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On the transformation of FOTF system to fractional order LTI model

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This article addresses modeling systems using fractional order derivatives, highlighting three basic approaches: differential equations, operator methods, and state space representations. Each approach carries different advantages and limitations in the context of time invariant linear systems (LTI) of fractional order. The focus of the article is on Fractional Order Transfer Functions, which represent a special subject of interest because they offer practical utility with available simulation libraries (e.g., CRONE, FOMCON, NiNteger) and approximation techniques (e.g., Oustaloup, CFE, Thiele, Padé).

This paper describes a transition between fractional order transfer functions (FOTF) and pseudo-rational representations of such systems. While the existing literature contains the basics of fractional differential equations and operator theory, it often omits explicit formulas for conversion between different representations. The paper fills a significant gap by proposing a novel algorithm for converting FOTF models to fractional LTI representations. Unlike previous works that implicitly assume such transitions exist, this paper provides formulas for coefficient transformations and demonstrates its effectiveness by minimizing the degree of the system.

Key words: fractional order system, fractional order transfer function, FOTF, fractional order LTI system, model transformation.

1. Introduction

The article considers an aspect of modeling systems using fractional order derivatives. This issue, as in classical control theory, has three basic branches of research: the use of differential equations, operator methods, and using state-space representations. Each of the above approaches has certain advantages and disadvantages. This article focuses on the transition between the fractional or-

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der transfer function (FOTF) model and the pseudo-rational description of the fractional order linear time-invariant (fractional order LTI) model.

Considerations on ordinary fractional differential equations can be found, for example, in [12], in the context of the properties of the fractional differential operators see [4, 22]. It is noteworthy that this type of approach, although usually quite complicated, allows to consider the widest range of problems. The second approach applied to describe real models is the use of fractional continuous-time linear systems, which are studied in many textbooks [9, 10]. The convenience of this approach is the existence of some analogues with classical matrix systems. It is worth mentioning such properties as controllability or observability of fractional systems, which are convenient to analyze in matrix form [11, 26, 28].

The final approach is the use of fractional order transfer function (FOTF), which is analyzed in textbooks such as [21]. The use of this kind of systems is very convenient due to the existence of a number of simulation packages/libraries in this area: CRONE, NiNteger (Non-integer), FOTF (Fractional Order Transfer Fuction), FOMCOM [13]. In addition, there is a whole range of approximation methods for transforming FOTF to a classical transfer function like: Oustaloup method [20], Refined Oustaloup method [29], CFE (Continued Fraction Expansion) [14], Thiele continued fraction (aka Matsuda method) [7, 15, 16], M-SBL [3], Carlson method, [2], Chareff method [14], Padé method [6, 7].

This article presents an issue related to the transition algorithms between the FOTF model and fractional order LTI. This matter is not analyzed in detail in the literature. Some publications basically just accept that the transition between systems exists and do not analyze the question of what specific formula can be applied to such transformation of coefficients. For example [9] presenting single-input single-output (SISO) linear fractional systems with the proper transfer function, however does not indicate from where to obtain the coefficients of operator s^{α}, this is due to the fact that the existence of these coefficients is obvious and is not necessary in the considerations.

Algorithms we are familiar with in this area include an approach from Petras' book [21], which uses formulas proposed by [17]. In addition, the [6] provides a transition algorithm for transfer functions with powers in the form of decimal fractions. The novelty of the article is the idea of a new transformation between systems and the proof that the proposal minimizes the global order of fractional order LTI.

The arrangement of the article is as follows. Different representations of systems with fractional derivatives are presented in Section 2. Then the transition algorithms proposed by [17] and [6] are pointed out in Section 3. Finally, the new proposal is indicated along with a proof of optimality provided in Section 4. The article was accompanied by examples.

2. Models for fractional order systems

Fractional order differential equation can represent some specific dynamic processes [9, 21, 22]. Consider the following form

$$\begin{aligned} a_n D_t^{\alpha_n} y(t) + a_{n-1} D_t^{\alpha_{n-1}} y(t) + \ldots + a_0 D_t^{\alpha_0} y(t) \\ = b_n D_t^{\beta_n} u(t) + b_{n-1} D_t^{\beta_{n-1}} u(t) + \ldots + b_0 D_t^{\beta_0} u(t), \end{aligned} \tag{1}$$

where $u(t)$ denotes the input and $y(t)$ stands for the output of the system. The non-integer order derivative operator can be formulated as:

$$_c D_t^{\alpha} = \left(\frac{d^{\alpha}}{dt^{\alpha}} \right), \tag{2}$$

where c and t denote the limits of the operation (usually $c = 0$), while $\alpha \in \mathbb{R}^+$ refers to the order of the derivative ($\alpha \in \mathbb{R}^-$ indicates the order of integration).

There is diversity in defining the operator (2). The most popular is the Riemann-Liouville definition, in control theory the Caputo definition is most often used, while for numerical purposes Grunwald-Letnikov is applied [13]. The Riemann–Liouville fractional derivative of order α is defined as

$$^{RL}_{c} D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int\limits_c^t (t-\tau)^{n-\alpha-1} f(\tau) \mathrm{d}\tau, \tag{3}$$

where $n = max(0, \lceil \alpha \rceil)$. While the Caputo fractional derivative, might be presented as

$$^{C}_{c} D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int\limits_c^t (t-\tau)^{n-\alpha-1} \frac{d^n}{d\tau^n} f(\tau) \mathrm{d}\tau. \tag{4}$$

It is worth noting that the above definitions are not equivalent, and the system designer, when deciding which definition to use, changes the interpretation of system (1).

Example 1. *Let* $f(t) = t^2 + 1$ *and* $\alpha = 0.5$, *it is easy to show that*

$$^{RL}_{0} D_t^{\alpha} f(t) = \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}} + \frac{1}{\sqrt{t\pi}}, \tag{5}$$

$$^{C}_{0} D_t^{\alpha} f(t) = \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}}. \tag{6}$$

Example 1 shows that even for the simplest functions, definitions can obtain different values. There is the following relationship between (3) and (4) [1]:

$$ {}^{RL}_{c}D^{\alpha}_{t} f(t) = {}^{C}_{c}D^{\alpha}_{t} f(t) + \sum_{j=0}^{n-1} \frac{(t-c)^{j-\alpha}}{\Gamma(j-\alpha+1)} f^{(j)}(c) \tag{7} $$

Let's also present the form of the Laplace transform for these two fractional derivative definitions. The Laplace transform of Riemann-Liouville fractional differential operator of order α is given by [24]

$$ \mathcal{L}\{{}^{RL}D^{\alpha}_{t} f(t)\} = s^{\alpha}F(s) + \sum_{k=0}^{n-1} s^{n-k-1} [{}^{RL}D^{k}_{t}\, {}^{RL}I^{n-\alpha}_{t} f(t)]_{t=0} \quad n-1 < \alpha < n \tag{8} $$

where ${}^{RL}I^{\alpha}_{t}$ is the Riemann–Liouville integral.

On the other hand, the Laplace transform of Caputo Fractional differential operator of order α is given by [24]

$$ \mathcal{L}\{{}^{C}D^{\alpha}_{t} f(t)\} = s^{\alpha}F(s) + \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0) \quad n-1 < \alpha < n \tag{9} $$

Caputo fractional derivative handles initial conditions more appropriately and is simpler (cf. (7)), so it is more convenient to use in modeling. In most practical applications, this definition is most often implemented. However, the choice of Caputo's fractional derivative is arbitrary and does not necessarily provide the best representation of the physical systems.

In literature, FOTF is most denoted as:

$$ G(s) = \frac{\sum_{j=0}^{m} b_j s^{\beta_j}}{\sum_{i=0}^{n} a_i s^{\alpha_i}}, \tag{10} $$

where a_i, b_j, are real constant coefficients with $i \in \{0, 1 \ldots, n\}$, $j \in \{0, 1 \ldots, m\}$, while the powers α_i, β_j, can be arranged in some chosen natural order $0 \leqslant \alpha_0 < \alpha_1 < \ldots < \alpha_n$ and $0 \leqslant \beta_0 < \beta_1 < \ldots < \beta_m$.

