtain

$$\|x(t, t_0, x_0)\| \leqslant kme^{\|\lambda\|_1}\|x_0\|e^{-\alpha(t-t_0)} + kme^{\|\lambda\|_1} \int_{t_0}^{t} e^{-\alpha(t-s)}\sigma(s)\,\mathrm{d}s. \tag{10}$$

On the other hand, it is proven in [18, Lemma 3] that if $\sigma \in L^p(\mathbb{R}_+, \mathbb{R}_+)$ with $p \in [1, \infty)$ or $\lim_{t\to\infty} \sigma(t) = 0$ then there exists $A \geqslant 0$ such that $\int_0^t e^{-\alpha(t-s)}\sigma(s)\,\mathrm{d}s \leqslant A$ for all $t \geqslant 0$. Consequently

$$\|x(t, t_0, x_0)\| \leqslant C\|x_0\|e^{-\alpha(t-t_0)} + \delta, \tag{11}$$

with $C = kme^{\|\lambda\|_1}$ and $\delta = AC$, which means that the ball B_δ is globally uniformly exponentially stable and the solution is defined on $[t_0, +\infty)$.

Theorem 4. *Assume that all conditions of Theorem 3 are satisfied except that condition (i) is replaced by*

$$\|F(t,x)\| \leqslant \lambda(t)\|x\|^l + \sigma(t), \quad \text{with } l \in (0,1),$$

where $\lambda \in L^p(\mathbb{R}_+, \mathbb{R}_+)$ with $p \in [1, \infty]$. Then, the system (1) is globally practically uniformly exponentially stable.

Proof. Under the assumptions of Theorem 4, and using (4) together with (5), one has for any $t \geqslant t_0 \geqslant 0$,

$$\|x(t, t_0, x_0)\|e^{\alpha(t-t_0)} \leqslant k\|x_0\| + k \int_{t_0}^{t} e^{\alpha(s-t_0)} \Big(\lambda(s)\|x(s, t_0, x_0)\|^l + \sigma(s)\Big)\,\mathrm{d}s$$
$$+ \sum_{t_0<t_i<t} ke^{\alpha(t_i-t_0)}\beta_i\|x(t_i - 0, t_0, x_0)\|,$$

or equivalently

$$V(t) \leqslant g(t) + k \int_{t_0}^{t} e^{(1-l)\alpha(s-t_0)}\lambda(s)V(s)^l\,\mathrm{d}s + \sum_{t_0<t_i<t} k\beta_i V(t_i - 0),$$

where $V(t) = \|x(t,t_0,x_0)\|e^{\alpha(t-t_0)}$ and $g(t) = k\|x_0\| + k\int_{t_0}^{t} e^{\alpha(s-t_0)}\sigma(s)\,\mathrm{d}s$. Applying Corollary 1, we obtain

$$V(t) \leqslant \prod_{t_0<t_i<t}(1+k\beta_i)\left[g(t)^{1-l} + (1-l)k\int_{t_0}^{t}\lambda(s)e^{(1-l)\alpha(s-t_0)}\,\mathrm{d}s\right]^{\frac{1}{1-l}}$$
$$\leqslant m\left[g(t)^{1-l} + (1-l)k\int_{t_0}^{t}\lambda(s)e^{(1-l)\alpha(s-t_0)}\,\mathrm{d}s\right]^{\frac{1}{1-l}}.$$

Using Lemma 2, it follows that

$$V(t) \leqslant m2^{\frac{l}{l-1}}\left[g(t) + \left(k(1-l)\int_{t_0}^{t}\lambda(s)e^{(1-l)\alpha(s-t_0)}\,\mathrm{d}s\right)^{\frac{1}{1-l}}\right]. \tag{12}$$

Now, suppose $p \in (1,\infty]$ and let $q = \frac{p}{p-1}$ be the conjugate of p. Using the Hölder inequality, we obtain

$$V(t) \leqslant m2^{\frac{l}{l-1}}\left[g(t) + \left(\frac{k^q(1-l)^q\|\lambda\|_p^q}{(1-l)\alpha q}\right)^{\frac{1}{(1-l)q}}\left(e^{(1-l)\alpha q(t-t_0)}-1\right)^{\frac{1}{(1-l)q}}\right]$$
$$\leqslant m2^{\frac{l}{l-1}}\left[g(t) + \left(c\|\lambda\|_p\alpha^{\frac{1-p}{p}}e^{\alpha(t-t_0)}\right)\right].$$

