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Invariant properties of positive linear systems with integer and fractional orders

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The invariant properties of the stability, reachability, observability and transfer matrices of positive linear continuous-time systems with integer and fractional orders are investigated. It is shown that the stability, reachability, observability and transfer matrix of positive linear systems are invariant under their integer and fractional orders.

Key words: invariance, positive, linear, system, stability, reachability, observability, transfer matrix

1. Introduction

A dynamical system is called fractional if it is described by fractional order differential or difference equation. The fundamentals of fractional calculus and fractional systems have been given in [23, 26–31]. The stability of fractional linear systems has been analyzed in [3–5].

In positive systems inputs, state variables and outputs take only nonnegative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. An overview of state of the art in positive systems theory is given in the monographs [2, 6, 13].

The determination of the matrices A, B, C, D of the state equations of linear systems for a given transfer matrices is called the realization problem. The

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realization problem has been investigated in [12, 26, 27, 32]. A tutorial on the positive realization problem has been given in the paper [1] and in the books [6, 13, 27]. The positive realization problem for linear systems with delays has been analyzed in [7, 8, 14, 20, 21, 27], for cone systems in [10] and positive stable realizations in [9, 15–17, 32]. The existence and determination of the set of Metzler matrices for given stable polynomials have been considered in [11]. The realization problem for positive 2D hybrid systems has been addressed in [19]. For fractional linear systems the realization problem has been considered in [18, 26, 32]. The relationship between controllability of standard and fractional linear systems has been investigated in [25].

In this paper the invariant properties of the stability, reachability , observability and transfer matrices of positive linear systems with integer and fractional orders will be investigated.

The paper is organized as follows. In Section 2 the invariance of stability of the positive linear systems with integer and fractional orders is investigated. The invariance of the reachability of the positive linear systems is analyzed in Section 3 and the observability in Section 4. The invariance of transfer matrices of positive linear systems is considered in Section 5. Concluding remarks are given in Section 6.

The following notation will be used: $\mathfrak{R}$ – the set of real numbers, $\mathfrak{R}^{n\times m}$ — the set of $n\times m$ real matrices, $\mathfrak{R}_+^{n\times m}$ – the set of $n\times m$ real matrices with nonnegative entries and $\mathfrak{R}_+^n = \mathfrak{R}_+^{n\times 1}$, M_n – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n\times n$ identity matrix.

2. Stability invariance of the orders of positive linear continuous-time systems

Consider the autonomous linear continuous-time linear system

$$\dot{x}(t) = Ax(t), \tag{1}$$

where $x(t) \in \mathfrak{R}^n$ is the state vector and $A \in \mathfrak{R}^{n\times n}$.

The system (1) is called (internally) positive if $x(t) \in \mathfrak{R}_n^n$, $t \geqslant 0$ for any initial conditions $x(0) \in \mathfrak{R}_+^n$.

A matrix $A = \left[a_{ij}\right] \in \mathfrak{R}^{n\times n}$ is called the Metzler matrix if $a_{ij} \geqslant 0$ for $i \neq j$.

Theorem 1. *[2, 6, 13] The system (1) is positive if and only if A is a Metzler matrix.*

The positive system (1) is called asymptotically stable (and the matrix A Hurwitz) if

$$\lim_{t\to\infty} x(t) = 0 \ \textit{for all} \ x(0) \in \mathfrak{R}_+^n . \tag{2}$$

The system (1) is asymptotically stable if and only if all eigenvalues s_k *of the matrix* A *are negative, i.e.* $\mathrm{Re}\, s_k < 0$ *for* $k = 1, \dots, n$.

Theorem 2. *[13] For the positive system (1) the following conditions are equivalent:*

1. *The positive system (1) is asymptotically stable (the Metzler matrix* A *is Hurwitz).*
2. *All coefficients of the characteristic polynomial*

$$\det\left[I_n s - A\right] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{3}$$

are positive, i.e. $a_i > 0$ *for* $i = 0, 1, \dots, n-1$.