Considering the fact that ${}_{c}D^{0}_{t} = 1$, the transfer function (10) ($\alpha_0 = 0$ and $\beta_0 = 0$) can be described by the formula

$$ G(s) = \frac{\sum_{j=1}^{m} b_j s^{\beta_j} + b_0}{\sum_{i=1}^{n} a_i s^{\alpha_i} + a_0}. \tag{11} $$

where a_0 and b_0 are free terms.

Most of the transfer functions considered in research have free terms, as well as the static gain of the system $k = \frac{b_0}{a_0}$, so this form is more convenient. Despite this fact, the majority of authors prefer the notation (10) (see, e.g., [3]).

Sometimes, for the analytical purposes, it may be valuable to transform the (10) system into a fractional continuous-time linear system [9]:

$$\bar{G}(s) = \frac{\sum_{j=1}^{\bar{m}} \bar{b}_j (s^\gamma)^j + \bar{b}_0}{\sum_{i=1}^{\bar{n}} \bar{a}_i (s^\gamma)^i + \bar{a}_0}, \tag{12}$$

Occasionally an additional condition is imposed: the coefficient $\bar{a}_{\bar{n}} = 1$, while $\bar{a}_i$ for $i \in \{0, .., \bar{n} - 1\}$. Furthermore, the orders of the numerator and denominator satisfy the following relations with regard to the system (10): $\bar{m} >> m$ and $\bar{n} >> n$. Most often, system (12) is treated as known and no formal transition from (10) to (12) is provided.

There are authors who refer to (12) as the pseudo-rational transfer function formula [8] and arrange it in the following expression

$$\bar{G}(\lambda) = \frac{\sum_{j=1}^{\bar{m}} \bar{b}_j (\lambda)^j + \bar{b}_0}{\sum_{i=1}^{\bar{n}} \bar{a}_i (\lambda)^i + \bar{a}_0}, \tag{13}$$

where $s^\gamma = \lambda$.

The pseudo-rational description of the fractional order linear time-invariant model can be formulated by a state-space equation

$$\begin{aligned} {}_0D_t^\gamma s(t) = \frac{d^\gamma x(t)}{dt^\gamma} &= Ax(t) + Bu(t), \quad 0 < \gamma < 1, \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{14}$$

An exemplary realization of the matrix A, B, C, D indicating a connection to the transfer function (12) can be described as [9] (assuming that $\bar{n} = \bar{m} + 1$)

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -\bar{a}_0 \\ 1 & 0 & 0 & \dots & 0 & -\bar{a}_1 \\ 0 & 1 & 0 & \dots & 0 & -\bar{a}_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -\bar{a}_{\bar{n}-1} \end{bmatrix}, \quad B = \begin{bmatrix} \bar{b}_0 \\ \bar{b}_1 \\ \bar{b}_2 \\ \vdots \\ \bar{b}_{\bar{n}-1} \end{bmatrix}, \quad C^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \quad D = \bar{b}_{\bar{n}}. \tag{15}$$

State-space representation is related to formula (12) by the following expression:

$$\bar{G}(s) = C\left[I_{\bar{n}} s^\gamma - A\right]^{-1} B + D. \tag{16}$$

System (14) can be extended to a multi-order fractional system [11]

$$\begin{bmatrix} {}_0D_t^{\gamma_1} x_1(t) \\ {}_0D_t^{\gamma_2} x_2(t) \\ \vdots \\ {}_0D_t^{\gamma_n} x_n(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \dots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix} u, \tag{17}$$

$$y(t) = \begin{bmatrix} C_1 & C_2 & \ldots & C_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}. \tag{18}$$

Whereas the corresponding transfer function is given by the formula: [26]:

$$\bar{G}(s) = \begin{bmatrix} C_1 & C_2 & \ldots & C_n \end{bmatrix} \begin{bmatrix} s^{\gamma_1} I - A_{11} & -A_{12} & \ldots & -A_{1n} \\ -A_{21} & s^{\gamma_2} I - A_{22} & \ldots & -A_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ -A_{n1} & -A_{n2} & \ldots & s^{\gamma_n} I - A_{nn} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix}. \tag{19}$$

3. Existing system transformations

The purpose of the article is to propose a formal transition from (10) to (12). The section will first present the transformation methods known in the literature. A novel solution will be presented in the next section.

For the purpose of algorithms, recall the necessary definitions.

Definition 1. *[6] Any (12) can be written as*

$$\widetilde{G}(s) = \frac{\bar{b}_0 + \widetilde{b}_1 s^{\widetilde{\beta}_1} + \ldots + \widetilde{b}_{\bar{m}} s^{\widetilde{\beta}_{\bar{m}}}}{\bar{a}_0 + \widetilde{b}_{\bar{m}+1} s^{\widetilde{\beta}_{\bar{m}+1}} + \ldots + \widetilde{b}_{\bar{m}+\bar{n}} s^{\widetilde{\beta}_{\bar{m}+\bar{n}}}}, \tag{20}$$

with the following change in the notation: $\widetilde{b}_i = \bar{b}_i$ *for* $1 < i \leqslant \bar{m}$, $\widetilde{b}_{\bar{m}+i} = \bar{a}_i$ *for* $\bar{m} + 1 \leqslant i \leqslant \bar{m} + \bar{n}$ *and* $\widetilde{\beta}_i = \bar{\beta}_i$ *for* $1 < i \leqslant \bar{m}$, $\widetilde{\beta}_{\bar{m}+i} = \alpha_i$ *for* $\bar{m} + 1 \leqslant i \leqslant \bar{m} + \bar{n}$.

Assume for simplicity that the power coefficients in (12) can be written as decimal fractions.

Remark 1. *[6] Since the power coefficients are positive decimal fractions the* $\widetilde{\beta}_i$ *take the form*

$$\widetilde{\beta}_1, \ldots, \widetilde{\beta}_{m+n} \in \frac{\mathbb{N}}{10^{\widetilde{k}}}, \tag{21}$$

where $\widetilde{k}$ *denotes the maximum number of decimal places,* $\widetilde{k}$ *and is the minimum value satisfying the condition* $\forall_{i \in m+n} \beta_i 10^{\widetilde{k}} \in \mathbb{N}$.

Definition 2. *[6] The Least Common Multiple (LCM) for decimal fractions is defined as follows*

$$LCM_{f10}(\widetilde{\beta}_1, \ldots, \widetilde{\beta}_{m+n}) = \frac{LCM(10^{\widetilde{k}} \widetilde{\beta}_1, \ldots, 10^{\widetilde{k}} \widetilde{\beta}_{m+n})}{10^{\widetilde{k}}}. \tag{22}$$

Definition 3. *[6] Any (20) can be written as*

$$\widetilde{G}(s) = \frac{\bar{b}_0 + \widetilde{b}_1 (s^{\hat{\gamma}})^{\hat{\beta}_1} + \ldots + \widetilde{b}_i (s^{\hat{\gamma}})^{\hat{\beta}_i}}{\bar{a}_0 + \widetilde{b}_{i+1} (s^{\hat{\gamma}})^{\hat{\beta}_{i+1}} + \ldots + \widetilde{b}_{m+n} (s^{\hat{\gamma}})^{\hat{\beta}_{m+n}}}. \tag{23}$$

Theorem 1. *[6] Any system (10) (presented in the form (20)) with decimal coefficients, can be transformed to (12) presented in the form (23)) by using*

$$\hat{\gamma} = \frac{\min_{\widetilde{\beta}}}{LCM_{f10}(\widetilde{\beta})} \cdot 10^{-\widetilde{k}}, \quad \hat{\beta}_i = \frac{\widetilde{\beta}_i}{\min_{\widetilde{\beta}}} LCM_{f10}(\widetilde{\beta}) \cdot 10^{\widetilde{k}} = \frac{\widetilde{\beta}_i}{\hat{\gamma}} \in \mathbb{N}, \tag{24}$$

$$\begin{gathered} \min_{\widetilde{\beta}} = \min_{i \in m+n} (\widetilde{\beta}_1, \ldots, \widetilde{\beta}_i, \ldots, \widetilde{\beta}_{m+n}), \\ LCM_{f10}(\widetilde{\beta}) = LCM_{f10}(\widetilde{\beta}_1, \ldots, \widetilde{\beta}_i, \ldots, \widetilde{\beta}_{m+n}). \end{gathered} \tag{25}$$

The proof can be found in [6].