Multiplying both sides by $e^{-\alpha(t-t_0)}$, we obtain

$$\|x(t,t_0,x_0)\| \leqslant km2^{\frac{l}{1-l}}\left(\|x_0\|e^{-\alpha(t-t_0)} + \int_{t_0}^{t}e^{-\alpha(t-s)}\sigma(s)\,\mathrm{d}s\right)$$
$$+\left(2^l m^{1-l}k\|\lambda\|_p\alpha^{\frac{1-p}{p}}\right)^{\frac{1}{1-l}}$$
$$\leqslant C\|x_0\|e^{-\alpha(t-t_0)} + r,$$

with $C = 2^{\frac{l}{1-l}}km$ and $\delta = CA + \left(2^l m^{1-l}k\|\lambda\|_p\alpha^{\frac{1-p}{p}}\right)^{\frac{1}{1-l}}$, which means that the ball B_r is globally uniformly exponentially stable and the solution is defined on $[t_0,+\infty)$.

Finally if $p = 1$ and $l \in (0,1)$, then $x^l \leqslant x$ and the result follows from Theorem 3.

4. Numerical examples

In what follows, we give some numerical examples to illustrate our theoretical study. The first example pertains to Theorem 1. The second example illustrates Theorem 2 and the third example is relative to Theorem 3. All the illustrations have been performed with the software Matlab.

In all the examples we take $I_i(x) = 0.9\|x\|$. It is clear that I_i satisfies the assumption (ii) of all the Theorems.

Example 1. Consider the following nonlinear system

$$\begin{cases} \dot{x}_1 = -x_1 + \dfrac{1}{1+t^2}\|x\|, & t \neq t_i, \\ \dot{x}_2 = -x_2 + \dfrac{t}{1+t^2}\|x\|, & t \neq t_i, \\ \Delta x = I_i(x), & t = t_i = 2, 5, 8, \ i = 1, 2, 3, \\ x_0 = (0.1, 0.3), \end{cases} \tag{13}$$

which can be written as

$$\begin{cases} \dot{x} = A(t)x + F(t,x), & t \neq t_i\,, \\ \Delta x = I_i(x), & t = t_i\,, \\ x_{t_0} = x_0, \end{cases}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ is the state of the system, $A = -I_2$ and

$$F(t,x) = \frac{\|x\|}{1+t^2}(1,t).$$

The linear system is globally uniformly exponentially stable. Moreover, we have

$$\begin{aligned} \|F(t,x)\|^2 &= F_1^2(t,x) + F_2^2(t,x) \\ &= \frac{1}{(1+t^2)}\|x\|^2. \end{aligned}$$

Hence $\|F(t,x)\| \leqslant \lambda(t)\|x\|$ with $\lambda(t) = \dfrac{1}{\sqrt{1+t^2}}\cdot$ All the assumptions of Theorem 1 are satisfied, and the system (13) is thus globally uniformly exponentially stable. Figure 1 shows the time evolution of the states (x_1, x_2) of the system (13) with the initial states $(x_{1,0}, x_{2,0}) = (0.1, 0.3)$. One can notice $t \mapsto \log\|x(t)\|$ remains below a line with negative slope for all time t, which confirms that the origin is exponentially stable as predicted by theory.

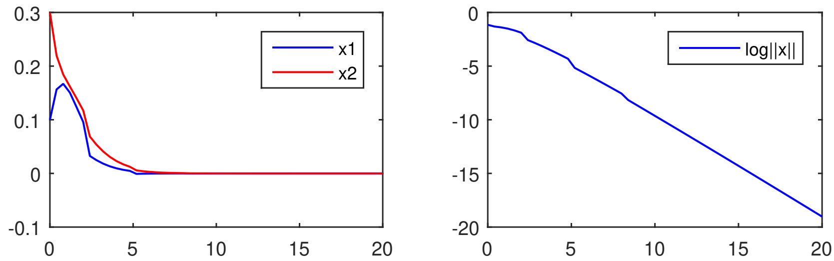

Figure 1: Time evolution of the states (x_1, x_2) (left) and $\|x\|$ (right) of the system (13). The parameters used are $t_0 = 0$ and $x_0 = (0.1, 0.3)$