3. *All principal minors* M_i, $i = 1, \dots, n$ *of the matrix* $-A$ *are positive, i.e.*

$$M_1 = \left|-a_{11}\right| > 0, \quad M_2 = \begin{vmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{vmatrix} > 0, \quad \dots, \quad M_n = \det\left[-A\right] > 0. \tag{4}$$

4. *There exists strictly positive vector* $\lambda = \begin{bmatrix} \lambda_1 & \cdots & \lambda_n \end{bmatrix}$,

$$\lambda_k > 0, \quad k = 1, \dots, n \;\; \textit{such that} \;\; A\lambda < 0. \tag{5}$$

Remark 1. *From (5) it follows that the positive system (1) is asymptotically stable only if all diagonal entries of* A *are negative.*

Consider the autonomous fractional linear continuous-time system

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t), \quad 0 < \alpha < 1, \tag{6}$$

where $x(t) \in \Re^n$ is the state vector and $A \in \Re^{n\times n}$, and

$$\frac{d^\alpha x(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{x}(\tau)}{(t-\tau)^\alpha} \mathrm{d}\tau, \quad \Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \mathrm{d}t, \tag{7}$$

is the Caputo definition of α order of $x(t)$.

The system (6) is called positive if $x(t) \in \Re^n_+$, $t \geqslant 0$ for any initial conditions $x(0) \in \Re^n_+$.

Theorem 3. *[23] The fractional system is positive if and only if* A *is a Metzler matrix.*

The positive system (6) is called asymptotically stable (and the matrix A *Hurwitz) if*

$$\lim_{t\to\infty} x(t) = 0 \;\; \textit{for all} \;\; x(0) \in \Re^n_+ . \tag{8}$$

The positive fractional system (6) is asymptotically stable if and only if the real parts of all eigenvalues s_k of the matrix A are negative, i.e. $\text{Re} s_k < 0$ for $k = 1, \ldots, n$ [23].

Theorem 4. *For the positive fractional system (6) the following conditions are equivalent:*

1. *The positive system (6) is asymptotically stable (the Metzler matrix A is Hurwitz).*
2. *All coefficients of the characteristic polynomial*

$$\det \left[I_n s - A \right] = s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 \tag{9}$$

are positive, i.e. $a_i > 0$ for $i = 0, 1, \ldots, n-1$.

3. *All principal minors M_i, $i = 1, \ldots, n$ of the matrix $-A$ are positive, i.e.*

$$M_1 = \left| -a_{11} \right| > 0, \quad M_2 = \begin{vmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{vmatrix} > 0, \quad \ldots, \quad M_n = \det \left[-A \right] > 0. \tag{10}$$

4. *There exists strictly positive vector $\lambda = \begin{bmatrix} \lambda_1 & \cdots & \lambda_n \end{bmatrix}$, $\lambda_k > 0$, $k = 1, \ldots, n$ such that*

$$A\lambda < 0. \tag{11}$$

Remark 2. *From (11) it follows that the positive fractional system (6) is asymptotically stable only if all diagonal entries of A are negative.*

From comparison of Theorems 1 and 2 with Theorems 3 and 4 we have the following important corollary respectively.

Corollary 1. *The stability of positive linear continuous-time systems is invariant under the their (integer and fractional) orders.*

These considerations can be extended to positive linear continuous-time systems with delays in state vectors.

3. Reachability invariance of the orders of positive linear continuous-time systems

Consider the standard continuous-time linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{12}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$ are the state and input vectors and $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$.