Example 2. *Let*

$$G_{ex}(s) = \frac{1 + s^{3.6}}{1 + s^{1.2} + s^{0.4}}, \tag{26}$$

using Theorem 1, it can be written

$$\hat{\gamma} = \frac{\min_{\widetilde{\beta}}}{LCM_{f10}(\widetilde{\beta})} \cdot 10^{-\widetilde{k}} = \frac{\min(3.6, 1.2, 0.4)}{\frac{LCM(3.6 \cdot 10, 1.2 \cdot 10, 0.4 \cdot 10)}{10^1}} 10^{-1} = \frac{1}{90}, \tag{27}$$

$$\hat{\beta}_1 = \frac{\widetilde{\beta}_1}{\hat{\gamma}} = \frac{3.6}{\frac{1}{90}} = 324, \quad \hat{\beta}_2 = \frac{\widetilde{\beta}_2}{\hat{\gamma}} = \frac{1.2}{\frac{1}{90}} = 108, \quad \hat{\beta}_3 = \frac{\widetilde{\beta}_3}{\hat{\gamma}} = \frac{0.4}{\frac{1}{90}} = 36, \tag{28}$$

so

$$\widetilde{G}_{ex}(s) = \frac{1 + (s^{\frac{1}{90}})^{324}}{1 + (s^{\frac{1}{90}})^{108} + (s^{\frac{1}{90}})^{36}}. \tag{29}$$

The idea of the above algorithm was to generalize the LCM of coefficients to decimal fractions, it turned out that this approach generates large powers of coefficients $\hat{\beta}_i$, which resulted from the need to ensure their value as natural numbers in each case.

Let's assume slightly broader, following [17], the power coefficients in (12) are rational numbers.

Remark 2. *Since the power coefficients are rational numbers, the* $\widetilde{\beta}_i$ *take the form*

$$\widetilde{\beta}_i = \frac{u_{\widetilde{\beta}_i}}{d_{\widetilde{\beta}_i}} \in \mathbb{Q}, \quad u_{\widetilde{\beta}_i} \in \mathbb{N}, \quad d_{\widetilde{\beta}_i} \in \mathbb{N}/\{0\}, \quad i \in \{1, \ldots, m+n\}, \tag{30}$$

where the corresponding coefficients $u_{\widetilde{\beta}_i}$ are the numerators and $d_{\widetilde{\beta}_i}$ are the denominators derived from the $\widetilde{\beta}_i$ coefficients. Therefore, the transfer function (23) can then also be presented in the following form

$$\widetilde{G}(s) = \frac{b_0 + \widetilde{b}_1 (s^{\hat{\gamma}})^{\frac{u_{\widetilde{\beta}_1}}{d_{\widetilde{\beta}_1}}} + \ldots + \widetilde{b}_i (s^{\hat{\gamma}})^{\frac{u_{\widetilde{\beta}_i}}{d_{\widetilde{\beta}_i}}}}{a_0 + \widetilde{b}_{i+1} (s^{\hat{\gamma}})^{\frac{u_{\widetilde{\beta}_{i+1}}}{d_{\widetilde{\beta}_{i+1}}}} + \ldots + \widetilde{b}_{m+n} (s^{\hat{\gamma}})^{\frac{u_{\widetilde{\beta}_{m+n}}}{d_{\widetilde{\beta}_{m+n}}}}}. \tag{31}$$

The most common approach presented in the literature is transformation:

Theorem 2. *[17, 21] Any system (10) (presented in the form (20)) with rational coefficients, can be transformed to (12) (presented in the form (23)) by using*

$$\hat{\gamma} = \frac{1}{LCM(d_{\widetilde{\beta}_i})}, \quad \hat{\beta}_i = \widetilde{\beta}_i \cdot LCM(d_{\widetilde{\beta}_i}) = \frac{\widetilde{\beta}_i}{\hat{\gamma}} \in \mathbb{N}. \tag{32}$$

The article [17], which demonstrates the original method, does not present a formal proof of the algorithm; only a reference to elementary properties of algebra appears. In addition, the authors present a slightly different approach than the one presented in this article. In [17], the authors define the so-called fractional order polynomial, which includes all coefficients of the numerator and denominator. For the system (20), it could take a form

$$\widetilde{P}(s) = \widetilde{b}_0 + \widetilde{b}_1 s^{\widetilde{\beta}_1} + \ldots + \widetilde{b}_{\bar{m}} s^{\widetilde{\beta}_{\bar{m}}} + \widetilde{b}_{\bar{m}+1} s^{\widetilde{\beta}_{\bar{m}+1}} + \ldots + \widetilde{b}_{\bar{m}+\bar{n}} s^{\widetilde{\beta}_{\bar{m}+\bar{n}}}, \tag{33}$$

which indicates some differences in formal aspects while it does not change the grounds of consideration.

Now let's try to consider the transfer function from Example 2 using Theorem 2.

Example 3. *Transfer function (26) can be represented as follows*

$$G_{ex}(s) = \frac{1 + s^{\frac{18}{5}}}{1 + s^{\frac{6}{5}} + s^{\frac{2}{5}}} \tag{34}$$

using Theorem 2, it can be written

$$\hat{\gamma} = \frac{1}{LCM(d_{\widetilde{\beta}_i})} = \frac{1}{LCM\,(5, 5, 5)} = \frac{1}{5}, \tag{35}$$

$$\hat{\beta}_1 = \frac{\widetilde{\beta}_1}{\hat{\gamma}} = \frac{\frac{18}{5}}{\frac{1}{5}} = 18, \quad \hat{\beta}_2 = \frac{\widetilde{\beta}_2}{\hat{\gamma}} = \frac{\frac{6}{5}}{\frac{1}{5}} = 6, \quad \hat{\beta}_3 = \frac{\widetilde{\beta}_3}{\hat{\gamma}} = \frac{\frac{2}{5}}{\frac{1}{5}} = 2, \tag{36}$$

hence

$$G_{ex}(s) = \frac{1 + (s^{\frac{1}{5}})^{18}}{1 + (s^{\frac{1}{5}})^6 + (s^{\frac{1}{5}})^2}. \tag{37}$$

The integer powers in Example 3 are smaller than those in Example 2, which will make the study of the system easier because it has a lower order. The issue addressed next is an attempt to answer the question of whether these powers can be reduced even further.

4. Main result

The section will present a new approach to transformation between the studied systems. For this purpose, some additional definitions will be provided.

The Least Common Multiple (LCM_f) and the Greatest Common Divisor (GCD_f) for fraction numbers are mathematical issues less known than their equivalents for integers. LCM_f and GCD_f formulations can be found in literature [23, 25], but they can rarely mention in standard mathematics textbooks. Notable properties can also be observed in mathematical forums such as [27]. Therefore, it was decided to present their definitions here.

Definition 4. *[23, 25] The Least Common Multiple (LCM) for fractional numbers is defined as follows*

$$\begin{aligned} LCM_f\left(\widetilde{\beta}_1, \ldots, \widetilde{\beta}_{m+n}\right) &= LCM_f\left(\frac{u_{\widetilde{\beta}_1}}{d_{\widetilde{\beta}_1}}, \ldots, \frac{u_{\widetilde{\beta}_{m+n}}}{d_{\widetilde{\beta}_{m+n}}}\right) \\ &= \frac{LCM(u_{\widetilde{\beta}_1}, \ldots, u_{\widetilde{\beta}_{m+n}})}{GCD(d_{\widetilde{\beta}_1}, \ldots, d_{\widetilde{\beta}_{m+n}})}. \end{aligned} \tag{38}$$

Note fractions should be in reduced form.