Example 2. Consider the following nonlinear system

$$\begin{cases} \dot{x}_1 = -x_1 + \dfrac{t}{1+t^2}\sqrt{\|x\|}, & t \neq t_i\,, \\ \dot{x}_2 = -x_2 + \frac{1}{1+t^2}\sqrt{\|x\|}, & t \neq t_i\,, \\ \Delta x = I_i(x), & t = t_i = 3, 7, 11,\ i = 1, 2, 3, [2pt] \\ x_0 = (-0.5, 0.5), & \end{cases} \tag{14}$$

which can be written as

$$\begin{cases} \dot{x} = A(t)x + F(t,x), & t \neq t_i\,, \\ \Delta x = I_i(x), & t = t_i\,, \\ x_{t_0} = x_0, & \end{cases}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ is the state of the system, $A = -I_2$ and

$$F(t,x) = \frac{\sqrt{\|x\|}}{1+t^2}(t, 1).$$

The linear system is globally uniformly exponentially stable. Moreover, we have

$$\begin{aligned} \|F(t,x)\|^2 &= F_1^2(t,x) + F_2^2(t,x) \\ &= \frac{1}{(1+t^2)}\|x\|. \end{aligned}$$

Hence $\|F(t,x)\| \leqslant \lambda(t)\sqrt{\|x\|}$ with $\lambda(t) = \frac{1}{\sqrt{1+t^2}}$. All the assumptions of Theorem 2 are satisfied, then the system (14) is globally uniformly practically exponentially stable. Figure 2 shows the time evolution of the states (x_1, x_2) of the system (13) with the initial states $(x_{1,0}, x_{2,0}) = (-0.5, 0.5)$. One can notice that the solution goes to $r = 0.0032$ as t goes to infinity. This confirms that the solution is practically stable.

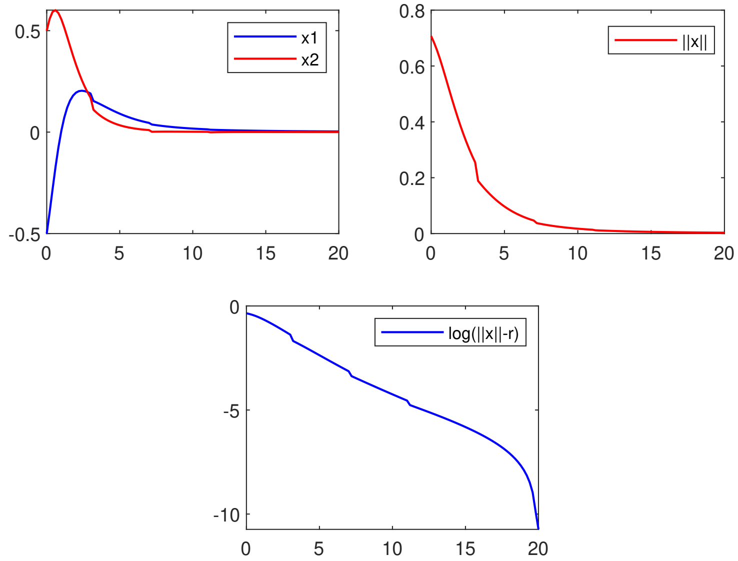

Figure 2: Time evolution of the states (x_1, x_2) , $\|x\|$ and $\log(\|x\| - r)$ of the system (14). The parameters used are $t_0 = 0$ and $x_0 = (-0.5, 0.5)$

Example 3. Consider the following system

$$\begin{cases} \dot{x}_1 = -x_1 + \dfrac{t}{1+t^2}\|x\| + \dfrac{1}{\log(2+t)}, & t \neq t_i\,, \\ \dot{x}_2 = -x_2 + \dfrac{1}{1+t^2}\|x\| + \dfrac{1}{\log(2+t)}, & t \neq t_i\,, \\ \Delta x = I_i(x), & t = t_i = 6, 8, 10,\ i = 1, 2, 3, \\ x_0 = (0.3, -0.2), \end{cases} \tag{15}$$

which can be written as

$$\begin{cases} \dot{x} = A(t)x + F(t, x), & t \neq t_i\,, \\ \Delta x = I_i(x), & t = t_i\,, \\ x_{t_0} = x_0, \end{cases}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ is the state of the system, $A = -I_2$ and

$$F(t, x) = \frac{\|x\|}{1+t^2}(t, 1) + \frac{1}{\log(2+t)}(1, 1).$$

The linear system is globally uniformly exponentially stable. Moreover, we have

$$\begin{aligned}
\|F(t,x)\|^2 &= F_1^2(t,x) + F_2^2(t,x) \\
&= \left(\frac{t\|x\|}{1+t^2} + \frac{1}{\log(2+t)}\right)^2 + \left(\frac{\|x\|}{1+t^2} + \frac{1}{\log(2+t)}\right)^2 \\
&\leqslant 2\left(\frac{t^2\|x\|^2}{(1+t^2)^2} + \frac{1}{(\log(2+t))^2} + \frac{\|x\|^2}{(1+t^2)^2} + \frac{1}{(\log(2+t))^2}\right) \\
&= 2\left(\frac{\|x\|^2}{1+t^2} + \frac{2}{(\log(2+t))^2}\right).
\end{aligned}$$