Definition 1. *[13, 23] The continuous-time linear system (12) is called (internally) positive if $x(t) \in \Re_+^n$ and all $u(t) \in \Re_+^m$, $t \geqslant 0$.*

Theorem 5. *[13, 23] The continuous-time linear system (12) positive if and only if*

$$A \in M_n\,, \quad B \in \Re_+^{n\times m}. \tag{13}$$

Definition 2. *[13, 23] The standard positive system (12) is called reachable in the time $[0, t_f]$, $t_f > 0$, if there exist an input $u(t) \in \Re_+^m$ for $t \in [0,\ t_f]$ which steers the state of system from $x(0) = 0$ to the given final state $x_f \in \Re_+^n$, i.e. $x(t_f) = x_f$.*

Theorem 6. *[13, 23] The standard positive system (12) is reachable in the time $[0,\ t_f]$ if and only if the reachability matrix*

$$R(t_f) = \int_0^{t_f} e^{A\tau} BB^T e^{A^T\tau}(\tau)\mathrm{d}\tau \in \Re_+^{n\times n} \tag{14}$$

is a monomial matrix.

The input $u(t) \in \Re_+^m$, $t \in [0,\ t_f]$ which steers the state of system from $x(0) = 0$ to the given final state $x_f \in \Re_+^n$, is given by

$$u(\tau) = B^T e^{A^T(t_f-\tau)} R^{-1}(t_f)x_f \in \Re_+^m\,, \quad \tau \in [0,\ t_f]. \tag{15}$$

Consider the fractional continuous-time linear system

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bu(t), \quad 0 < \alpha < 1, \tag{16}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$ are the state and input vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$ and the Caputo derivative of $x(t)$ is defined by (7).

Definition 3. *[13, 23] The fractional positive system (16) is called reachable in the time $[0,\ t_f]$, $t_f > 0$, if there exist an input $u(t) \in \Re_+^m$ for $t \in [0,\ t_f]$ which steers the state of system from $x(0) = 0$ to the given final state $x_f \in \Re_+^n$, i.e. $x(t_f) = x_f$.*

Theorem 7. *The fractional positive system (16) is reachable in the time $[0,\ t_f]$ if and only if the reachability matrix*

$$\overline{R}(t_f) = \int_0^{t_f} \Phi(\tau)BB^T\Phi^T(\tau)\mathrm{d}\tau \in \Re_+^{n\times n} \tag{17}$$

is a monomial matrix, where $\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma\left[(k+1)\alpha\right]}$.

The input $u(t)$ *which steers the state of the system from* $x(0) = 0$ *to* $x_f = x(t_f) \in \Re^n$ *is given by*

$$u(\tau) = B^T \Phi^T (t_f - \tau) \overline{R}^{-1}(t_f) x_f \in \Re_+^m, \quad \tau \in [0,\ t_f]. \tag{18}$$

Proof. It is well-known [23] that $\overline{R}^{-1}(t_f) \in \Re_+^{n\times n}$ if and only if the matrix (17) is monomial. Substituting (18) into

$$x(t_f) = \int_0^{t_f} \Phi(t_f - \tau) B u(\tau) \mathrm{d}\tau \tag{19}$$

we obtain

$$\begin{aligned} x(t_f) &= \int_0^{t_f} \Phi(t_f - \tau) B B^T \Phi^T (t_f - \tau) \overline{R}^{-1}(t_f) x_f \mathrm{d}\tau \\ &= \int_0^{t_f} \Phi(\tau) B B^T \Phi^T(\tau) \mathrm{d}\tau \overline{R}^{-1}(t_f) x_f = x_f\,. \end{aligned} \tag{20}$$

Therefore, the input (18) steers the state of the system from $x(0) = 0$ to $x(t_f) = x_f$. □

Theorem 8. *The fractional positive continuous-time linear system is reachable in the time* $[0,\ t_f]$ *if and only if the standard positive continuous-time linear system (12) is reachable in the same interval* $[0,\ t_f]$.