Definition 5. *[23, 25] The Greatest Common Divisor (GCD) for fractional numbers is defined as follows*

$$\begin{aligned} GCD_f(\widetilde{\beta}_1, \ldots, \widetilde{\beta}_{m+n}) &= GCD_f\left(\frac{u_{\widetilde{\beta}_1}}{d_{\widetilde{\beta}_1}}, \ldots, \frac{u_{\widetilde{\beta}_{m+n}}}{d_{\widetilde{\beta}_{m+n}}}\right) \\ &= \frac{GCD(u_{\widetilde{\beta}_1}, \ldots, u_{\widetilde{\beta}_{m+n}})}{LCM(d_{\widetilde{\beta}_1}, \ldots, d_{\widetilde{\beta}_{m+n}})}. \end{aligned} \tag{39}$$

Below, a certain lemma will be presented and proven, which will be needed for further consideration.

Lemma 1. *If there is at least one coefficient $\widetilde{\beta}_i < 1$ in the set $\{\widetilde{\beta}_1, \ldots, \widetilde{\beta}_{m+n}\}$ then the*

$$GCD_f(\widetilde{\beta}_1, \ldots, \widetilde{\beta}_{m+n}) < 1, \tag{40}$$

(this is only a necessary condition).

Proof. Note that the numerator and denominator belong to the following intervals (due to elementary algebra):

$$GCD(u_{\widetilde{\beta}_1}, \ldots, u_{\widetilde{\beta}_{m+n}}) \in \left(1, \min(u_{\widetilde{\beta}_1}, \ldots, u_{\widetilde{\beta}_{m+n}})\right) \tag{41}$$

$$LCM(d_{\widetilde{\beta}_1}, \ldots, d_{\widetilde{\beta}_{m+n}}) \in \left(\max(d_{\widetilde{\beta}_1}, \ldots, d_{\widetilde{\beta}_{m+n}}), \prod_{i\in 1,\ldots,m+n} d_{\widetilde{\beta}_i}\right) \tag{42}$$

Let there be a fixed order of all elements of the denominator and elements of the numerator, which can be written as $d_{\widetilde{\beta}_1} \leqslant d_{\widetilde{\beta}_2} \leqslant \ldots \leqslant d_{\widetilde{\beta}_{m+n}} < u_{\widetilde{\beta}_1} \leqslant u_{\widetilde{\beta}_2} \leqslant \ldots \leqslant u_{\widetilde{\beta}_{m+n}}$ then $\min(u_{\widetilde{\beta}_1}, \ldots, u_{\widetilde{\beta}_{m+n}}) = u_{\widetilde{\beta}_1}$, $\max(d_{\widetilde{\beta}_1}, \ldots, d_{\widetilde{\beta}_{m+n}}) = d_{\widetilde{\beta}_{m+n}}$ so $d_{\widetilde{\beta}_{m+n}} < u_{\widetilde{\beta}_1}$ therefore the intervals $(1, u_{\widetilde{\beta}_1})$ and $(d_{\widetilde{\beta}_{m+n}}, \prod_{i\in 1,\ldots,m+n} d_{\widetilde{\beta}_i})$ overlap and it is not known whether their quotient is less, or greater than 1.

But if we would exchange places at least two elements, e.g., $d_{\widetilde{\beta}_1} \leqslant d_{\widetilde{\beta}_2} \leqslant \ldots \leqslant u_{\widetilde{\beta}_1} < d_{\widetilde{\beta}_{m+n}} \leqslant u_{\widetilde{\beta}_2} \leqslant \ldots \leqslant u_{\widetilde{\beta}_{m+n}}$ then the intervals $(1, u_{\widetilde{\beta}_1}) < (d_{\widetilde{\beta}_{m+n}}, \prod_{i\in 1,\ldots,m+n} d_{\widetilde{\beta}_i})$ therefore the quotient (39) is less than 1. The situation of this kind of arrangement will occur if at least one value of the numerator is less than the denominator, so one factor $\widetilde{\beta}_i$ less than 1 is enough, which ends the proof.

It should be emphasized that Lemma 1 is only a necessary condition; the fact that the result of GCD_f is less than 1 does not mean that at least one factor is less than 1. Consider the following example.

Example 4. *GCD_f calculations (all fractions in reduced form):*

$$GCD_f\left(\frac{22}{15}, \frac{33}{20}, \frac{77}{10}\right) = \frac{GCD(22,33,77)}{LCM(15,20,10)} = \frac{11}{60} < 1 \tag{43}$$

$$GCD_f\left(\frac{22}{5}, \frac{33}{10}, \frac{77}{10}\right) = \frac{GCD(22,33,77)}{LCM(5,10,10)} = \frac{11}{10} > 1 \tag{44}$$

From a practical point of view, in most cases, usually fractions reduce each other in some way, and do not contain prime factorization arranged similarly as in example (44). Our observations indicate that the situation when fractions whose GCD_f is greater than 1 is rather exceptional.

This can be checked as follows. Let S denote the set of all fractions defined $\{x/y : 1 \leqslant x \leqslant N,\ 1 \leqslant y \leqslant N\}$, while let S_u denote set of all unique fractions from S, obtained using the formula $GCD_f(x_1/y_1, x_2/y_2)$, $x_1/y_1 \in S$, $x_2/y_2 \in S$. In addition, S_1 represents all fractions from the set S_u that are greater than 1. The cardinality of the sets has been denoted as $|S_u|$, $|S_1|$ respectively. The frequency of GCD_f fractions greater than 1 is examined in Table 1.

Table 1: Results of a study on the frequency of GCD_f greater than 1 for two fractions from the set S

N	50	60	70	80	90	100	150
$\|S_1\|$	773	1101	1493	1965	2479	2633	6857
$\|S_u\|$	16028	26303	40862	62936	88498	109951	394130
$\|S_1\|/\|S_u\|$	4.82281%	4.18583%	3.65376%	3.12222%	2.80119%	2.3947%	1.73978%

It is worth noting that the cardinality of the set $|S_1|$ is described by sequence A015614 in the OEIS [19].

Remark 3. *[5] Linearity of GCD. For all positive integers l, m, and n*

$$GCD(l \cdot m, l \cdot n) = l \cdot GCD(m, n). \tag{45}$$

Using the above definitions and remarks, in this article we propose a more different form of transformation.

Theorem 3. *Any system (10) (presented in the form (20)) with rational coefficients, can be transformed to (12) (presented in the form (23)) by using*

$$\hat{\gamma} = GCD_f(\widetilde{\beta}) \in \mathbb{R}^+, \quad \hat{\beta}_i = \frac{\widetilde{\beta}_i}{GCD_f(\widetilde{\beta})} = \frac{\widetilde{\beta}_i}{\hat{\gamma}} \in \mathbb{N}, \tag{46}$$

where

$$GCD_f(\widetilde{\beta}) = GCD_f(\widetilde{\beta}_1, \ldots, \widetilde{\beta}_i, \ldots, \widetilde{\beta}_{m+n}). \tag{47}$$

Fractions $\widetilde{\beta}_i$ should be in reduced form.

Proof. The possibility of establishing (23) from (20) is elementary

$$\widetilde{b}_i \left(s^{\hat{\gamma}}\right)^{\hat{\beta}_i} = \widetilde{b}_i \left(s^{GCD_f \widetilde{\beta}}\right)^{\frac{\widetilde{\beta}_i}{GCD_f \widetilde{\beta}}} = \widetilde{b}_i s^{\widetilde{\beta}_i}. \tag{48}$$

Now let us prove that $\hat{\beta}_i \in \mathbb{N}$ is

$$\hat{\beta}_i = \frac{\widetilde{\beta}_i}{GCD_f(\widetilde{\beta})} = \underbrace{\frac{u_{\widetilde{\beta}_i}}{GCD(u_{\widetilde{\beta}_1}, \ldots, u_{\widetilde{\beta}_{m+n}})}}_{\mathbb{N}} \cdot \underbrace{\frac{LCM(d_{\widetilde{\beta}_1}, \ldots, d_{\widetilde{\beta}_{m+n}})}{d_{\widetilde{\beta}_i}}}_{\mathbb{N}} \in \mathbb{N} \tag{49}$$

The first factor is natural because $GCD(u_{\widetilde{\beta}_1}, \ldots, u_{\widetilde{\beta}_{m+n}})$ is a divisor of each $u_{\widetilde{\beta}_i}$, while the second factor is natural because $LCM(d_{\widetilde{\beta}_1}, \ldots, d_{\widetilde{\beta}_{m+n}})$ is a multiple of each $d_{\widetilde{\beta}_i}$.