Hence $\|F(t,x)\| \leqslant \lambda(t)\|x\| + \sigma(t)$ with $\lambda(t) = \dfrac{\sqrt{2}}{\sqrt{1+t^2}}$ and $\sigma(t) = \dfrac{2}{\log(2+t)}$. All the assumptions of Theorem 3 are satisfied, then the system (15) is globally uniformly practically exponentially stable. Figure 3 shows the time evolution of the states (x_1, x_2) of the system (15) with the initial states $(x_{1,0}, x_{2,0}) = (0.3, -0.2)$. One can notice that the solution goes to $r = 0.4845$ as t goes to infinity. This confirms that the solution is practically stable.

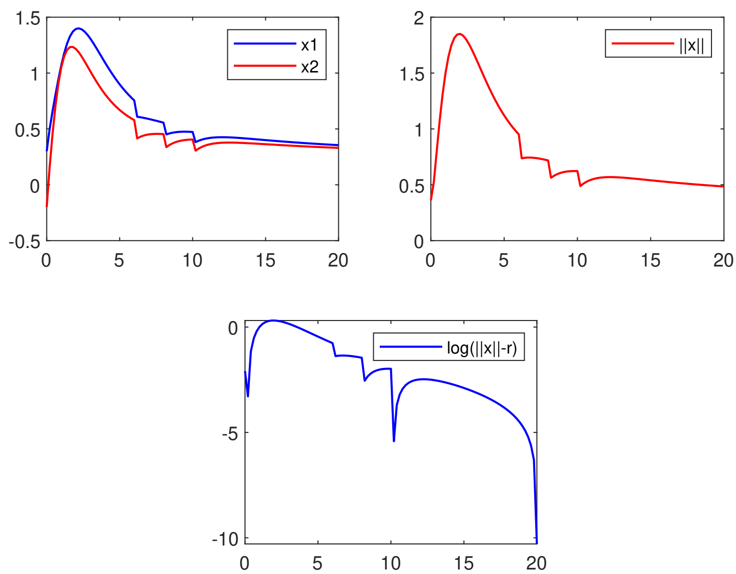

Figure 3: Time evolution of the states (x_1, x_2) , $\|x\|$ and $\log(\|x\| - r)$ of the system (15). The parameters used are $t_0 = 0$ and $x_0 = (0.3, -0.2)$

4.1. Application to Hopfield neural networks

In this subsection, we present a numerical example illustrating the applicability of Theorem 3 to a two-dimensional impulsive Hopfield neural network subject to external perturbations [28].

We consider the following system:

$$\begin{cases} \dot{x}(t) = -Dx(t) + Wf(x(t)) + u(t), & t \neq t_i, \\ \Delta x(t) = I_i(x(t)), & t = t_i, \\ x(0) = x_0 \in \mathbb{R}^2, \end{cases} \tag{16}$$

where $x(t) = (x_1(t), x_2(t))^\top$ denotes the state vector.

The parameter matrices are chosen as

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad W = \frac{1}{1+t^2}\begin{pmatrix} 0.5 & 0.3 \\ 0.2 & 0.4 \end{pmatrix},$$

and the activation function is defined componentwise by

$$f(x) = \big(\tanh(x_1), \tanh(x_2)\big)^\top.$$

The impulse moments are given by

$$t_i = 5, 11, 16, \ i = 1, 2, 3, ,$$

and the impulsive operator is defined as

$$I_k(x) = 1.2\|x\|.$$

Moreover, the external perturbation is taken as

$$u(t) = \frac{1}{2+t}(1,1)^\top.$$

System (16) can be rewritten in the form

$$\dot{x}(t) = Ax(t) + F(t, x(t)),$$

where

$$A = -D = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}.$$

The linear system $\dot{x} = Ax$ is uniformly exponentially stable since

$$\|e^{At}\| \leqslant e^{-2t}, \qquad t \geqslant 0.$$

Hence, condition (3.2) of Theorem 3 is satisfied with $k = 1$ and $\alpha = 2$.