Proof. Note that the reachability matrices (14) and (17) of the standard positive system (12) and of fractional positive system (16) differ only by the transition matrices e^{At} for standard system and $\Phi(t)$ (defined by (17)) for fractional system. Using the well-known Cayley-Hamilton theorem or the Lagrange-Sylvester formula [12, 23, 24] it is possible to write the transition matrices in the forms

$$e^{At} = \sum_{k=0}^{n-1} c_k(t) A^k \tag{21}$$

and

$$\Phi(t) = \sum_{k=0}^{n-1} \overline{c}_k(t) A^k, \tag{22}$$

where $c_k(t)$ and $\overline{c}_k(t)$ for $k = 0, 1, \dots, n-1$ are nonzero linearly independent functions of time t [22].

Therefore, the reachability matrix (17) is monomial if and only if the reachability matrix (14) is monomial. By Theorems 6 and 7 the fractional positive system (16) is reachable in the time $[0,\ t_f]$ if and only if the standard positive system (12) is reachable in the interval $[0,\ t_f]$. □

Therefore, from Theorem 8 we have the following important corollary.

Corollary 2. *The reachability of positive linear continuous-time systems is invariant under the their (integer and fractional) orders.*

4. Observability invariance of the orders of positive continuous-time linear systems

Consider the standard continuous-time linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{23a}$$

$$y(t) = Cx(t), \tag{23b}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ are the state input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$.

Definition 4. *[13] The system (23) is called (internally) positive if and only if $x \in \Re_+^n$, $y(t) \in \Re_+^p$, $t \geqslant 0$ for any $u(t) \in \Re_+^m$, $t \geqslant 0$ and all initial conditions $x(0) \in \Re_+^n$.*

Theorem 9. *[13] The system (23) is positive if and only if*

$$A \in M_n, \quad B \in \Re_+^{n\times m}, \quad C \in \Re_+^{p\times n}. \tag{24}$$

Definition 5. *The positive system (23) is called (strongly) observable in the interval of $[0,\ t_f]$ if knowing the input $u(t)$ and output $y(t)$ for $[0,\ t_f]$ it is possible to find the unique $x(0) \in \Re_+^n$ of the system.*

Theorem 10. *The positive system (23) is observable in the interval $[0,\ t_f]$ if and only if the matrix*

$$W_f = \int_0^{t_f} e^{A^T t} C^T C e^{At} \mathrm{d}t \tag{25}$$

is monomial.

Proof. Assuming $B = 0$ and premultiplying the equation

$$y(t) = Ce^{At}x(0) \tag{26}$$

by $e^{At}C^T$ we obtain

$$e^{A^Tt}C^TCe^{At}x(0) = e^{A^Tt}C^Ty(t). \tag{27}$$

Integrating (27) on the interval $[0,\ t_f]$ we obtain

$$\int_0^{t_f} e^{A^Tt}C^TCe^{At}\mathrm{d}t x(0) = \int_0^{t_f} e^{A^Tt}C^Ty(t)\mathrm{d}t \tag{28}$$

and

$$x(0) = W_f^{-1}\int_0^{t_f} e^{A^Tt}C^Ty(t)\mathrm{d}t \in \Re_+^n \tag{29}$$

if and only if the matrix (25) is monomial. □

Consider the fractional continuous-time linear system

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bu(t), \quad 0 < \alpha < 1, \tag{30a}$$

$$y(t) = Cx(t), \tag{30b}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$ and the Caputo derivative of $x(t)$ is defined by (7).

Definition 6. *[13, 23] The fractional system (30) is called (internally) positive if* $x \in \Re_+^n$, $y(t) \in \Re_+^p$, $t \geqslant 0$ *for any* $u(t) \in \Re_+^m$, $t \geqslant 0$ *and all initial conditions* $x(0) \in \Re_+^n$.