Consider the following example.

Example 5. *Using the transfer function* (34) *from Example 3, it can be written using Theorem 3:*

$$\hat{\gamma} = GCD_f(\widetilde{\beta}) = GCD_f\left(\frac{18}{5}, \frac{6}{5}, \frac{2}{5}\right) = \frac{2}{5}, \tag{50}$$

$$\hat{\beta}_1 = \frac{\widetilde{\beta}_1}{GCD_f(\widetilde{\beta})} = \frac{\frac{18}{5}}{\frac{2}{5}} = 9, \quad \hat{\beta}_2 = \frac{\widetilde{\beta}_2}{GCD_f(\widetilde{\beta})} = \frac{\frac{6}{5}}{\frac{2}{5}} = 3, \\ \hat{\beta}_3 = \frac{\widetilde{\beta}_3}{GCD_f(\widetilde{\beta})} = \frac{\frac{2}{5}}{\frac{2}{5}} = 1, \tag{51}$$

thus

$$G_{ex}(s) = \frac{1 + (s^{\frac{2}{5}})^9}{1 + (s^{\frac{2}{5}})^3 + (s^{\frac{2}{5}})^1}. \tag{52}$$

The transition from system (10) with rational coefficients to (12) can be implemented in infinitely many ways. However, from a practical point of view, it is best to make the coefficient $\hat{\gamma} \in \mathbb{R}^+$ as large as possible and the coefficients $\hat{\beta}_i \in \mathbb{N}$ as small as possible. This is due to the need for the degree of the analyzed system to be as small as possible. Note that in Example 5 the degree cannot be reduced any further, because the lowest $\hat{\beta}_i$ coefficient is 1. However, it is not always possible to present such a formula (12) that at least one of the beta coefficients will be 1. The limitation that all coefficients should belong to the domain of natural numbers must be kept in mind. This problem will be analyzed in more detail below.

The following theorem is true.

Theorem 4. *Let the set Γ denote the set of admissible coefficients γ of the system* (12) *shown in the form* (23) *($\gamma \in \Gamma \subset \mathbb{R}^+$) and let for each $i \in \{1, 2, \ldots, m+n\}$ the corresponding B_i denote the sets of admissible values of the coefficients β_i ($\beta_i \in B_i \subset \mathbb{N}$).*

It is true that the sets Γ and B_i have the constraints $\sup(\Gamma) = \hat{\gamma}$ and for any i $\inf(B_i) = \hat{\beta}_i$, with $\hat{\gamma}$, $\hat{\beta}_i$ given by the formulas (46) *derived from Theorem 3.*

Proof. Notice that the coefficients $\hat{\gamma}$ and $\hat{\beta}_i$ are related to each other

$$\hat{\gamma} \cdot \hat{\beta}_i = const \cdot \frac{\widetilde{\beta}_i}{const} = \widetilde{\beta}_i\,, \tag{53}$$

where $const$ denotes any arbitrary constant and fractions $\widetilde{\beta}_i$ are in reduced form.

Therefore, when the coefficient $\hat{\gamma}$ is largest then the corresponding $\hat{\beta}_i$ is smallest – so only one condition (smallest or largest) needs to be checked.

The coefficients $\hat{\beta}_i$ are smallest when they cannot be reduced - which means that it is enough to check whether the

$$GCD(\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_{m+n}) = 1. \tag{54}$$

Rewrite the formula using (49), then it is

$$\begin{aligned} &GCD(\ldots, \hat{\beta}_i, \ldots) \\ &\quad = GCD\left(\ldots, \frac{u_{\widetilde{\beta}_i}}{GCD(u_{\widetilde{\beta}_1}, \ldots, u_{\widetilde{\beta}_{m+n}})} \cdot \frac{LCM(d_{\widetilde{\beta}_1}, \ldots, d_{\widetilde{\beta}_{m+n}})}{d_{\widetilde{\beta}_i}}, \ldots\right). \end{aligned} \tag{55}$$

Then using the unique prime factorizations of the integers [18]

$$a = \prod_j p_j^{e_j(a)}, \tag{56}$$

where p_j are consecutive prime numbers and $e_j(a)$ the power factor at p_j.

Then there is

$$GCD\left(\ldots, \frac{\prod\limits_j p_j^{e_j(u_{\widetilde{\beta}_i})}}{\prod\limits_j p_j^{\max(e_j(u_{\widetilde{\beta}_1}),\ldots,e_j(u_{\widetilde{\beta}_{m+n}}))}} \cdot \frac{\prod\limits_j p_j^{\min(e_j(d_{\widetilde{\beta}_1}),\ldots,e_j(d_{\widetilde{\beta}_{m+n}}))}}{\prod\limits_j p_j^{e_j(d_{\widetilde{\beta}_i})}}, \ldots\right), \tag{57}$$

using the linearity of GCD (Remark 3)

$$\frac{\prod_j p_j^{\min(e_j(d_{\widetilde{\beta}_1}),\ldots,e_j(d_{\widetilde{\beta}_{m+n}}))}}{\prod_j p_j^{\max(e_j(u_{\widetilde{\beta}_1}),\ldots,e_j(u_{\widetilde{\beta}_{m+n}}))}} \cdot GCD\left(\ldots, \frac{\prod_j p_j^{e_j(u_{\widetilde{\beta}_i})}}{\prod_j p_j^{e_j(d_{\widetilde{\beta}_i})}}, \ldots\right), \tag{58}$$

using GCD's properties, it is possible to write

$$\frac{\prod_j p_j^{\min(e_j(d_{\widetilde{\beta}_1}),\ldots,e_j(d_{\widetilde{\beta}_{m+n}}))}}{\prod_j p_j^{\max(e_j(u_{\widetilde{\beta}_1}),\ldots,e_j(u_{\widetilde{\beta}_{m+n}}))}} \cdot \frac{\prod_j p_j^{\max(e_j(u_{\widetilde{\beta}_1}),\ldots,e_j(u_{\widetilde{\beta}_{m+n}}))}}{\prod_j p_j^{\min(e_j(d_{\widetilde{\beta}_1}),\ldots,e_j(d_{\widetilde{\beta}_{m+n}}))}} = 1 \tag{59}$$

which completes the proof.

The following properties follow directly from the above theorem.

Remark 4. *The form of the system resulting from Theorem 3 has the smallest possible coefficients $\hat{\beta}_i$ and the largest possible $\hat{\gamma}$. Thus, in this sense, it is the* *most optimal representation of the system* (10) *and transfer function of the form* (12) *with lower order cannot be found.*

Remark 5. *The smallest possible order after the transformation of transfer function* (10) *to form* (12) *has the order of the numerator*

$$\bar{m} = \widetilde{\beta}_{m,max} = \max_{i \in \{1,\ldots,m\}} (\widetilde{\beta}_1, \ldots, \widetilde{\beta}_i, \ldots, \widetilde{\beta}_m), \tag{60}$$

and the denominator

$$\bar{n} = \widetilde{\beta}_{n,max} = \max_{i \in \{m+1,\ldots,n\}} (\widetilde{\beta}_{m+1}, \ldots, \widetilde{\beta}_i, \ldots, \widetilde{\beta}_{m+n}). \tag{61}$$

Remark 6. *If there is at least one* $\widetilde{\beta}_i$ *less than* 1 *then the coefficient* $\hat{\gamma}$ *belongs to the interval* $(0, 1]$ *(if* $\exists_{i \in \{1,\ldots,m+n\}}\ \widetilde{\beta}_i < 1 \rightarrow \hat{\gamma} \in (0, 1]$*).*

The property follows directly from Lemma 1.