Using the Lipschitz continuity of the hyperbolic tangent function, we have

$$\|f(x)\| \leqslant \|x\|,$$

which implies

$$\|Wf(x)\| \leqslant \|W\|\,\|x\| \leqslant \frac{0.8}{1+t^2}\,\|x\|.$$

Consequently, the nonlinear term satisfies

$$\|F(t,x)\| \leqslant \frac{0.8}{1+t^2}\,\|x\| + \frac{\sqrt{2}}{2+t}.$$

Thus, the growth condition of Theorem 3.3 holds with

$$\lambda(t) = \frac{0.8}{1+t^2} \in L^1(\mathbb{R}^+), \qquad \sigma(t) = \frac{\sqrt{2}}{2+t} \in L^2(\mathbb{R}^+),\ p > 1.$$

Furthermore, since

$$\|I_k(x)\| \leqslant 1.2\|x\|,$$

the impulsive effects satisfy the boundedness condition required in Theorem 3.3.

Therefore, all assumptions of Theorem 3 are fulfilled, and the impulsive Hopfield neural network (16) is globally uniformly practically exponentially stable. Figure 4 shows the time evolution of the states (x_1, x_2) of the system (15) with

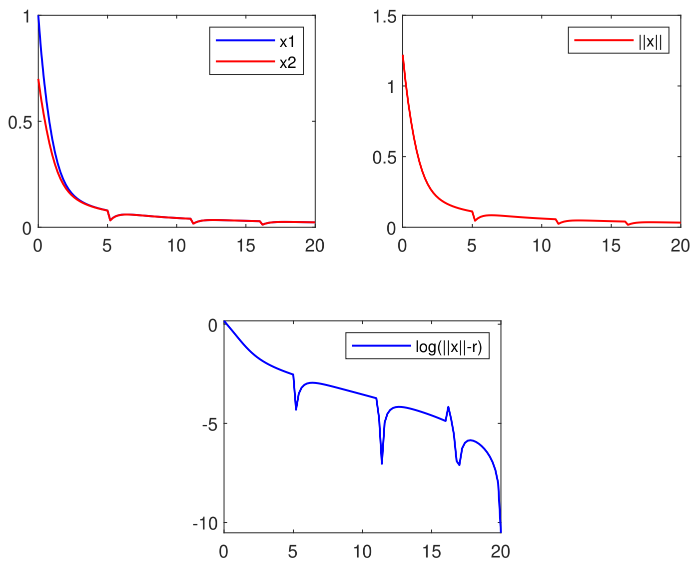

Figure 4: Time evolution of the states (x_1, x_2), $\log\|x\|$ and $\log(\|x\| - r)$ of the system (16). The parameters used are $t_0 = 0$ and $x_0 = (1, 0.7)$

the initial states $(x_{1,0}, x_{2,0}) = (1, 0.7)$. One can notice that the solution goes to $r = 0.0329$ as t goes to infinity. This confirms that the solution is practically stable.

4.2. Summary

This work addresses the stability analysis of nonlinear impulsive differential systems. The primary objective is to establish sufficient conditions that guarantee the global exponential stability and global practical exponential stability of solutions for such systems. A key methodological feature is the use of Gronwall-Bellman-type inequalities to derive these stability criteria. This approach offers a practical advantage over traditional Lyapunov methods, as it avoids the often challenging task of explicitly constructing a suitable Lyapunov function. The theoretical findings are validated through several numerical examples and simulations, demonstrating their effectiveness. Furthermore, to illustrate the practical applicability of the derived conditions, the results are successfully implemented to analyze the stability of a Hopfield neural network. The novelty can be summarized as follows: deriving stability criteria via Gronwall-Bellman inequalities as an alternative to the classical Lyapunov approach and establishing criteria for practical exponential stability in perturbed impulsive systems. Therefore, the paper provides a readily applicable framework for ensuring the stability of a broad class of nonlinear systems subject to impulsive effects.

5. Conclusion

In this paper, we addressed the stability analysis problem for nonlinear impulsive differential equations. We derived some sufficient conditions that ensure the global exponential stability and the global practical exponential stability of the solutions. The results we obtained can easily be applied in practice since they are based on the Gronwall-Bellman inequalities rather than the classical Lyapunov methods that require the knowledge of a Lyapunov function. Some examples and simulation results are provided to illustrate the applicability of the main results. Finally, we applied the results to a Hopfield neural network model.

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