Theorem 11. *The fractional system (30) is positive if and only if*

$$A \in M_n, \quad B \in \Re_+^{n\times m}, \quad C \in \Re_+^{p\times n}. \tag{31}$$

Definition 7. *The fractional positive system (30) is called observable in the interval* $[0,\ t_f]$ *if knowing the input* $u(t)$ *and output* $y(t)$ *for* $[0,\ t_f]$ *it is possible to find the unique* $x(0) \in \Re_+^n$ *of the system.*

Theorem 12. *The solution of the equation (30a) has the form*

$$x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)\mathrm{d}\tau, \tag{32a}$$

where

$$\Phi_0(t) = \sum_{i=0}^\infty \frac{A^it^{i\alpha}}{\Gamma(i\alpha+1)}, \quad \Phi(t) = \sum_{i=0}^\infty \frac{A^it^{(i+1)\alpha-1}}{\Gamma\left[(i+1)\alpha\right]}. \tag{32b}$$

Theorem 13. *The positive fractional system (30) is observable in the interval* $[0,\ t_f]$ *if and only if the matrix*

$$W_\alpha = \int_0^{t_f} \Phi_0^T(t) C^T C \Phi_0(t)\,\mathrm{d}t \tag{33}$$

is monomial.

Proof. Using (32a) for $B = 0$ and (30) and premultiplying the equation

$$y(t) = C\Phi_0 x(0) \tag{34}$$

by $\Phi_0^T(t)C^T$ we obtain

$$\Phi_0^T(t) C^T C \Phi_0(t) x(0) = \Phi_0^T(t) C^T y(t). \tag{35}$$

Integrating (35) on the interval $[0,\ t_f]$ we obtain

$$\int_0^{t_f} \Phi_0^T(t) C^T C \Phi_0(t)\,\mathrm{d}t x(0) = \int_0^{t_f} \Phi_0^T(t) C^T y(t)\,\mathrm{d}t \tag{36}$$

and

$$x(0) = W_\alpha^{-1} \int_0^{t_f} \Phi_0^T(t) C^T y(t)\,\mathrm{d}t \tag{37}$$

if and only if the matrix (33) is monomial. □

Theorem 14. *The positive fractional system (30) is observable in the interval* $[0,\ t_f]$ *if and only if the positive system (23) is observable in the same interval* $[0,\ t_f]$.

Proof. Using (32b) we obtain

$$\Phi_0(t) = \sum_{i=0}^{\infty} \frac{A^i t^{i\alpha}}{\Gamma(i\alpha+1)} \in \Re_+^{n\times n} \quad \text{for}\ \ t \geqslant 0\ \ \text{and}\ \ 0 < \alpha < 1 \tag{38}$$

if and only if

$$e^{At} = \sum_{i=0}^{\infty} \frac{A^i t^i}{i!} \in \Re_+^{n\times n}, \qquad t \geqslant 0. \tag{39}$$

From (38) it follows that $\Phi_0(t) \in \Re_+^{n\times n}$, $t \geqslant 0$ if and only if $e^{At} \in \Re_+^{n\times n}$, $t \geqslant 0$ and this implies that the matrix (33) is monomial if and only if the matrix (24) is monomial. Therefore, the positive system (30) is observable if and only if the positive system (23) is observable. □

Therefore, from Theorem 14 we have the following important corollary.

Corollary 3. *The observability of positive linear continuous-time systems is invariant under their (integer and fractional) orders.*

5. Transfer matrix invariance of positive continuous-time linear systems

Consider the positive continuous-time system (23).

The transfer matrix of the system (23) is given by

$$T(s) = C\left[I_n s - A\right]^{-1} B. \tag{40}$$

The matrices (24) are called the positive realization of the transfer matrix (40) if they satisfy (40) and it is called asymptotically stable realization if the matrix A is an asymptotically stable Metzler matrix (Hurwitz Metzler matrix).

Theorem 15. *[6, 13, 23] The positive realization (24) is asymptotically stable if and only if all coefficients of the polynomial*

$$p_A(s) = \det\left[I_n s - A\right] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{41}$$

are positive, i.e. $a_i > 0$ *for* $i = 0, 1, \ldots, n-1$.