In many practical systems transformations (cf. [9]), it is required that the coefficient $\hat{\gamma} \in (0, 1]$ regardless of the value of the $\widetilde{\beta}_i$ coefficients. To add this constraint, Theorem 3 must be expanded to the following form.

Theorem 5. *Any system* (10) *(presented in the form* (20)*) with rational coefficients, can be transformed to* (12) *(presented in the form* (23)*) by using*

$$\hat{\gamma} = \frac{GCD_f(\widetilde{\beta})}{\lceil GCD_f(\widetilde{\beta}) \rceil} \in (0, 1], \quad \hat{\beta}_i = \frac{\widetilde{\beta}_i \lceil GCD_f(\widetilde{\beta}) \rceil}{GCD_f(\widetilde{\beta})} = \frac{\widetilde{\beta}_i}{\hat{\gamma}} \in \mathbb{N}, \tag{62}$$

where the symbol $\lceil\ \rceil$ *indicates the ceiling, in addition all fractions* $\widetilde{\beta}_i$ *should be in reduced form.*

Proof. The possibility of establishing (23) from (20) is elementary

$$\widetilde{b}_i (s^{\hat{\gamma}})^{\hat{\beta}_i} = \widetilde{b}_i \left(s^{\frac{GCD_f(\widetilde{\beta})}{\lceil GCD_f(\widetilde{\beta}) \rceil}} \right)^{\frac{\widetilde{\beta}_i \lceil GCD_f(\widetilde{\beta}) \rceil}{GCD_f(\widetilde{\beta})}} = \widetilde{b}_i s^{\widetilde{\beta}_i}. \tag{63}$$

Similarly, as in the proof of Theorem 3

$$\begin{aligned} \hat{\beta}_i &= \frac{\widetilde{\beta}_i \lceil GCD_f(\widetilde{\beta}) \rceil}{GCD_f(\widetilde{\beta})} \\ &= \underbrace{\frac{u_{\widetilde{\beta}_i}}{GCD(u_{\widetilde{\beta}_1}, \ldots, u_{\widetilde{\beta}_{m+n}})}}_{\mathbb{N}} \cdot \underbrace{\frac{LCM(d_{\widetilde{\beta}_1}, \ldots, d_{\widetilde{\beta}_{m+n}})}{d_{\widetilde{\beta}_i}}}_{\mathbb{N}} \cdot \underbrace{\lceil GCD_f(\widetilde{\beta}) \rceil}_{\mathbb{N}} \in \mathbb{N}. \end{aligned} \tag{64}$$

In addition, since any fraction of type $\frac{a}{\lceil a \rceil}$ where $a = \mathbb{N}/\{0\}$ belongs to $(0, 1]$ then $\hat{\gamma} \in (0, 1]$.

There is an analogue of the Theorem 4 but that for $\hat{\gamma} \in (0, 1]$

Theorem 6. *Let the set* Γ *denote the set of admissible coefficients* γ *of the system* (12) *shown in the form* (23) *(*$\gamma \in \Gamma \subset (0, 1]$*) and and let* B_i *be defined as in Theorem 4.*

It is true that the sets Γ *and* B_i *have the constraints* $\sup(\Gamma) = \hat{\gamma}$ *and for any i* $\inf(B_i) = \hat{\beta}_i$*, with* $\hat{\gamma}$*,* $\hat{\beta}_i$ *given by the formulas* (62) *derived from Theorem 5.*

Proof. Let's consider two cases, extending the proof of Theorem 4.

Note that if $\hat{\gamma} < 1$, the problem reduces to the situation in 4. This is because if $GCD_f(\widetilde{\beta}) < 1$ then $\lceil GCD_f(\widetilde{\beta}) \rceil = 1$ therefore $\hat{\gamma} = \frac{GCD_f(\widetilde{\beta})}{\lceil GCD_f(\widetilde{\beta}) \rceil} = GCD_f(\widetilde{\beta}) \in (0, 1]$, i.e., the transformation described by formula (62) goes into (46). Thus, optimality is assured by Theorem 4.

Meanwhile, if $\hat{\gamma} > 1$, the coefficient should be modified so that it falls within the interval $(0, 1]$. Thus, this problem can be described as follows – having an $\hat{\gamma} > 1$, by what natural number should it be divided to make $\frac{\hat{\gamma}}{a} < 1$. Since the coefficients $\hat{\beta}_i$ must remain natural, it is, also, that $a \in \mathbb{N}/\{0\}$. The next step is to select a so that $\hat{\gamma} = \frac{GCD_f(\widetilde{\beta})}{a}$ a is as large as possible and falls within the interval $(0, 1]$. It is obvious that if $\frac{a}{\lceil a \rceil} < 1$ then also $\frac{a}{\lceil a \rceil + i} < 1$ for any $i \in \mathbb{N}/\{0\}$, while $\frac{a}{\lceil a \rceil - i} > 1$, so the smallest natural number providing the largest possible $\frac{a}{\lceil a \rceil} < 1$ is $\lceil a \rceil$, hence $\hat{\gamma} = \frac{GCD_f(\widetilde{\beta})}{\lceil GCD_f(\widetilde{\beta}) \rceil}$ – which completes the proof.

Now let's extend the consideration of systems (10) (in the form (20)) to those that have real coefficients $\widetilde{\beta}_i \in \mathbb{R}$.

Theorem 7. *Not all systems with real coefficients can be transformed from the form* (10) *to* (12).

Proof. The proof follows directly from the properties of numbers. To find the natural coefficients of $\hat{\beta}_i$, it is required that all the coefficients of $\widetilde{\beta}_i$ are commensurable, because only then can their ratio be represented as a rational number.

Formally, it can be rephrased as follows. Let $\widetilde{\beta}_i$ and $\widetilde{\beta}_j$ be incommensurable then referring to (53) can be written

$$\frac{\widetilde{\beta}_i}{\widetilde{\beta}_j} \neq \frac{\hat{\gamma} \cdot \hat{\beta}_i}{\hat{\gamma} \cdot \hat{\beta}_j} \tag{65}$$

therefore $\hat{\beta}_i$ and $\hat{\beta}_j$ cannot simultaneously be natural numbers.

From the above considerations, certain properties follow.

Remark 7. *System* (10) *with real coefficients can be transformed to* (12) *if and only if all $\widetilde{\beta}_i$ coefficients are commensurable.*

Assume that the power coefficients in (12) are real and all commensurable.

Remark 8. *Since the power coefficients are positive reals and they are commensurable, all $\widetilde{\beta}_i$ take the form*

$$\widetilde{\beta}_1, \ldots, \widetilde{\beta}_{m+n} = \widetilde{c} \cdot \widetilde{\kappa}_1, \ldots, \widetilde{c} \cdot \widetilde{\kappa}_{m+n} \in \mathbb{I} \cdot \mathbb{Q} \tag{66}$$

where $\mathbb{I}$ set of all irrationals, $\widetilde{c} \in \mathbb{I}$ irrational constant, and $\widetilde{\kappa}_i$ is the corresponding coefficient such that $\widetilde{\kappa}_i = \frac{\widetilde{\beta}_1}{\widetilde{c}}$.

Definition 6. *Any* (20) *with real commensurable coefficients in form* (66)*, can be rewritten as*

$$G(s) = \frac{b_0 + \widetilde{b}_1 s^{\widetilde{c} \cdot \widetilde{\kappa}_1} + \ldots + \widetilde{b}_i s^{\widetilde{c} \cdot \widetilde{\kappa}_1}}{a_0 + \widetilde{b}_{i+1} s^{\widetilde{c} \cdot \widetilde{\kappa}_{i+1}} + \ldots + \widetilde{b}_{m+n} s^{\widetilde{c} \cdot \widetilde{\kappa}_{m+n}}}. \tag{67}$$

The equivalent of Theorem 3, for real numbers, can be presented as follows.