The positive realization problem can be stated as follows. Given a proper transfer matrix $T(s)$ find its positive realization (24).

Theorem 16. *[27] If (24) is a positive realization of (40) then the matrices*

$$\overline{A} = PAP^{-1}, \quad \overline{B} = PB, \quad \overline{C} = CP^{-1} \tag{42}$$

are also a positive realization of (40) if the matrix $P \in \mathfrak{R}_+^{n\times n}$ *is a monomial matrix.*

Now let us consider the positive fractional system (30). The transfer matrix of the system (30) is given by

$$T(\lambda) = C\left[I_n \lambda - A\right]^{-1} B, \quad \lambda = s^\alpha. \tag{43}$$

The positive realization problem for fractional system (30) can be stated in a similar way as for the positive system (23) substituting $\lambda = s^\alpha$.

Theorem 17. *If the matrix* $A \in M_n$ *is Hurwitz and* $B \in \mathfrak{R}_+^{n\times m}$, $C \in \mathfrak{R}_+^{p\times n}$ *then all coefficients of the transfer matrix (40) are positive.*

Proof is given in [27].

Example 1. Consider the positive linear system (23) with the matrices

$$A = \begin{bmatrix} -3 & 1 & 1 \\ 0 & -4 & 2 \\ 2 & 1 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}. \tag{44}$$

By Theorem 15 the matrix A is Hurwitz since all coefficients of its characteristic polynomial

$$p_A(s) = \det\left[I_3 s - A\right] = \begin{vmatrix} s+3 & -1 & -1 \\ 0 & s+4 & -2 \\ -2 & -1 & s+5 \end{vmatrix} = s^3 + 12s^2 + 43s + 42 \tag{45}$$

are positive.

The transfer matrix of the positive system with (44) has the form

$$\begin{aligned} T(s) &= C\left[I_3 s - A\right]^{-1} B = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s+3 & -1 & -1 \\ 0 & s+4 & -2 \\ -2 & -1 & s+5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} \\ &= \frac{1}{s^3 + 12s^2 + 43s + 42} \begin{bmatrix} 4s + 16 & s^2 + 10s + 19 \end{bmatrix}. \end{aligned} \tag{46}$$

All coefficients of the transfer matrix (46) are positive.

Theorem 18. *There exists a positive asymptotically stable realization (24) of (40) only if its coefficients are positive.*

Proof is given in [26].

Similar results can be obtained for the positive fractional linear systems (30) [26].

Theorem 19. *If the matrix $A \in M_n$ is Hurwitz and $B \in \Re_+^{n\times m}$, $C \in \Re_+^{p\times n}$ of the positive fractional system (30) then all coefficients of its transfer matrix (40) are positive.*

Theorem 20. *[26] There exists a positive asymptotically stable realization (31) of (43) only if its coefficients are positive.*

Therefore, from the above considerations we have the following important corollary.

Corollary 4. *The positivity of the coefficients of the transfer matrices of the positive linear continuous-time systems is invariant under their (integer and fractional) orders.*

Note that the above considerations can be easily extended to the standard and positive when the output equation has the form

$$y(t) = Cx(t) + Du(t), \tag{47}$$

where $D \in \Re_{+}^{p\times m}$.

6. Concluding remarks

The invariant properties of the stability, reachability, observability and transfer matrices of positive linear continuous-time systems with integer and fractional orders have been investigated. It has been shown that:

1. The stability of positive linear systems is invariant under their integer and fractional orders (Theorems 2 and 4).
2. The reachability of positive linear systems is invariant under their integer and fractional orders (Theorem 8 and Corollary 2).
3. The observability of positive linear systems is invariant under their integer and fractional orders (Theorem 13 and Corollary 3).
4. The transfer matrix of positive linear systems is invariant under their integer and fractional orders (Theorems 17, 19 and Corollary 4).

The considerations can be extended to positive linear discrete-time systems.

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