Theorem 8. *Any system* (10) *(presented in the form* (67)*) with real commensurable coefficients in the form* (66)*, can be transformed to* (12) *(presented in the form* (23)*) by using*

$$\hat{\gamma} = GCD_f(\widetilde{\kappa}) \cdot \widetilde{c} \in \mathbb{R}^+, \quad \hat{\beta}_i = \frac{\widetilde{\kappa}_i}{GCD_f(\widetilde{\kappa})} = \frac{\widetilde{\kappa} \cdot \widetilde{c}}{\hat{\gamma}} = \frac{\widetilde{\beta}_i}{\hat{\gamma}} \in \mathbb{N}, \tag{68}$$

where

$$GCD_f(\widetilde{\kappa}) = GCD_f(\widetilde{\kappa}_1, \ldots, \widetilde{\kappa}_i, \ldots, \widetilde{\kappa}_{m+n}), \tag{69}$$

fractions $\widetilde{\kappa}_i$ should be in reduced form.

The proof is almost identical to the proof of Theorem 3, which is why it was omitted.

As an analogue, Theorem 5 for the real numbers can be written as follows.

Theorem 9. *Any system* (10) *(presented in the form* (67)*) with real commensurable coefficients in the form* (66)*, can be transformed to* (12) *(presented in the form* (23)*) by using*

$$\hat{\gamma} = \frac{GCD_f(\widetilde{\kappa}) \cdot \widetilde{c}}{\left\lceil GCD_f(\widetilde{\kappa}) \cdot \widetilde{c} \right\rceil} \in (0, 1], \quad \hat{\beta}_i = \frac{\widetilde{\kappa}_i \cdot \left\lceil GCD_f(\widetilde{\kappa}) \cdot \widetilde{c} \right\rceil}{GCD_f(\widetilde{\kappa})} = \frac{\widetilde{\kappa} \cdot \widetilde{c}}{\hat{\gamma}} = \frac{\widetilde{\beta}_i}{\hat{\gamma}} \in \mathbb{N}, \tag{70}$$

fractions $\widetilde{\kappa}_i$ should be in reduced form.

The proof was omitted due to similarity with the proof of Theorem 5.

Let's consider an example illustrating how the Theorem 8 and Theorem 9 work.

Example 6. *Let there be a system given by the formula*

$$G_{ex}(s) = \frac{1 + s^{\frac{18}{5}\sqrt{7}}}{1 + s^{\frac{6}{5}\sqrt{7}} + s^{\frac{2}{5}\sqrt{7}}}, \tag{71}$$

for which

$$GCD_f(\widetilde{\kappa}) = GCD_f\left(\frac{18}{5}, \frac{6}{5}, \frac{2}{5}\right) = \frac{2}{5}, \tag{72}$$

then according to Theorem 8 and (68) one can write

$$\hat{\gamma} = GCD_f(\widetilde{\kappa}) \cdot \widetilde{c} = \frac{2}{5} \cdot \sqrt{7} > 1, \tag{73}$$

$$\hat{\beta}_1 = \frac{\widetilde{\kappa}_1}{GCD_f(\widetilde{\kappa}) \cdot \widetilde{c}} = \frac{\frac{18}{5}\sqrt{7}}{\frac{2}{5}\sqrt{7}} = 9, \tag{74}$$

$$\hat{\beta}_2 = \frac{\widetilde{\kappa}_2}{GCD_f(\widetilde{\kappa}) \cdot \widetilde{c}} = \frac{\frac{6}{5}\sqrt{7}}{\frac{2}{5}\sqrt{7}} = 3, \tag{75}$$

$$\hat{\beta}_3 = \frac{\widetilde{\kappa}_3}{GCD_f(\widetilde{\kappa}) \cdot \widetilde{c}} = \frac{\frac{2}{5}\sqrt{7}}{\frac{2}{5}\sqrt{7}} = 1, \tag{76}$$

which gives the final form of the transformed system

$$G_{ex}(s) = \frac{1 + (s^{\frac{2\sqrt{7}}{5}})^9}{1 + (s^{\frac{2\sqrt{7}}{5}})^3 + (s^{\frac{2\sqrt{7}}{5}})^1}. \tag{77}$$

On the other hand, according to Theorem 9 and equation (70),

$$\hat{\gamma} = \frac{GCD_f(\widetilde{\kappa}) \cdot \widetilde{c}}{\lceil GCD_f(\widetilde{\kappa}) \cdot \widetilde{c} \rceil} = \frac{\frac{2}{5} \cdot \sqrt{7}}{\left\lceil \frac{2}{5} \cdot \sqrt{7} \right\rceil} = \frac{\frac{2}{5} \cdot \sqrt{7}}{2} = \frac{\sqrt{7}}{5} < 1, \tag{78}$$

$$\hat{\beta}_1 = \frac{\widetilde{\kappa}_1 \cdot \lceil GCD_f(\widetilde{\kappa}) \cdot \widetilde{c} \rceil}{GCD_f(\widetilde{\kappa}) \cdot \widetilde{c}} = \frac{\frac{18}{5}\sqrt{7} \cdot 2}{\frac{2}{5}\sqrt{7}} = 18, \tag{79}$$

$$\hat{\beta}_2 = \frac{\widetilde{\kappa}_2 \cdot \lceil GCD_f(\widetilde{\kappa}) \cdot \widetilde{c} \rceil}{GCD_f(\widetilde{\kappa}) \cdot \widetilde{c}} = \frac{\frac{6}{5}\sqrt{7} \cdot 2}{\frac{2}{5}\sqrt{7}} = 6, \tag{80}$$

$$\hat{\beta}_3 = \frac{\widetilde{\kappa}_3 \cdot \lceil GCD_f(\widetilde{\kappa}) \cdot \widetilde{c} \rceil}{GCD_f(\widetilde{\kappa}) \cdot \widetilde{c}} = \frac{\frac{2}{5}\sqrt{7} \cdot 2}{\frac{2}{5}\sqrt{7}} = 2 \tag{81}$$

is calculated, which gives the final form of the transformed system

$$G_{ex}(s) = \frac{1 + (s^{\frac{\sqrt{7}}{5}})^{18}}{1 + (s^{\frac{\sqrt{7}}{5}})^6 + (s^{\frac{\sqrt{7}}{5}})^2}. \tag{82}$$

For Theorems 8 and 9, it is possible to formulate analogous Theorems to 4 and 6.

Theorem 10. *Let the set* Γ *denote the set of admissible coefficients* γ *of the system* (12) *shown in the form* (67) *(*$\gamma \in \Gamma \subset \mathbb{R}^+$*) and let for each* $i \in \{1, 2, \ldots, m+n\}$ *the corresponding* B_i *denote the sets of admissible values of the coefficients* β_i *(*$\beta_i \in B_i \subset \mathbb{N}$*).*

It is true that the sets Γ *and* B_i *have the constraints* $\sup(\Gamma) = \hat{\gamma}$ *and for any* i $\inf(B_i) = \hat{\beta}_i$*, with* $\hat{\gamma}$*,* $\hat{\beta}_i$ *given by the formulas* (68) *derived from Theorem 8.*

Theorem 11. *Let the set* Γ *denote the set of admissible coefficients* γ *of the system* (12) *shown in the form* (67) *(*$\gamma \in \Gamma \subset (0, 1]$*) and let* B_i *be defined as in Theorem 10.*

It is true that the sets Γ *and* B_i *have the constraints* $\sup(\Gamma) = \hat{\gamma}$ *and for any* i $\inf(B_i) = \hat{\beta}_i$*, with* $\hat{\gamma}$*,* $\hat{\beta}_i$ *given by the formulas* (70) *derived from Theorem 9.*

Proof. It is enough to note that in (68) and (70) in the coefficient $\hat{\gamma}$ there is a constant $\widetilde{c}$ therefore it can be written

$$\hat{\gamma} \cdot \hat{\beta}_i = const \cdot \widetilde{c} \cdot \frac{\widetilde{\beta}_i}{const} = const \cdot \widetilde{c} \cdot \frac{\widetilde{\kappa}_i}{const \cdot \widetilde{c}} \tag{83}$$

Thus, the reasoning of the proofs of Theorems 4 and 6 can be repeated for $\widetilde{\kappa}_i \in \mathbb{Q}$.

5. Conclusion

The paper explores existing algorithms that allow transformations of FOTF systems (10) to Fractional Continuous-Time Linear Systems (12). The research succeeded in obtaining new transformations between these systems, focusing on situations when the coefficients are fractions, and real numbers. In addition, it was possible to prove that the indicated methods are the best due to the possible generated order of the system (12). In addition, it was shown if the coefficients of the system's powers do not commensurate, it cannot be transformed to the form (12). All results are summarized in Table 2.

Table 2: Summary of properties of the obtained transformations between systems

Algorithm	Coef. of trans. func.	Range $\hat{\gamma}$	Eq. transform.	Optimal lowest $\hat{\beta}_i \in \mathbb{N}$ in range $\hat{\gamma}$)
Theorem 1 from [6]	Finite decimal	(0, 1]	(24)	No
Theorem 2 from [17]	Rational	(0, 1]	(32)	No
Theorem 3	Rational	$\mathbb{R}^+$	(46)	yes (Theorem 4)
Theorem 5	Rational	(0, 1]	(62)	yes (Theorem 6)
Theorem 8	Real commensurable	$\mathbb{R}^+$	(68)	yes (Theorem 10)
Theorem 9	Real commensurable	(0, 1]	(70)	yes (Theorem 11)
don't exist (Remark 7)	Real incommensurable	–	–	–

References

[1] T. Abdeljawad: On Riemann and Caputo fractional differences. *Computers & Mathematics with Applications*, **62**(3), (2011), 1602–1611. DOI: 10.1016/j.camwa.2011.03.036

[2] G. Carlson and C. Halijak: Approximation of fractional capacitors $(1/s)(1/n)$ by a regular Newton process. *IEEE Transactions on Circuit Theory*, **11**(2), (1964), 210–213. DOI: 10.1109/TCT.1964.1082270

[3] F.N. Deniz, B.B. Alagoz, N. Tan and M. Koseoglu: Revisiting four approximation methods for fractional order transfer function implementations: Stability preservation, time and frequency response matching analyses. *Annual Reviews in Control*, **49**, (2020), 239–257. DOI: 10.1016/j.arcontrol.2020.03.003

[4] K. Diethelm and N. Ford: Analysis of fractional differential equations. *Journal of Mathematical Analysis and Applications*, **265**, (2002), 229–248. DOI: 10.1006/jmaa.2000.7194

[5] M. Fiore and I. Leslie: *Discrete Mathematics. Course materials.* University of Cambridge, 2016. URL: https://www.cl.cam.ac.uk/teaching/1617/DiscMath/ (visited on 04/25/2025).

[6] J. Jagodziński and M. Lampasiak: On some properties of Padé approximations for fractional order transfer function (FOTF). In *2024 28th International Conference on Methods and Models in Automation and Robotics (MMAR)*, Miedzyzdroje, Poland, (2024), 340–345. DOI: 10.1109/MMAR62187.2024.10680833

[7] J. Jagodziński and M. Lampasiak: Padé approximation method for fractional-order systems. In *IEEE 25th International Carpathian Control Conference (ICCC)*, Krynica Zdrój, Poland, (2024), 1–6. DOI: 10.1109/ICCC62069.2024.10569957

[8] W. Jakowluk: *Methods and Tools in CAD*. Chapter: Fractional-order LTI system identification using integer-order state-space model. Publishing House of Bialystok University of Technology, 2021, pp. 75–86.

[9] T. Kaczorek: *Selected Problems of Fractional Systems Theory*. Springer-Verlag Berlin Heidelberg, 2011.

[10] T. Kaczorek and Ł. Sajewski: *The Realization Problem for Positive and Fractional Systems. Studies in Systems, Decision and Control 1*. Springer, International Publishing, Switzerland, 2014.

[11] T. Kaczorek: Positive linear systems consisting of n subsystems with different fractional orders. *IEEE Transactions on Circuits and Systems I: Regular Papers*, **58**(6), (2011), 1203–1210. DOI: 10.1109/TCSI.2010.2096111

[12] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo: *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, North-Holland, 2006. DOI: 10.1016/S0304-0208(06)80006-X

[13] K. Kothari, U. Mehta and R. Prasad: Fractional-order system modeling and its applications. *Journal of Engineering Science and Technology Review*, **12**, (2019), 1–10. DOI: 10.25103/jestr.126.01

[14] B.T. Krishna: Studies on fractional order differentiators and integrators: A survey. *Signal Processing*, **91**(3), (2011), 386–426. DOI: 10.1016/j.sigpro.2010.06.022

[15] M. Lampasiak and J. Jagodziński: Analysis of methods for approximating fractional order systems based on zero and pole values. *IEEE 26th International Carpathian Control Conference (ICCC 2025)*, Starý Smokovec, Slovakia, (2025). DOI: 10.1109/ICCC65605.2025.11022819

[16] K. Matsuda and H. Fujii: H(infinity) optimized wave-absorbing control – Analytical and experimental results. *Journal of Guidance, Control, and Dynamics*, **16**(6), (1993), 1146–1153. DOI: 10.2514/3.21139

[17] F. Merrikh-Bayat and M. Afshar: Extending the root-locus method to fractional-order systems. *Journal of Applied Mathematics*, **2008**(1), (2008). DOI: 10.1155/2008/528934

[18] R.S. Millman, P.J. Shiue and E.B. Kahn: *Problems and Proofs in Numbers and Algebra*. Springer International Publishing, Switzerland, 2015. DOI: 10.1007/978-3-319-14427-6

[19] OEIS Foundation Inc.: Entry A015614 in The On-Line Encyclopedia of Integer Sequences. 2025. URL: https://oeis.org/A015614

[20] A. Oustaloup, F. Levron, B. Mathieu and F.M. Nanot: Frequency-band complex noninteger differentiator: characterization and synthesis. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, **47**(1), (2000), 25–39. DOI: 10.1109/81.817385

[21] I. Petras: *Fractional-Order Nonlinear Systems: Modeling, Analysis and Simulation*. Springer, Berlin Heidelberg, 2011. DOI: 10.1007/978-3-642-18101-6

[22] I. Podlubny: *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, Methods of Their Solution and Some of Their Applications*. Academic Press (Elsevier), San Diego/Amsterdam, 1998.

[23] R.V. Praveen: *Quantitative Aptitude and Reasoning*. PHI Learning Pvt. Ltd., Delhi, 2016.

[24] A. Shaikh and B. Sontakke: Properties of Caputo operator and its applications to linear fractional differential equations. *International Journal of Engineering Research and Applications*, **5**(5), (2015), 22–27.

[25] M.K. Srivastav: H.C.F and L.C.M – A case study of completeness property of real numbers. *International Journal of Mathematics Trends and Technology*, **30**(1), (2016), 30–33. DOI: 10.14445/22315373/IJMTT-V30P505

[26] M. Tavakoli and M. Stabatabaei: Controllability and observability analysis of continuous-time multi-order fractional systems. *Multidimensional Systems and Signal Processing*, **28**, (2017), 427–450.

[27] A. Tavares: Answer: GCD and LCM of fractions. (2011). URL: https://math.stackexchange.com/questions/44836/gcd-and-lcm-of-fractions (visited on 04/25/2025).

[28] D. Xu, Y. Li and W. Zhou: Controllability and observability of fractional linear systems with two different orders. In *Hindawi Publishing Corporation Thee Scientific World Journal*, (2014). DOI: 10.1155/2014/618162

[29] D. Xue, C. Zhao and Y. Chen: A modified approximation method of fractional order system. In *2006 International Conference on Mechatronics and Automation*, Luoyang, Henan, China, (2006), 1043–1048. DOI: 10.1109/ICMA.2006.257769